**Approximate solutions of boundary value problems of fractional order by using sinc-Galerkin method**

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**Abstract:** The aim of the present study is to obtain approximate solutions of fractional order linear two-point boundary value problem which are generalizations of classical boundary value problems by using sinc-Galerkin method. The fractional derivatives are defined in the Caputo sense using frequently in fractional calculus. The method is tested on some problems with homogeneous and nonhomogeneous boundary conditions and comparisons are made with the exact solutions and numerical solutions obtained by Haar Wavelet method. Numerical and graphical results show that the sinc-Galerkin method is a very effective and powerful tool in solving such problems.

**Keywords:** Fractional order boundary value problem, sinc-Galerkin method, Caputo derivative.

1. **Introduction**

Fractional calculus, which might be considered as an extension of classical calculus, are as old as the classical calculus and fractional differential equations have been frequently used to describe many scientific phenomena in earthquake engineering, biomedical engineering, image processing, signal processing and physics.

Many numerical methods are developed because of the obtaining exact solution of ordinary differential equations of fractional order is more difficult than the obtaining one of ordinary differential equations of integer order. These methods include Haar wavelet method [1], Adomian decomposition method [2], spline collocation method [3], least squares finite-element method [4], variational iteration method [5], generalized differential transform method [6], extrapolation method [7], Chebyshev wavelet method [8] and Legendre wavelet method [9].

In this paper, we use sinc-Galerkin method to obtain approximate solutions of fractional linear two-point boundary value problem

\[ \frac{C}{a}D_x^\alpha y + y = f(x), \quad a < x < b, \quad 1 < \alpha < 2 \]

with boundary conditions

\[ y(a) = y(b) = 0 \]

where \( \frac{C}{a}D_x^\alpha \) is Caputo fractional derivative operator.

The paper is organized as follows. Section 2 presents basic theorems for fractional calculus and sinc-Galerkin method. In Section 3, we use the sinc-Galerkin method to obtain an approximate solution of a general fractional two-point boundary value problem. In Section 4, we present two examples in order to illustrate the effective and accuracy of the present

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method. The obtained results are compared with the Haar wavelet method in the table and graphical forms.

2. Preliminaries

2.1 Fractional Calculus

In this section, firstly we present the definitions of the Riemann-Liouville and the Caputo of fractional derivative. Later, we give the definition of the integration by parts of fractional order by using these definitions.

Definition 2.1\[10\] Let \( f : [a, b] \rightarrow \mathbb{R} \) be a function, \( \alpha \) a positive real number, \( n \) the integer satisfying \( n - 1 \leq \alpha < n \), and \( \Gamma \) the Euler gamma function. Then,

1. The left and right Riemann-Liouville fractional integrals of order \( \alpha \) of a function \( f(x) \) are given as

\[
\begin{align*}
\mathcal{I}_a^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \, dt, \\
\mathcal{I}_b^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) \, dt,
\end{align*}
\]

respectively.

2. The left and right Riemann-Liouville fractional derivatives of order \( \alpha \) of \( f(x) \) are given as

\[
\begin{align*}
\mathcal{D}_a^\alpha f(x) &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t) \, dt, \\
\mathcal{D}_b^\alpha f(x) &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b (t-x)^{n-\alpha-1} f(t) \, dt,
\end{align*}
\]

respectively.

3. The left and right Caputo fractional derivatives of order \( \alpha \) of \( f(x) \) are given as

\[
\begin{align*}
\mathcal{C}_a^\alpha f(x) &= \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t) \, dt, \\
\mathcal{C}_b^\alpha f(x) &= \frac{1}{\Gamma(n-\alpha)} \int_x^b (-1)^n (t-x)^{n-\alpha-1} f^{(n)}(t) \, dt,
\end{align*}
\]

respectively.

In private, if \( f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0 \), then

\[
\mathcal{C}_a^\alpha f(x) = \mathcal{D}_a^\alpha f(x)
\]

and if \( f(b) = f'(b) = \cdots = f^{(n-1)}(b) = 0 \), then

\[
\mathcal{C}_b^\alpha f(x) = \mathcal{D}_b^\alpha f(x).
\]
Now we can write the definition of integration by parts of fractional order by using the relations given in (2.1)-(2.4).

**Definition 2.2** [10] If $0 < \alpha < 1$ and $f$ is a function such that $f(a) = f(b) = 0$, we can write

$$\int_a^b g(x) C_a^\alpha D_b f(x) \, dx = \int_a^b f(x) C_a^\alpha D_b g(x) \, dx$$

and

$$\int_a^b g(x) C_a^\alpha D_b f(x) \, dx = \int_a^b f(x) C_a^\alpha D_b g(x) \, dx.$$

**Theorem 2.1** [3] Let be $\alpha > 0$ and $n \in \mathbb{N}$ such that $n - 1 < \alpha \leq n$ and $f(x) \in C^n[a, b]$, then

$$C_a^\alpha D_b^n f(x) = a I_{n-\alpha} x f^{(n)}(x).$$

**Theorem 2.2** [3] Let be $\alpha > 0$. If $f$ is continuous, then

$$a D_a^\alpha a I_{\alpha} f(x) = f(x).$$

### 2.2 Sinc basis functions properties and quadrature interpolations

In this section, we recall notations and definitions of the sinc function and derive useful formulas that are important for this paper.

#### The Sinc basis functions

**Definition 2.3** [14] The function defined all $z \in \mathbb{C}$ by

$$\text{sinc}(z) = \begin{cases} \frac{\sin(\pi z)}{\pi z}, & z \neq 0, \\ 1, & z = 0. \end{cases} \tag{2.6}$$

is called the sinc function.

**Definition 2.4** [14] Let $f$ be a function defined on $\mathbb{R}$ and let $h > 0$. Define the series

$$C(f, h)(x) = \sum_{k=-\infty}^{\infty} f(kh) \text{sinc} \left( \frac{x - kh}{h} \right) \tag{2.7}$$

where from (2.6)

$$S(k, h)(x) = \text{sinc} \left( \frac{x - kh}{h} \right) = \begin{cases} \frac{\sin(\pi z/k h)}{\pi z/k h}, & x \neq kh, \\ 1, & x = kh. \end{cases}$$

Whenever the series in (2.7) converges it is called the Whittaker cardinal function of $f$. They are based on the infinite strip $D_s$ in the complex plane

$$D_s \equiv \left\{ w = a + iv : |v| < d \leq \frac{\pi}{2} \right\}.$$ 

In general, approximations can be constructed for infinite, semi-infinite and finite intervals. Define the function

$$w = \phi(z) = \ln \left( \frac{z}{1-z} \right) \tag{2.8}$$

which is a conformal mapping from $D_E$, the eye-shaped domain in the $z$-plane, onto the infinite strip $D_s$, where

$$D_E = z = \left\{ x + iy : \arg \left( \frac{z}{1-z} \right) < d \leq \frac{\pi}{2} \right\}.$$
This is shown in Figure 1. For the sinc-Galerkin method, the basis functions are derived from the composite translated sinc functions

\[ S_k(z) = S(k, h)(z) \alpha \phi(z) = \text{sinc} \left( \frac{\phi(z) - kh}{h} \right) \]

for \( z \in D_E \). The function \( z = \phi^{-1}(w) = \frac{-\pi^w}{1 + \pi^w} \) is an inverse mapping of \( w = \phi(z) \). We may define the range of \( \phi^{-1} \) on the real line as

\[ \Gamma = \{ \phi^{-1}(u) \in D_E : -\infty < u < \infty \} \]

the evenly spaced nodes \( \{kh\}_{k=-\infty}^{\infty} \) on the real line. The image which corresponds to these nodes is denoted by

\[ x_k = \phi^{-1}(kh) = \frac{e^{kh}}{1 + e^{kh}}. \]

**Sinc function interpolation and quadratures**

**Definition 2.5** [11] Let \( D_E \) be a simply connected domain in the complex plane \( C \), and let \( \partial D_E \) denote the boundary of \( D_E \). Let \( a, b \) be points on \( \partial D_E \) and \( \phi \) be a conformal map \( D_E \) onto \( D_S \) such that \( \phi(a) = -\infty \) and \( \phi(b) = \infty \). If the inverse map of \( \phi \) is denoted by \( \phi^{-1} \), define

\[ \Gamma = \{ \phi^{-1}(u) \in D_E : -\infty < u < \infty \} \]

and \( z_k = \phi(kh) \), \( k = \pm 1, \pm 2 \).

**Definition 2.6** [11] Let \( B(D_E) \) be the class of functions \( F \) that are analytic in \( D_E \) and satisfy

\[ \int_{\psi(L + u)} |F(z)| \, dz \to 0, \text{ as } u = \mp \infty, \]

where

\[ L = \{ iy : |y| < d \leq \frac{\pi}{2} \}; \]

and those on the boundary of \( D_E \) satisfy

\[ T(F) = \int_{\partial D_E} |F(z)| \, dz < \infty. \]

**Theorem 2.5** [11] Let \( \Gamma \) be \((0, 1)\), \( F \in B(D_E) \), then for \( h > 0 \) sufficiently small,

\[ \int_{\Gamma} F(z) \, dz - h \sum_{j=\infty}^{\infty} \frac{F(z_j)}{\phi(z_j)} = i \frac{1}{2} \int_{\partial D} F(z) k(\phi(h)) \, dz \equiv I_F \]  

(2.9)

where
For the sinc-Galerkin method, the infinite quadrature rule must be truncated to a finite sum. The following theorem indicates the conditions under which an exponential convergence results.

**Theorem 2.6** [11] If there exist positive constants $\alpha$, $\beta$ and $C$ such that

$$
\left| \frac{F(x)}{\phi(x)} \right| \leq C \begin{cases} 
e 0, & x \in \psi((-\infty, \infty)) \\
\ne 0, & x \in \psi((0, \infty)) 
\end{cases} 
$$

then the error bound for the quadrature rule (2.9) is

$$
\left| \int_\Gamma F(x) dx - h \sum_{j=-M}^{N} \frac{F(x_j)}{\phi(x_j)} \right| \leq C \left( e^{-\frac{\alpha}{\beta} h} + e^{-\frac{\alpha}{\beta} N h} \right) + |F|.
$$

(2.11)

The infinite sum in (2.9) is truncated with the use of (2.10) to arrive at the inequality (2.11). Making the selections

$$
h = \frac{\pi d}{\alpha M},
$$

$$
N \equiv \left\lfloor \frac{\alpha M}{\beta + 1} \right\rfloor
$$

where $\lfloor \cdot \rfloor$ is an integer part of the statement and $M$ is the integer value which specifies the grid size, then

$$
\int_\Gamma F(x) dx = h \sum_{j=-M}^{N} \frac{F(x_j)}{\phi(x_j)} + O \left( e^{-(\pi a M)^2} \right).
$$

We used these theorems to approximate the integrals that arise in the formulation of the discrete systems corresponding to a second-order boundary value problem.

### 3 The sinc-Galerkin Method

Consider linear two-point boundary value problem

$$
a^\alpha D_x^\alpha y + y = f(x), \quad a < x < b, \quad 1 < \alpha < 2
$$

(3.1)

with boundary conditions

$$
y(a) = y(b) = 0
$$

(3.2)

where $aD_x^\alpha$ is Caputo fractional derivative operator. According to Theorem 2.1, equation (3.1) is written by

$$
a^\alpha D_x^\alpha y + y = f(x), \quad a < x < b, \quad 0 < 2 - \alpha < 1
$$

(3.3)

If we apply $aD_x^{2-\alpha}$ to both sides of equation (3.3) and use Theorem 2.2, we have

$$
y'' + aD_x^\beta y = g(x), \quad a < x < b, \quad 0 < \beta < 1
$$

(3.4)

where $g(x) = aD_x^{2-\alpha}f(x)$ and $\beta = 2 - \alpha$. Also, by using boundary conditions and equality (2.5), equation (3.4) can be written as

$$
y'' + aD_x^\beta y = g(x), \quad a < x < b, \quad 0 < \beta < 1
$$

(3.5)

An approximate solution for $y(x)$ is represented by the formula

$$
y_n(x) = \sum_{j=-M}^{M} c_j S_j(x), \quad n = 2M + 1,
$$

(3.6)

where $S_j(x)$ is the function $S(j, h)\phi(x)$ for some fixed step size $h$. The unknown coefficients $c_j$ in (3.6) are determined by orthogonalizing the residual with respect to the basis functions, i.e.
< y'', S_k > + < \frac{c_j}{\phi_j} D_k^\beta y, S_k > = < g(x), S_k >. \quad (3.7)

The inner product used for the sinc-Galerkin method is defined by

< f, \eta > = \int_a^b f(x) \eta(x) w(x) dx.

where \( w(x) \) is weight function and it is convenient to take

\[ w(x) = \frac{1}{\phi(x)} \]

for the case of second-order problems.

**Lemma 3.1** [15] Let \( \phi \) be the conformal one-to-one mapping of the simply connected domain \( D_E \) onto \( D_S \), given by (2.8). Then

\[
\delta^{(0)}_{jk} = \left[ S(j, h) \phi(x) \right]_{x=x_k} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}
\]

\[
\delta^{(1)}_{jk} = h \frac{d}{d\phi} \left[ S(j, h) \phi(x) \right]_{x=x_k} = \begin{cases} 0, & j = k \\ \left( -1 \right)^{j-k}, & j \neq k \end{cases}
\]

\[
\delta^{(2)}_{jk} = h^2 \frac{d^2}{d\phi^2} \left[ S(j, h) \phi(x) \right]_{x=x_k} = \begin{cases} -\frac{\pi^2}{3} \left( 2 \left( -1 \right)^{j-k} \right), & j \neq k \end{cases}
\]

The method of approximating the integrals in (3.6) begins by integrating by parts to transfer all derivatives from \( y \) to \( S_k \). The following theorems which can easily prove by using Lemma 3.1 and Definition 2.2 are used to solve Equation (3.1).

**Theorem 3.2** [12] The following relations hold

\[
< y'', S_k > \approx h^2 \sum_{j=-M}^{M} \frac{y(x_j)}{\phi(x_j)} \delta^{(0)}_{jk} g_{2,j}, \quad (3.8)
\]

and

\[
< g, S_k > \approx h^2 \frac{g(x_k)}{\phi(x_k)} w(x_k), \quad (3.9)
\]

where

\[
g_{2,2} = w\left( \phi^2 \right), \quad g_{2,1} = w^2 \phi'' + 2w' \phi', \quad g_{2,0} = w''.
\]

**Theorem 3.3** For \( 0 < \beta < 1 \), the following relations hold:

\[
< \frac{c_j}{\phi_j} D_k^\beta y(x), S_k > \approx -\frac{h}{\Gamma(1-\beta)} \sum_{j=-M}^{M} \frac{y(x_j)}{\phi(x_j)} \frac{d}{dx} \left[ h_L \sum_{r=-L}^{L} \frac{x_r - x}{\xi(x_r)} \frac{K(x_r)}{\xi(x_r)} \right]_{x=x_j}. \quad (3.10)
\]

where \( K(x) = S_k(x) w(x) \), \( \xi(t) = \ln \left( \frac{1}{t^2} \right) \) and \( h_L = \pi/\sqrt{L} \).

**Proof:** See [16].

Replacing each term of (3.7) with the approximation defined in (3.8) – (3.10), replacing \( y(x_j) \) by \( c_j \) and dividing by \( h \) we obtain the following theorem.
Theorem 3.4 If the assumed approximate solution of the boundary-value problem (3.5) is (3.6), then the discrete sinc–Galerkin system for the determination of the unknown coefficients \( \{c_j\}_{j=-M}^M \) is given by

\[
\sum_{j=-M}^{M} \left( \frac{2}{M} \sum_{i=0}^{M} \theta_{ik} g_{ij}(x_j) \right) c_j = \frac{1}{1 - \beta} \phi_i(x_j) \int_{-L}^{L} (x_j - x)^{-\beta} K(x_j, x) \phi'(x) \text{d}x \left|_{x=x_j} \right.
\]

where \( K(x) = S_k(x) w(x) \), \( \xi(t) = \ln \left( \frac{t - \alpha}{\beta} \right) \).

4 Examples
In this section, two problems that have homogeneous and nonhomogeneous boundary conditions will be tested by using the present method. In all the examples, we take \( d = \pi/2 \), \( \alpha = 1/2 \), \( N = M \).

Example 4.1 [17] Consider linear fractional boundary value problem

\[
\frac{\alpha}{\beta} D_{x}^{1.2} y(x) + \frac{3}{57} y(x) = f(x),
\]

subject to the nonhomogeneous boundary conditions

\[
y(0) = 0, \quad y(1) = \frac{1}{\Gamma(3.2)}
\]

where \( f(x) = x + \frac{3 x^2}{\Gamma(3.2)} \). The exact solution of this problem is \( y(x) = \frac{x^2}{\Gamma(3.2)} \). First we convert the nonhomogeneous boundary conditions to homogeneous conditions by considering the transformation \( x = \frac{y(x)}{\frac{1}{\Gamma(3.2)}} \). This change of variable yields the following boundary value problem

\[
\frac{\alpha}{\beta} D_{x}^{1.2} u(x) + \frac{3}{57} u(x) = g(x),
\]

with homogeneous boundary conditions

\[
u(0) = 0, \quad u(1) = 0
\]

where

\[
g(x) = x + \frac{3}{57} \frac{x^2}{\Gamma(3.2)} - \frac{x}{\Gamma(3.2)}
\]

According to Theorem 2.1, equation (4.1) can be written as

\[
\frac{\alpha}{\beta} D_{x}^{0.8} u''(x) + \frac{3}{57} u(x) = g(x)
\]

Acting with \( D_{x}^{0.8} \) on both sides of this equation and using Theorem 2.2, we obtain

\[
\frac{\alpha}{\beta} D_{x}^{0.8} u''(x) + \frac{3}{57} D_{x}^{0.8} u(x) = r(x)
\]

where \( r(x) = \frac{\alpha}{\beta} D_{x}^{0.8} g(x) = 0.217825 \times 4.89143 r^{0.2} + 0.194517 r^{1.4} \)

Finally, by using boundary conditions and (2.5), equation (4.2) can be written

\[
\frac{\alpha}{\beta} D_{x}^{0.8} u''(x) + \frac{3}{57} D_{x}^{0.8} u(x) = r(x)
\]

subject to the homogeneous boundary conditions

\[
u(0) = 0, \quad u(1) = 0
\]

The numerical solutions which are obtained by using the sinc-Galerkin method (SGM) for this problem are presented in Table 1. In addition to, in Table 2, the solutions are compared with the numerical solutions computed by using Haar Wavelet method (HWM). Also the graphs of exact and approximate solutions for different values of Land Mare
presented in Figure 2.

Table 1 Numerical results for \( L = 5, \ M = 5 \)

<table>
<thead>
<tr>
<th>x</th>
<th>Exact Sol.</th>
<th>Num. Sol.</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.002603</td>
<td>0.003024</td>
<td>0.000421008</td>
</tr>
<tr>
<td>0.2</td>
<td>0.011960</td>
<td>0.011040</td>
<td>0.000920051</td>
</tr>
<tr>
<td>0.3</td>
<td>0.029183</td>
<td>0.028462</td>
<td>0.000721595</td>
</tr>
<tr>
<td>0.4</td>
<td>0.054954</td>
<td>0.055164</td>
<td>0.000210129</td>
</tr>
<tr>
<td>0.5</td>
<td>0.089785</td>
<td>0.090523</td>
<td>0.000738097</td>
</tr>
<tr>
<td>0.6</td>
<td>0.134093</td>
<td>0.134454</td>
<td>0.000361033</td>
</tr>
<tr>
<td>0.7</td>
<td>0.188230</td>
<td>0.187585</td>
<td>0.000645323</td>
</tr>
<tr>
<td>0.8</td>
<td>0.252506</td>
<td>0.251336</td>
<td>0.001169680</td>
</tr>
<tr>
<td>0.9</td>
<td>0.327195</td>
<td>0.327288</td>
<td>0.000092972</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2 Numerical results for \( L = 40, \ M = 100 \)

<table>
<thead>
<tr>
<th>x</th>
<th>Exact Sol.</th>
<th>Num. Sol.</th>
<th>Error(SGM)</th>
<th>Error(HWM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.002603</td>
<td>0.002609</td>
<td>6.29*10^-6</td>
<td>1.53*10^-6</td>
</tr>
<tr>
<td>0.2</td>
<td>0.011960</td>
<td>0.011969</td>
<td>8.95*10^-6</td>
<td>1.52 *10^-7</td>
</tr>
<tr>
<td>0.3</td>
<td>0.029183</td>
<td>0.029192</td>
<td>8.66*10^-6</td>
<td>8.07 *10^-7</td>
</tr>
<tr>
<td>0.4</td>
<td>0.054954</td>
<td>0.054961</td>
<td>6.43*10^-6</td>
<td>6.31*10^-7</td>
</tr>
<tr>
<td>0.5</td>
<td>0.089785</td>
<td>0.089789</td>
<td>3.23*10^-6</td>
<td>5.19*10^-7</td>
</tr>
<tr>
<td>0.6</td>
<td>0.134093</td>
<td>0.134093</td>
<td>2.46*10^-7</td>
<td>1.82*10^-6</td>
</tr>
<tr>
<td>0.7</td>
<td>0.188230</td>
<td>0.188227</td>
<td>2.83*10^-6</td>
<td>2.43*10^-6</td>
</tr>
<tr>
<td>0.8</td>
<td>0.252506</td>
<td>0.252502</td>
<td>4.40*10^-6</td>
<td>3.11*10^-6</td>
</tr>
<tr>
<td>0.9</td>
<td>0.327195</td>
<td>0.327191</td>
<td>4.35*10^-6</td>
<td>3.96*10^-6</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 2. Graphs of exact and approximate solutions for different values of \( L \) and \( M \): left figure for \( L = 5 \) and \( M = 5 \) and right figure for \( L = 40 \) and \( M = 100 \).

Example 4.2 [13] Consider linear fractional boundary value problem
\[ C_0 \mathcal{D}^{1.5}_x y(x) + y(x) = f(x) \]  
(4.3)

subject to the homogeneous boundary conditions

\[ y(0) = 0, \ y(1) = 0 \]

where \( f(x) = x^5 - x^4 + \frac{128}{\sqrt{\pi}} x^{3.5} - \frac{64}{5\sqrt{\pi}} x^{2.5} \). The exact solution of this problem is \( y(x) = x^4 (x - 1) \). According to Theorem 2.1, equation (4.3) can be written

\[ \mathcal{D}^{0.5}_x y(x) + y(x) = f(x) \]

Acting with \( \mathcal{D}^{0.5}_x \) on both sides of this equation and using Theorem 2.2, we obtain

\[ y''(x) + \mathcal{D}^{0.5}_x y(x) = g(x) \]  
(4.4)

where \( g(x) = \mathcal{D}^{0.5}_x f(x) = \frac{256}{63\sqrt{\pi}} x^{4.5} - \frac{128}{35\sqrt{\pi}} x^{3.5} + 20x^3 - 12x^2 \).

Finally, by using boundary conditions and (2.5), equation (4.4) can be written

\[ y''(x) + C_0 \mathcal{D}^{0.5}_x y(x) = g(x) \]

subject to the homogeneous boundary conditions

\[ y(0) = 0, \ y(1) = 0 \]

The numerical solutions which are obtained by using the sinc-Galerkin method (SGM) for this problem are presented in Table 3 and Table 4. In addition to, in Figure 3, the graphs of exact and approximate solutions for different values of \( \text{Land} \ M \) are presented.

**Table 3** Numerical results for \( L = 5, \ M = 5x \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>Exact Sol.</th>
<th>Num. Sol.</th>
<th>Error</th>
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Table 4 Numerical results for $L=40$, $M=100$

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<th>Error</th>
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</table>

Figure 3. Graphs of exact and approximate solutions for different values of $L$ and $M$: left figure for $L = 5$ and $M = 5$ and right figure for $L = 40$ and $M = 100$.

5 Conclusion
In this study, we use sinc-Galerkin method to obtain approximate solutions of two-point boundary value problems for linear fractional differential equations with constant coefficients. In order to illustrate the effective and accuracy of the present method, it is applied to some special examples in the literature and the obtained results are compared with exact solutions and also with the solutions obtained by the Haar wavelet method. As a result, it is shown that sinc-Galerkin method is very effective and reliable for obtaining approximate solutions of fractional boundary value problems.

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References


