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Some Hermite-Hadamard inequalities for *beta*-convex and its fractional applications

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Received: 24 April 2015, Revised: 23 May 2015, Accepted: 20 July 2015
Published online: 29 October 2015

Abstract: In this paper, the authors defined a new concept so called *beta*-convex function and compared with the other definitions of convexity. They also established some inequalities of Hadamard type via ordinary and Riemann-Liouville integral.

Keywords: Hermite-Hadamard inequalities, convex function, *beta*-convex function, *fractional* integral inequalities, beta function.

1 Introduction

The Hermite-Hadamard inequality asserts that the mean value of a continuous convex function $f:[\nu,\upsilon]\subseteq\mathbb{R}\to\mathbb{R}$ lies between the value of f at the midpoint of the interval $[\nu,\upsilon]$ and the arithmetic mean of the values of f at the endpoints of this interval, that is,

$$f\left(\frac{\nu+\upsilon}{2}\right)\leq\frac{1}{\upsilon-\nu}\int_{\nu}^{\upsilon}f(x)\,dx\leq\frac{f(\nu)+f(\upsilon)}{2}. \tag{1}$$

Moreover, each side of this double inequality characterizes convexity in the sense that a real-valued continuous function f defined on an interval I is convex if its restriction to each compact subinterval $[\nu,\upsilon]\subset I$ verifies the left hand side of (1) (equivalently, the right hand side of (1)). If f is a positive concave function, then the inequality is reversed. See [1] and [21] for details.

Definition 1. [17] *A function $f:I\subset\mathbb{R}\to\mathbb{R}$ is said to be convex on I if the inequality*

$$f(t\nu+(1-t)\upsilon)\leq tf(\nu)+(1-t)f(\upsilon) \tag{2}$$

holds for all $\nu,\upsilon\in I$ and $t\in[0,1]$. We say that f is concave if $-f$ is convex.

Definition 2. [8] *We say that a map $f:I\subset\mathbb{R}\to\mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and, for all $\nu,\upsilon\in I$ with $\nu<\upsilon$ and $t\in[0,1]$, satisfies the following inequality*

$$f(t\nu+(1-t)\upsilon)\leq f(\nu)+f(\upsilon). \tag{3}$$

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Definition 3. [14] *A function* $f : I \subset \mathbb{R} \to \mathbb{R}$ *is said to be Godunova-Levin function, if*

$$f(t\nu + (1-t)\upsilon) \leq \frac{f(\nu)}{t} + \frac{f(\upsilon)}{1-t} \tag{4}$$

holds for all $\nu, \upsilon \in I$ *with* $\nu < \upsilon$ *and* $t \in [0,1]$.

For some useful details and extensions of Godunova-Levin functions, see [10], [11], [12], [14], [16], [22], [23], [26].

Definition 4. [22] *A function* $f : I \subset \mathbb{R} \to \mathbb{R}$ *is said to be s-Godunova-Levin function of first kind, if*

$$f(t\nu + (1-t)\upsilon) \leq \frac{f(\nu)}{t^s} + \frac{f(\upsilon)}{1-t^s} \tag{5}$$

for all $\nu, \upsilon \in I$ *with* $\nu < \upsilon$ *and* $t \in [0,1]$, $s \in (0,1]$.

It is obvious that for $s = 1$ the definition of s-Godunova-Levin functions of first kind collapses to the definition of Godunova-Levin functions.

Our next definition is established by Dragomir [10],[11].

Definition 5. [10],[11] *A function* $f : I \subset \mathbb{R} \to \mathbb{R}$ *is said to be s-Godunova-Levin functions of second kind, if*

$$f(t\nu + (1-t)\upsilon) \leq \frac{f(\nu)}{t^s} + \frac{f(\upsilon)}{(1-t)^s} \tag{6}$$

holds for all $\nu, \upsilon \in I$, $s \in (0,1]$ *and* $t \in (0,1)$.

It is obvious that for $s = 0$, s-Godunova-Levin functions of second kind reduces to the definition of $P(I)$. If $s = 1$, it then reduces to Godunova-Levin functions.

Definition 6. [24] *A function* $f : \mathbb{R}_+ \to \mathbb{R}$, *where* $\mathbb{R}_+ = [0,\infty)$, *is said to be s-convex in the first sense if*

$$f(\alpha\nu + \beta\upsilon) \leq \alpha^s f(\nu) + \beta^s f(\upsilon) \tag{7}$$

for all $\nu,\ \upsilon \in [0,\infty)$, $\alpha,\beta \geq 0$ *with* $\alpha^s + \beta^s = 1$ *and for some fixed* $s \in (0,1]$. *This class of functions is denoted by* K_s^1 .

Definition 7. [3] *A function* $f : \mathbb{R}_+ \to \mathbb{R}$, *where* $\mathbb{R}_+ = [0,\infty)$, *is said to be s-convex in the second sense if*

$$f(\alpha\nu + \beta\upsilon) \leq \alpha^s f(\nu) + \beta^s f(\upsilon) \tag{8}$$

for all $\nu,\ \upsilon \in [0,\infty)$, $\alpha,\beta \geq 0$ *with* $\alpha + \beta = 1$ *and for some fixed* $s \in (0,1]$. *This class of functions is denoted by* K_s^2.

This definition of s-convexity considered by Breckner, where the problem when the rational s-convex functions are s-convex was considered. Also, we note that, it can be easily seen that for $s = 1$, s-convexity (in both senses) reduces to the ordinary convexity of functions defined on $[0,+\infty)$.

Definition 8. [31] *A function* $f : I \to \mathbb{R}$ *is said to be tgs-convex on I if inequality*

$$f(t\nu + (1-t)\upsilon) \leq t(1-t)[f(\nu) + f(\upsilon)] \tag{9}$$

holds for all $\nu, \upsilon \in I$ and $t \in (0,1)$. We say that f is tgs-concave if $-f$ is tgs-convex.

Definition 9. [32] *Let $h : J \to R$ be a non-negative function, $h \neq 0$. We say that $f : I \to \mathbb{R}$ is an h-convex function, or that f belongs to the class $S\nu(h,I)$, if f is non-negative and for all ν, $\upsilon \in I$, $\alpha \in (0,1)$ we have*

$$f(\alpha\nu + (1-\alpha)\upsilon) \leq h(\alpha) f(\nu) + h(1-\alpha) f(\upsilon). \tag{10}$$

If inequality (10) is reversed, then f is said to be h-concave, i.e. $f \in SV(h,I)$.

Obviously, if $h(\alpha) = \alpha$, then all non-negative convex functions belong to $SX(h,I)$ and all nonnegative concave functions belong to $SV(h,I)$; if $h(\alpha) = \frac{1}{\alpha}$, then $SX(h,I) = Q(I)$; if $h(\alpha) = 1$, then $SX(h,I) \supseteq P(I)$; and if $h(\alpha) = \alpha^s$, where $s \in (0,1)$, then $SX(h,I) \supseteq K_s^2$.

Remark 1. Let h be a non-negative function such that

$$h(\alpha) \geq \alpha$$

for all $\alpha \in (0,1)$. For example, the function $h_k(\nu) = \nu^k$ where $k \leq 1$ and $\nu > 0$ has that property. If f is a non-negative convex function on I , then for ν, $\upsilon \in I$, $\alpha \in (0,1)$ we have

$$f(\alpha\nu + (1-\alpha)\upsilon) \leq \alpha f(\nu) + (1-\alpha) f(\upsilon) \leq h(\alpha) f(\nu) + h(1-\alpha) f(\upsilon).$$

So, $f \in SX(h,I)$. Similarly, if the function h has the property: $h(\alpha) \leq \alpha$ for all $\alpha \in (0,1)$, then any non-negative concave function f belongs to the class $SV(h,I)$.

Definition 10. ***(Beta function)*** *In mathematics, the beta function, also called the Euler integral of the first kind, is a special function defined by*

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \;\; x,y > 0. \tag{11}$$

The beta function is symmetric, meaning that

$$\beta(x,y) = \beta(y,x).$$

When x and y are positive integers, it follows from the definition of the gamma function Γ that:

$$\beta(x,y) = \frac{(x-1)!\,(y-1)!}{(x+y-1)!}$$

It has many other forms, including:

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{12}$$

$$\beta(x,y) = 2\int_0^{\pi/2} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}\,d\theta, \;\; Re(x) > 0, Re(y) > 0$$

$$\beta(x,y)=\int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt,\ Re(x)>0, Re(y)>0$$
$$\beta(x,y)=\sum_{n=0}^{\infty}\frac{\binom{n-y}{n}}{x+n},$$
$$\beta(x,y)=\frac{x+y}{xy}\prod_{n=1}^{\infty}(1+\frac{xy}{n(x+y+n)})^{-1}.$$

In this paper, we defined a new concept so called *beta*-convex function and compared with the other definitions of convexity. We also established some inequalities of Hadamard type via ordinary and Riemann-Liouville integral.

Definition 11. *A function $f:I\subset\mathbb{R}\to\mathbb{R}$ is said to be beta-convex on I, if inequality*

$$f(tu+(1-t)v)\le t^p(1-t)^q f(u)+t^q(1-t)^p f(v) \tag{13}$$

holds for all $u,v\in I$, and $t\in[0,1]$, where $p,q>-1$. We say that f is beta-concave if $(-f)$ is beta-convex.

Remark 2. In above definition $(p,q)=\{(0,0),(1,0),(-1,0),(-s,0),(s,0),(1,1)\}$, we obtain $P(I)$, ordinary convex, Godunova-Levin function, s-Godunova-Levin function in the second sense, s-convex function in the second sense, tgs-convex function, respectively.

Remark 3. If we take $h(t)=t^p(1-t)^q$ in Definition 9, Definition 9 reduces to Definition 11. Namely, Definition of *beta*-convex function may be regarded as a special case of h-convex function (see [32]).

2 Main results

Throughout this section we assume that α has the values that do not undefined the gamma function and the result. Now, we begin with the theorem via Hadamard type inequality of *beta*-convex function.

Theorem 1. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a beta-convex function, $p,q>-1$ and $a,b\in I$ with $a<b$. The following double inequality holds:*

$$2^{p+q-1}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}[f(a)+f(b)] \tag{14}$$

If f is a positive beta-concave function, then the inequality is reversed.

Proof. Using *beta*-convexity of f on I we have

$$f(ta+(1-t)b)\le t^p(1-t)^q f(a)+t^q(1-t)^p f(b) \tag{15}$$

Integrating both sides of (15), we get

$$\begin{aligned}\int_0^1 f(ta+(1-t)b)\,dt&\le\int_0^1[t^p(1-t)^q f(a)+t^q(1-t)^p f(b)]\,dt\\&=f(a)\int_0^1 t^p(1-t)^q dt+f(b)\int_0^1 t^q(1-t)^p dt\\&=f(a)\beta(p+1,q+1)+f(b)\beta(p+1,q+1)\end{aligned}$$

$$
\begin{aligned}
&= \beta(p+1,q+1)[f(a)+f(b)] \\
&= \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}[f(a)+f(b)].
\end{aligned}
$$

Using (15) and substituting $a = ta + (1-t)b$, $b = (1-t)a + t$, we obtain

$$
\begin{aligned}
f\left(\frac{x+y}{2}\right) &= f\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right) \\
&\leq \frac{1}{2^{p+q}}[f(ta+(1-t)b) + f((1-t)a+tb)].
\end{aligned}
$$

Integrating both sides we have

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2^{p+q}}\left[\int_0^1 f(ta+(1-t)b)\,dt + \int_0^1 f((1-t)a+tb)\,dt\right] \\
&= \frac{1}{2^{p+q}}\left[\frac{1}{b-a}\int_a^b f(x)\,dx + \frac{1}{b-a}\int_a^b f(x)\,dx\right] \\
&= \frac{1}{2^{p+q-1}(b-a)}\int_a^b f(x)\,dx.
\end{aligned}
$$

which completes the proof.

Remark 4. **1)** If we choose $p = 0$, $q = 0$ in (14), then

$$
\frac{1}{(b-a)}\int_a^b f(x)\,dx \leq [f(a)+f(b)].
$$

2) If we choose $p = 0$, $q = 1$ or $p = 1$, $q = 0$ in (14), then (14) reduces to (1).

3) If we choose $p = 1$, $q = 1$ in (14), then (14) reduces

$$
2f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)}\int_a^b f(x)\,dx \leq \left[\frac{f(a)+f(b)}{6}\right].
$$

See [31].

4) If we choose $p = s$, $q = 0$ and $s \in (0,1]$ in (14), then (14) reduces

$$
2^{s-1}f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{s+1}.
$$

See [13].

5) If we choose $p = 2$, $q = 2$ in (14), then

$$
\frac{1}{(b-a)}\int_a^b f(x)\,dx \leq \left[\frac{f(a)+f(b)}{30}\right].
$$

6) If we choose $p = 10$, $q = 10$ in (14), then

$$
\frac{1}{(b-a)}\int_a^b f(x)\,dx \leq \left[\frac{f(a)+f(b)}{3879876}\right].
$$

Theorem 2. *Let f and g be real valued, nonnegative and beta-convex functions on $[a,b]$ and $p,q>-1$. Then, we have*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \frac{\Gamma(2p+1)\Gamma(2q+1)}{\Gamma(2p+2q+2)}M(a,b)+\frac{\Gamma(p+q+1)\Gamma(p+q+1)}{\Gamma(2p+2q+2)}N(a,b) \tag{16}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$, $N(a,b)=f(a)g(b)+f(b)g(a)$.

Proof. Using *beta*-convexity of f and g on $[a,b]$, we have

$$f(ta+(1-t)b) \le t^p(1-t)^q f(a)+t^q(1-t)^p f(b) \tag{17}$$
$$g(ta+(1-t)b) \le t^p(1-t)^q g(a)+t^q(1-t)^p g(b). \tag{18}$$

From (17) and (18) we obtain

$$f(ta+(1-t)b)g(ta+(1-t)b) \le [t^p(1-t)^q f(a)+t^q(1-t)^p f(b)]\times[t^p(1-t)^q g(a)+t^q(1-t)^p g(b)] \tag{19}$$

Integrating both sides of (19), we have

$$\begin{aligned}\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt &\le \int_0^1 [t^p(1-t)^q f(a)+t^q(1-t)^p f(b)]\times[t^p(1-t)^q g(a)+t^q(1-t)^p g(b)]dt\\ &\le \int_0^1 t^{2p}(1-t)^{2q} f(a)g(a)dt+\int_0^1 t^{2q}(1-t)^{2p} f(b)g(b)dt\\ &+\int_0^1 t^{p+q}(1-t)^{p+q} f(b)g(a)dt+\int_0^1 t^{p+q}(1-t)^{p+q} f(a)g(b)dt\end{aligned}$$

so

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \beta(2p+1,2q+1)[f(a)g(a)+f(b)g(b)]+\beta(p+q+1,p+q+1)[f(b)g(a)+f(a)g(b)].$$

Using (12) we get the desired result.

Remark 5. **1)** If we choose $p=0$, $q=0$ in (16), then

$$\frac{1}{(b-a)}\int_a^b f(x)g(x)dx \le [M(a,b)+N(a,b)].$$

2) If we choose $p=0$, $q=1$ or $p=1$, $q=0$ in (16), then it reduces [25, Theorem 1.].

3) If we choose $p=s$, $q=0$ in (16), then

$$\frac{1}{(b-a)}\int_a^b f(x)g(x)dx \le \frac{1}{2s+1}M(a,b)+\frac{(s!)^2}{(2s+1)!}N(a,b).$$

4) If we choose $p = 1$, $q = 1$ in (16), then it reduces [31, Theorem 3].

5) If we choose $p = 2$, $q = 2$ in (16), then

$$\frac{1}{(b-a)}\int_a^b f(x)g(x)\,dx \le \left[\frac{M(a,b)+N(a,b)}{630}\right].$$

6) If we choose $p = 10$, $q = 10$ in (16), then

$$\frac{1}{(b-a)}\int_a^b f(x)g(x)\,dx \le \left[\frac{M(a,b)+N(a,b)}{5651\,707\,681\,620}\right].$$

Theorem 3. *Let f and g be real valued, nonnegative and beta-convex functions on $[a,b]$ and $p,q > -1$. Then, we have*

$$\begin{aligned} 2^{2(p+q)-1} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \le & \frac{1}{(b-a)}\int_a^b f(x)g(x)\,dx \\ & + \frac{\Gamma(p+q+1)\Gamma(p+q+1)}{\Gamma(2p+2q+2)} M(a,b) \\ & + \frac{\Gamma(2p+1)\Gamma(2q+1)}{\Gamma(2p+2q+2)} N(a,b). \end{aligned} \tag{20}$$

Proof. By substituting $a = ta + (1-t)b$, $b = (1-t)a + tb$ we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right) \\ g\left(\frac{a+b}{2}\right) &= g\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right). \end{aligned}$$

Since f and g are *beta*-convex on $[a,b]$, then we obtain

$$\begin{aligned} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) &= f\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right) \times g\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right) \\ &\le \frac{1}{2^{p+q}}[f(ta+(1-t)b) + f((1-t)a+tb)] \times \frac{1}{2^{p+q}}[g(ta+(1-t)b) + g((1-t)a+tb)] \\ &= \frac{1}{2^{2p+2q}}[f(ta+(1-t)b)g(ta+(1-t)b) + f((1-t)a+tb)g((1-t)a+tb)] \\ &+ \frac{1}{2^{2p+2q}}\{[t^p(1-t)^q f(a) + t^q(1-t)^p f(b)][(1-t)^p t^q g(a) + (1-t)^q t^p g(b)] \\ &+ [(1-t)^p t^q f(a) + (1-t)^q t^p f(b)][t^p(1-t)^q g(a) + t^q(1-t)^p g(b)]\} \\ &= \frac{1}{2^{2p+2q}}[f(ta+(1-t)b)g(ta+(1-t)b) + f((1-t)a+tb)g((1-t)a+tb)] \\ &+ \frac{1}{2^{2p+2q-1}}\left[(1-t)^{p+q}t^{p+q}(f(a)g(a) + f(b)g(b))\right. \\ &\left.+ t^{2p}(1-t)^{2q}(f(a)g(b) + f(b)g(a))\right]. \end{aligned} \tag{21}$$

Again as explained in the proof of inequality (16) given above we integrate both sides of (21) over $[0,1]$ and obtain

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) &\le \frac{1}{2^{2p+2q}}\left[\int_0^1 f(ta+(1-t)b)\,g(ta+(1-t)b)\,dt\right.\\
&\left.+\int_0^1 f((1-t)a+tb)\,g((1-t)a+tb)\,dt\right]\\
&+\frac{1}{2^{2p+2q-1}}\left[[f(a)g(a)+f(b)g(b)]\int_0^1 (1-t)^{p+q}t^{p+q}dt\right.\\
&\left.+[f(a)g(b)+f(b)g(a)]\int_0^1 t^{2p}(1-t)^{2q}dt\right]\\
&\le \frac{1}{2^{2p+2q-1}}\int_0^1 f(ta+(1-t)b)\,g(ta+(1-t)b)\,dt\\
&+\frac{1}{2^{2p+2q-1}}\left[\beta(p+q+1,p+q+1)M(a,b)+\beta(2p+1,2q+1)N(a,b)\right]\\
&\le \frac{1}{2^{2p+2q-1}}\left[\frac{1}{(b-a)}\int_a^b f(x)g(x)\,dt\right.\\
&+\frac{\Gamma(p+q+1)\Gamma(p+q+1)}{\Gamma(2p+2q+2)}M(a,b)\\
&\left.+\frac{\Gamma(2p+1)\Gamma(2q+1)}{\Gamma(2p+2q+2)}N(a,b)\right].
\end{aligned}
$$

Now multiplying both sides by $2^{2p+2q-1}$ we get (20).

Remark 6.

1) If we choose $p=0$, $q=1$ or $p=1$, $q=0$ in (20), then it reduces [25, Theorem 1].

2) If we choose $p=1$, $q=1$ in (20), then it reduces [31, Theorem 4].

Theorem 4. *Let f and g be real valued, nonnegative and beta-convex functions on $[a,b]$. The following inequality holds:*

$$
\begin{aligned}
&\frac{\Gamma(2p+2q+2)}{2\Gamma(2p+1)\Gamma(2q+1)(b-a)^2}\times\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)\,g(tx+(1-t)y)\,dtdydx \qquad (22)\\
&\le \frac{1}{b-a}\int_a^b f(x)g(x)\,dx+\frac{\Gamma(p+q+1)\Gamma(p+q+1)}{\Gamma(2p+1)\Gamma(2q+1)(b-a)^2}\left[\frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}\right]^2[M(a,b)+N(a,b)]
\end{aligned}
$$

where $M(a,b)$ and $N(a,b)$ are as defined in Theorem 2.

Proof. Using *beta*-convexity of f and g on $[a,b]$ we have

$$f(tx+(1-t)y)\le t^p(1-t)^q f(x)+t^q(1-t)^p f(y) \qquad (23)$$

$$g(tx+(1-t)b)\le t^p(1-t)^q g(x)+t^q(1-t)^p g(y) \qquad (24)$$

for $x,y\in[a,b]$ and $t\in[0,1]$. Multiplying both sides of (23) and (24) and integrating over $[0,1]$, we have

$$\int_0^1 f(tx+(1-t)y)\,g(tx+(1-t)y)\,dt \le \int_0^1 t^{2p}(1-t)^{2q} f(x)g(x)\,dt + \int_0^1 t^{2q}(1-t)^{2p} f(y)g(y)\,dt \tag{25}$$
$$+\int_0^1 t^{p+q}(1-t)^{p+q}\left[f(x)g(y)+f(y)g(x)\right]dt$$
$$=\beta(2p+1,2q+1)\left[f(x)g(x)+f(y)g(y)\right]+\beta(p+q+1,p+q+1)\left[f(x)g(y)+f(y)g(x)\right].$$

Integrating both sides of (25) on $[a,b]\times[a,b]$, we obtain

$$\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)\,g(tx+(1-t)y)\,dtdydx \le \beta(2p+1,2q+1)\left[\int_a^b\int_a^b f(x)g(x)\,dxdy+\int_a^b\int_a^b f(y)g(y)\,dydx\right]$$
$$+\beta(p+q+1,p+q+1)\times\left[\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(y)\,dy\right)+\left(\int_a^b f(y)\,dy\right)\left(\int_a^b g(x)\,dx\right)\right] \tag{26}$$

By using the right half of the Hadamard's inequality given in (14) on the right side of (26), we have

$$\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)\,g(tx+(1-t)y)\,dtdydx \le 2\beta(2p+1,2q+1)(b-a)\int_a^b f(x)g(x)\,dx$$
$$+2(b-a)^2\beta(p+q+1,p+q+1)\times\beta^2(p+1,q+1)\left[f(a)+f(b)\right]\left[g(a)+b(b)\right]$$
$$=2\frac{\Gamma(2p+1)\Gamma(2q+1)}{\Gamma(2p+2q+2)}(b-a)\int_a^b f(x)g(x)\,dx+2\frac{\Gamma(p+q+1)\Gamma(p+q+1)}{\Gamma(2p+2q+2)}$$
$$\times\left[\frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}\right]^2\left[M(a,b)+N(a,b)\right].$$

Now multiplying both sides by $\frac{\Gamma(2p+2q+2)}{2\Gamma(2p+1)\Gamma(2q+1)(b-a)^2}$, we get required inequality.

Remark 7.

1) If we choose $p=0$, $q=1$ or $p=1$, $q=0$ in (22), then it reduces [25, Theorem 2].

2) If we choose $p=1$, $q=1$ in (22), then it reduces [31, Theorem 5].

Theorem 5. *Let f and g be real valued, nonnegative and beta-convex functions on $[a,b]$. The following inequality holds:*

$$\frac{\Gamma(2p+2q+2)}{\Gamma(2p+1)\Gamma(2q+1)(b-a)} \tag{27}$$
$$\times\int_a^b\int_0^1 f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)dtdx$$
$$\le\frac{1}{(b-a)}\int_a^b f(x)g(x)\,dx+\frac{1}{2^{2p+2q}}\left[M(a,b)+N(a,b)\right]$$
$$+\frac{\Gamma(p+q+1)\Gamma(p+q+1)}{\Gamma(2p+1)\Gamma(2q+1)2^{p+q-1}}\frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}\left[M(a,b)+N(a,b)\right],$$

where $M(a,b)$ and $N(a,b)$ are as defined in Theorem 2.

Proof. Using *beta*-convexity of f and g on $[a,b]$ we have

$$f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)\leq t^{p}(1-t)^{q}f(x)+t^{q}(1-t)^{p}f\left(\frac{a+b}{2}\right) \tag{28}$$

$$g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)\leq t^{p}(1-t)^{q}g(x)+t^{q}(1-t)^{p}g\left(\frac{a+b}{2}\right) \tag{29}$$

for $x\in[a,b]$ and $t\in[0,1]$. From (28) and (29), we obtain

$$\begin{aligned} & f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) \\ & \leq t^{2p}(1-t)^{2q}f(x)g(x)+t^{2q}(1-t)^{2p}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ & +t^{p+q}(1-t)^{p+q}\left[f(x)g\left(\frac{a+b}{2}\right)+f\left(\frac{a+b}{2}\right)g(x)\right]. \end{aligned} \tag{30}$$

As explained in the proof of inequality (16) given above, if we integrate both sides of (30) over $[0,1]$, then

$$\begin{aligned} & \int_0^1 f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)dt \\ & \leq \int_0^1 t^{2p}(1-t)^{2q}f(x)g(x)\,dt+\int_0^1 t^{2q}(1-t)^{2p}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)dt \\ & +\int_0^1 t^{p+q}(1-t)^{p+q}\left[f(x)g\left(\frac{a+b}{2}\right)+f\left(\frac{a+b}{2}\right)g(x)\right]dt \\ & =\beta(2p+1,2q+1)\left[f(x)g(x)+f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\right] \\ & +\beta(p+q+1,p+q+1)\left[f(x)g\left(\frac{a+b}{2}\right)+f\left(\frac{a+b}{2}\right)g(x)\right]. \end{aligned}$$

Integrating both sides on $[a,b]$, we have

$$\begin{aligned} & \int_a^b\int_0^1 f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)dtdx \\ & \leq \beta(2p+1,2q+1)\int_a^b f(x)g(x)\,dx \\ & +\beta(2p+1,2q+1)(b-a)f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ & +\beta(p+q+1,p+q+1)\left[g\left(\frac{a+b}{2}\right)\int_a^b f(x)\,dx+f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx\right]. \end{aligned}$$

From Definition 11 and Theorem 2, we have

$$\begin{aligned}
&\int_a^b\int_0^1 f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)dtdx \\
&\leq \beta(2p+1,2q+1)\int_a^b f(x)g(x)dx \\
&+\beta(2p+1,2q+1)(b-a)f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\
&+\beta(p+q+1,p+q+1) \\
&\times\left[g\left(\frac{a+b}{2}\right)\int_a^b f(x)dx+f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx\right] \\
&\leq \frac{\Gamma(2p+1)\Gamma(2q+1)}{\Gamma(2p+2q+2)}\int_a^b f(x)g(x)dx \\
&+\frac{\Gamma(2p+1)\Gamma(2q+1)(b-a)}{\Gamma(2p+2q+2)2^{2p+2q}}[f(a)+f(b)][g(a)+g(b)] \\
&+\frac{\Gamma(p+q+1)\Gamma(p+q+1)}{\Gamma(2p+2q+2)}\frac{(b-a)}{2^{p+q-1}}\frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}[f(a)+f(b)][g(a)+g(b)].
\end{aligned}$$

Now dividing both sides by $\frac{\Gamma(2p+1)\Gamma(2q+1)(b-a)}{\Gamma(2p+2q+2)}$, we get required inequality.

Remark 8. 1) If we choose $p=0$, $q=1$ or $p=1$, $q=0$ in (27), then it reduces [25, Theorem 2].

2) If we choose $p=1$, $q=1$ in (27), then it reduces [31, Theorem 6].

Theorem 6. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a beta-convex function on $[a,b]$ with $a<b$. Then*

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx \leq{}& \frac{\Gamma(2p+1)\Gamma(2q+1)}{\Gamma(2p+2q+2)}\left[[f(a)]^2+[f(b)]^2\right] \\
&+\frac{2\Gamma(p+q+1)\Gamma(p+q+1)}{\Gamma(2p+2q+2)}f(a)f(b).
\end{aligned} \tag{31}$$

Proof. Since f is *beta*-convex function on I, we have

$$f(ta+(1-t)b)\leq t^p(1-t)^q f(a)+t^q(1-t)^p f(b)$$

and

$$f((1-t)a+tb)\leq t^q(1-t)^p f(a)+t^p(1-t)^q f(b)$$

for all $a,b\in I$ and $t\in[0,1]$. One can see

$$\frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx=\int_0^1 f(ta+(1-t)b)f((1-t)a+tb)dt.$$

Using the elementary inequality $G(p,q) \leq K(p,q)\,(p,q \geq 0 \text{ real})$ and making the change of variable, we get

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)\,f(a+b-x)\,dx \\
&\leq \frac{1}{2}\int_0^1 \left\{[f(ta+(1-t)b)]^2+[f((1-t)a+tb)]^2\right\}dt \\
&\leq \frac{1}{2}\int_0^1 \left\{[t^p(1-t)^q f(a)+t^q(1-t)^p f(b)]^2\right. \\
&\left.+[t^q(1-t)^p f(a)+t^p(1-t)^q f(b)]^2\right\}dt \\
&=\int_0^1 [t^p(1-t)^q f(a)+t^q(1-t)^p f(b)]^2\,dt \\
&=[f(a)]^2\int_0^1 t^{2p}(1-t)^{2q}\,dt \\
&+2f(a)\,f(b)\int_0^1 t^{p+q}(1-t)^{p+q}\,dt+[f(b)]^2\int_0^1 t^{2q}(1-t)^{2p}\,dt \\
&=[f(a)]^2\,\beta(2p+1,2q+1) \\
&+2f(a)\,f(b)\,\beta(p+q+1,p+q+1)+[f(b)]^2\,\beta(2q+1,2p+1) \\
&=\frac{\Gamma(2p+1)\,\Gamma(2q+1)}{\Gamma(2p+2q+2)}\left[[f(a)]^2+[f(b)]^2\right] \\
&+\frac{2\Gamma(p+q+1)\,\Gamma(p+q+1)}{\Gamma(2p+2q+2)}f(a)\,f(b).
\end{aligned}$$

So, we get required inequality.

3 Fractional integral inequalities of β-convex functions

In this section, we prove some inequalities of Hermite-Hadamard type via fractional integral for *beta*-convex functions.

Definition 12. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt,\ \ x>a \tag{32}$$

and

$$J^{\alpha}_{b-}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (t-x)^{\alpha-1}f(t)\,dt,\ \ b>x \tag{33}$$

respectively where $\Gamma(\alpha)=\int_0^{\infty}e^{-u}u^{\alpha-1}du$. Here is $J^0_{a+}f(x)=J^0_{b-}f(x)=f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. For some recent results connected with fractional integral inequalities see [2],[4]-[7], [9], [28]-[30].

Theorem 7. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function with $a < b$ and $L_1[a,b]$. If f is a beta-convex function on $[a,b]$, then the following inequalities for fractional integrals holds:*

$$\begin{aligned}
2^{p+q-1}f\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)\right] \\
&\leq \frac{\alpha}{2}\left[\frac{\Gamma(p+\alpha)\,\Gamma(q+1)}{\Gamma(p+q+\alpha+1)}+\frac{\Gamma(q+\alpha)\,\Gamma(p+1)}{\Gamma(p+q+\alpha+1)}\right][f(a)+f(b)]
\end{aligned} \tag{34}$$

with $\alpha > 0$.

Proof. Since f is *beta*-convex on $[a,b]$, we have

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2^{p+q}}, \tag{35}$$

for $x,y \in [a,b]$. Now, let $x = ta+(1-t)b$ and $y = (1-t)a+tb$ with $t \in [0,1]$. Then we obtain by (35) that;

$$2^{p+q} f\left(\frac{a+b}{2}\right) \le f(ta+(1-t)b) + f((1-t)a+tb) \tag{36}$$

for all $t \in [0,1]$. Multiplying both sides of (36) by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we get

$$\begin{aligned}\frac{2^{p+q}}{\alpha} f\left(\frac{a+b}{2}\right) &\le \int_0^1 t^{\alpha-1} f(ta+(1-t)b)\,dt + \int_0^1 t^{\alpha-1} f((1-t)a+tb)\,dt \\ &= \frac{1}{(b-a)^{\alpha}} \int_a^b (b-u)^{\alpha-1} f(\mu)\,d\mu + \frac{1}{(b-a)^{\alpha}} \int_a^b (a-v)^{\alpha-1} f(\kappa)\,d\kappa \\ &= \frac{\Gamma(\alpha)}{(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right].\end{aligned} \tag{37}$$

Multiplying both sides of (37) by $\frac{\alpha}{2}$, then

$$2^{p+q-1} f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right]. \tag{38}$$

For the proof of the second inequality in (34), we first note that if f is a *beta*-convex function, then, for $t \in [0,1]$, it yields

$$f(ta+(1-t)b) \le t^p (1-t)^q f(a) + t^q (1-t)^p f(b)$$

and

$$f((1-t)a+tb) \le (1-t)^p t^q f(a) + (1-t)^q t^p f(b).$$

By adding these inequalities we have

$$f(ta+(1-t)b) + f((1-t)a+tb) \le \left[t^p (1-t)^q + t^q (1-t)^p\right] [f(a)+f(b)]. \tag{39}$$

Then multiplying both sides of (39) by $t^{\alpha-1}$, and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}\int_0^1 t^{\alpha-1} f(ta+(1-t)b)\,dt + \int_0^1 t^{\alpha-1} f((1-t)a+tb)\,dt &\le \int_0^1 \left[t^{p+\alpha-1}(1-t)^q + t^{q+\alpha-1}(1-t)^p\right][f(a)+f(b)]\,dt \\ &= \left[\beta(p+\alpha,q+1) + \beta(q+\alpha,p+1)\right][f(a)+f(b)]\end{aligned}$$

i.e.

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \le \frac{\alpha}{2} \left[\frac{\Gamma(p+\alpha)\Gamma(q+1)}{\Gamma(p+q+\alpha+1)} + \frac{\Gamma(q+\alpha)\Gamma(p+1)}{\Gamma(p+q+\alpha+1)}\right] [f(a)+f(b)] \tag{40}$$

From (38) and (40), we complete the proof.

Remark 9. If in Theorem 7, we let $\alpha=1$, then (34) reduces to (14).

Remark 10. If we choose $p=0$, $q=1$ or $p=1$, $q=0$ in (34), then it reduces [28, Theorem 2].

Theorem 8. *Let f and g be real-valued, nonnegative and beta-convex functions on $[a,b]$. Then for all $a,b>0$, $\alpha>0$, we have*

$$\frac{1}{(b-a)^{\alpha}}J_{a+}^{\alpha}\left[f(b)g(b)\right]\leq\frac{f(a)g(a)}{\Gamma(\alpha)}\frac{\Gamma(2p+\alpha)\Gamma(2q+1)}{\Gamma(2p+2q+\alpha+1)}+\frac{f(b)g(b)}{\Gamma(\alpha)}\frac{\Gamma(2q+\alpha)\Gamma(2p+1)}{\Gamma(2p+2q+\alpha+1)}+\frac{N(a,b)}{\Gamma(\alpha)}\frac{\Gamma(p+q+\alpha)\Gamma(p+q+1)}{\Gamma(2p+2q+\alpha+1)}. \tag{41}$$

and

$$2^{2p+2q}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\leq\frac{1}{(b-a)^{\alpha}}J_{a+}^{\alpha}\left[f(b)g(b)+f(b)g(a)\right]+\frac{1}{(b-a)^{\alpha}}J_{b-}^{\alpha}\left[f(a)g(a)+f(a)g(b)\right]. \tag{42}$$

where $M(a,b)$ and $N(a,b)$ are as defined in Theorem 2.

Proof. Since f and g are *beta*-convex functions on $[a,b]$, we have

$$f(ta+(1-t)b)\leq t^{p}(1-t)^{q}f(a)+t^{q}(1-t)^{p}f(b) \tag{43}$$

and

$$g(ta+(1-t)b)\leq t^{p}(1-t)^{q}g(a)+t^{q}(1-t)^{p}g(b). \tag{44}$$

for all $t\in[0,1]$. From (43)-(44), we obtain

$$f(ta+(1-t)b)g(ta+(1-t)b)\leq t^{2p}(1-t)^{2q}f(a)g(a)+t^{2q}(1-t)^{2p}f(b)g(b)+t^{p+q}(1-t)^{p+q}f(b)g(a)+t^{p+q}(1-t)^{p+q}f(a)g(b). \tag{45}$$

Now multiplying both sides of (45) by $\frac{t^{\alpha-1}}{\Gamma(\alpha)}$, then integrating the resulting inequality with respect to t over $[0,1]$, we get

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}f(ta+(1-t)b)g(ta+(1-t)b)\,dt\\&\leq\frac{f(a)g(a)}{\Gamma(\alpha)}\int_0^1 t^{2p+\alpha-1}(1-t)^{2q}dt+\frac{f(b)g(b)}{\Gamma(\alpha)}\int_0^1 t^{2q+\alpha-1}(1-t)^{2p}dt\\&+\frac{f(b)g(a)+f(a)g(b)}{\Gamma(\alpha)}\int_0^1 t^{p+q+\alpha-1}(1-t)^{p+q}dt\\&=\frac{f(a)g(a)}{\Gamma(\alpha)}\frac{\Gamma(2p+\alpha)\Gamma(2q+1)}{\Gamma(2p+2q+\alpha+1)}+\frac{f(b)g(b)}{\Gamma(\alpha)}\frac{\Gamma(2q+\alpha)\Gamma(2p+1)}{\Gamma(2p+2q+\alpha+1)}\\&=\frac{N(a,b)}{\Gamma(\alpha)}\frac{\Gamma(p+q+\alpha)\Gamma(p+q+1)}{\Gamma(2p+2q+\alpha+1)}.\end{aligned} \tag{46}$$

Then we have

$$\frac{1}{(b-a)^{\alpha}} J_{a+}^{\alpha}[f(b)g(b)] \leq \frac{f(a)g(a)}{\Gamma(\alpha)} \frac{\Gamma(2p+\alpha)\Gamma(2q+1)}{\Gamma(2p+2q+\alpha+1)} + \frac{f(b)g(b)}{\Gamma(\alpha)} \frac{\Gamma(2q+\alpha)\Gamma(2p+1)}{\Gamma(2p+2q+\alpha+1)} + \frac{N(a,b)}{\Gamma(\alpha)} \frac{\Gamma(p+q+\alpha)\Gamma(p+q+1)}{\Gamma(2p+2q+\alpha+1)}. \tag{47}$$

Since f and g are *beta*-convex functions on $[a,b]$, then for all $t\in[0,1]$, we obtain

$$\begin{aligned} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) &= f\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right)g\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right) \\ &\leq \frac{1}{2^{p+q}}[f(ta+(1-t)b)+f((1-t)a+tb)] \\ &\times \frac{1}{2^{p+q}}[g(ta+(1-t)b)+g((1-t)a+tb)] \\ &= \frac{1}{2^{2p+2q}}\{f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb) \\ &+f((1-t)a+tb)g(ta+(1-t)b)+f(ta+(1-t)b)g((1-t)a+tb)\} \end{aligned} \tag{48}$$

If we integrate both sides of (48) over $[0,1]$ and use (47), we obtain

$$\begin{aligned} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) &\leq \frac{1}{2^{2p+2q}}\frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}f(ta+(1-t)b)g(ta+(1-t)b)\,dt \\ &+\frac{1}{2^{2p+2q}}\frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}f((1-t)a+tb)g((1-t)a+tb)\,dt \\ &+\frac{1}{2^{2p+2q}}\frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}f((1-t)a+tb)g(ta+(1-t)b)\,dt \\ &+\frac{1}{2^{2p+2q}}\frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}f(ta+(1-t)b)g((1-t)a+tb)\,dt \\ &= \frac{1}{2^{2p+2q}}\frac{1}{(b-a)^{\alpha}}J_{a+}^{\alpha}[f(b)g(b)+f(b)g(a)]+J_{b-}^{\alpha}[f(a)g(a)+f(a)g(b)] \end{aligned}$$

From which we obtain (42).

Remark 11. If we let $\alpha=1$ in (41), then it reduces to (16).

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