Upper bound of the second Hankel determinant for a subclass of analytic functions

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Abstract: In the present investigation an upper bound of second Hankel determinant \( a_2 a_4 - a_3^2 \) for the functions belonging to the class \( S_{\alpha}^* (\alpha; A, B) \) is studied.

Keywords: Analytic functions, Subordination, Schwarz function, Second Hankel determinant.

1 Introduction

Let \( A \) be the class of analytic functions of the form

\[
f(z) = z + \sum_{i=2}^{\infty} a_i z^i
\]

in the unit disc \( E = \{z : |z| < 1\} \).

By \( S \), we denote the class of functions \( f(z) \in A \) and univalent in \( E \).

\( U \) denotes the class of Schwarzian functions

\[
w(z) = \sum_{i=1}^{\infty} p_i z^i
\]

which are analytic in the unit disc \( E = \{z : |z| < 1\} \) and satisfying the conditions \( w(0) = 0 \) and \( |w(z)| < 1 \).

For two functions \( f \) and \( g \) which are analytic in \( E, f \) is said to be subordinate to \( g \) (symbolically \( f \prec g \)) if there exists a Schwarz function \( w(z) \in U \), such that \( f(z) = g(w(z)) \).

\( S_{\alpha}^* (\alpha; A, B) \) denote the subclass of functions \( f(z) \in A \) and satisfying the condition

\[
(1-\alpha) \frac{2 zf'(z)}{f(z) - f(-z)} + \alpha \frac{2( zf'(z))'}{(f(z) - f(-z))} \leq \frac{1 + Az}{1 + Bz}, -1 \leq B < A \leq 1, 0 \leq \alpha \leq 1, z \in E.
\]  

The following observations are obvious:

(i) \( S_{\alpha}^* (\alpha; 1, -1) = S_{\alpha}^* (\alpha) \), the class of \( \alpha \)-starlike functions with respect to symmetric points.
(ii) \( S^*_1(0;1,-1) = S^*_1 \), the class of starlike functions with respect to symmetric points introduced by Sakaguchi [11].

(iii) \( S^*_1(1;1,-1) = K_1 \), the class of convex functions with respect to symmetric points introduced by Das and Singh [1].

(iv) \( S^*_1(0;A,B) = S^*_1(A,B) \), the subclass of starlike functions with respect to symmetric points introduced and studied by Goel and Mehrok [2].

(v) \( S^*_1(1;A,B) = K_1(A,B) \), the subclass of convex functions with respect to symmetric points.

In 1976, Noonan and Thomas [9] stated the \( q \)th Hankel determinant of \( f(z) \) for \( q \geq 1 \) and \( n \geq 1 \) as

\[
H_q(n) = \begin{vmatrix}
 a_n & a_{n+1} & \cdots & a_{n+q} \\
 a_{n+1} & \ddots & \vdots & \vdots \\
 \vdots & \ddots & \ddots & \vdots \\
 a_{n+q} & \cdots & a_{n+2q-2} \\
\end{vmatrix}.
\]

For our discussion in this paper, we consider the Hankel determinant in the case of \( q = 2 \) and \( n = 2 \), known as second Hankel determinant:

\[
H_2(2) = \begin{vmatrix}
 a_2 & a_3 \\
 a_3 & a_4 \\
\end{vmatrix} = [a_2a_4 - a_3^2],
\]

and obtain an upper bound to the functional \( H_2(2) \) for \( f(z) \in S^*_1(\alpha;A,B) \). Earlier Janteng et al.([3],[4],[5]), Mehrok and Singh [8], Singh ([12],[13]) and many others have obtained sharp upper bounds of \( H_2(2) \) for different classes of analytic functions.

2 Preliminary Results

Let \( P \) be the family of all functions \( p \) analytic in \( E \) for which \( \Re(p(z)) > 0 \) and

\[
p(z) = 1 + p_1z + p_2z^2 + ...
\]

for \( z \in E \).

**Lemma 2.1.** [10] If \( p \in P \), then \( |p_k| \leq 2(k = 1, 2, 3, ...) \).

**Lemma 2.2.** [6,7] If \( p \in P \), then

\[
2p_2 = p_1^2 + (4 - p_1^2)x,
\]

\[
4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,
\]

for some \( x \) and \( z \) satisfying \( |x| \leq 1, |z| \leq 1 \) and \( p_1 \in [0, 2] \).

3 Main Result

**Theorem 3.1.** If \( f \in S^*_1(\alpha;A,B) \), then
\[ |a_3a_4 - a_1^2| \leq \frac{(A-B)^2}{4(1+2\alpha)^2}. \]  

(3.1)

**Proof.** If \( f(z) \in S^*(\alpha; A, B) \), then there exists a Schwarz function \( \omega(z) \in U \) such that

\[
(1-\alpha) \frac{2zf'(z)}{f(z)-f(-z)} + \alpha \frac{2(zf'(z))'}{(f(z)-f(-z))} = \phi(\omega(z)),
\]

(3.2)

where

\[
\phi(z) = \frac{1+Az}{1+Be} = 1 + (A-B)z + B(A-B)z^2 + B^2(A-B)z^3 + ...
\]

(3.3)

Define the function \( p_1(z) \) by

\[
p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1z + c_2z^2 + c_3z^3 + ...
\]

(3.4)

Since \( w(z) \) is a Schwarz function, we see that \( \text{Re}(p_1(z)) > 0 \) and \( p_1(0) = 1 \). Define the function \( h(z) \) by

\[
h(z) = (1-\alpha) \frac{2zf'(z)}{f(z)-f(-z)} + \alpha \frac{2(zf'(z))'}{(f(z)-f(-z))} = 1 + b_1z + b_2z^2 + b_3z^3 + ...
\]

(3.5)

In view of the equations (3.2), (3.4) and (3.5), we have

\[
h(z) = \phi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right) = \phi \left( \frac{c_1z + c_2z^2 + c_3z^3 + ...}{2 + c_1z + c_2z^2 + c_3z^3 + ...} \right)
\]

\[
= \phi \left( \frac{1}{2} c_1z + \frac{1}{2} \left( c_2 - \frac{c_1^2}{2} \right) z^2 + \frac{1}{2} \left( c_3 - c_1c_2 + \frac{c_1^3}{4} \right) z^3 + ... \right)
\]

\[
= 1 + \frac{B_c c_1}{2} z + \left[ \frac{B_3}{2} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{B_c c_1^2}{4} \right] z^2 + \left[ \frac{B_3}{2} \left( c_3 - c_1c_2 + \frac{c_1^3}{4} \right) + \frac{B_c c_1}{2} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{B_c c_1^3}{8} \right] z^3 + ...
\]

Thus,

\[
b_1 = \frac{B_c c_1}{2}; b_2 = \frac{B_3}{2} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{B_c c_1^2}{4} \text{ and } b_3 = \frac{B_3}{2} \left( c_3 - c_1c_2 + \frac{c_1^3}{4} \right) + \frac{B_c c_1}{2} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{B_c c_1^3}{8}.
\]

(3.6)

Using (3.3) and (3.5) in (3.6), we obtain
\[ a_2 = \frac{(A - B)c_i}{4(1 + \alpha)}, \]
\[ a_3 = \frac{(A - B)}{8(1 + 2\alpha)} \left[ 2c_i^2 - (B + 1)c_i^3 \right], \]
\[ a_4 = \frac{(A - B)}{64(1 + \alpha)(1 + 2\alpha)(1 + 3\alpha)} \left[ \begin{array}{c} 8(1 + \alpha)(1 + 2\alpha)c_i \\ + [8(1 + \alpha)^3 (1 + 3\alpha) - (1 + 2\alpha)(1 + 5\alpha) - 4(1 + \alpha)(1 + 2\alpha)^2]B \\ + (B + 1)[(2(1 + \alpha)(1 + 2\alpha) + (1 + 5\alpha)]B - (1 + 5\alpha)A + 2(1 + \alpha)(1 + 2\alpha)]c_i^3 \end{array} \right] \] - (3.7)

(3.7) yields,
\[ a_3 a_4 - a_i^3 = \frac{(A-B)^2}{C(\alpha)} \left[ 2Lc_i^4 (4c_i) + Mc_i^4 (2c_i) - Nc_i^4 - 4R(4c_i^3) \right] \] - (3.8)

where \( C(\alpha) = 256(1 + \alpha)^2 (1 + 2\alpha)^2 (1 + 3\alpha), \)

\[ L = (1 + \alpha)(1 + 2\alpha)^2, \]
\[ M = (1 + 2\alpha)(1 + 5\alpha)A + \left[ 8(1 + \alpha)^3 (1 + 3\alpha) - (1 + 2\alpha)(1 + 5\alpha) - 4(1 + \alpha)(1 + 2\alpha)^2 \right]B \]
\[ + \left[ 8(1 + \alpha)^2 (1 + 3\alpha) - 4(1 + \alpha)(1 + 2\alpha)^2 \right], \]
\[ N = (B + 1) \left[ (1 + 2\alpha)(1 + 5\alpha)A + 4(1 + \alpha)^2 (1 + 3\alpha) - 2(1 + \alpha)(1 + 2\alpha)^2 - (1 + 2\alpha)(1 + 5\alpha) \right]B \]
\[ + \left[ 4(1 + \alpha)^2 (1 + 3\alpha) - 2(1 + \alpha)(1 + 2\alpha)^2 \right] \]

and

\[ R = (1 + 3\alpha)(1 + \alpha)^2. \]

Using Lemma 2.1 and Lemma 2.2 in (3.8), we obtain
\[ a_3 a_4 - a_i^3 = \frac{(A-B)^2}{C(\alpha)} \left[ - \left( (1 + 2\alpha)(1 + 5\alpha)AB + \left[ 4(1 + \alpha)^2 (1 + 3\alpha) - 2(1 + \alpha)(1 + 2\alpha)^2 - (1 + 2\alpha)(1 + 5\alpha) \right]B \right] \right] c_i^4 \]
\[ + \left( (1 + 2\alpha)(1 + 5\alpha)A + \left[ 8(1 + \alpha)^3 (1 + 3\alpha) - (1 + 2\alpha)(1 + 5\alpha) - 4(1 + \alpha)(1 + 2\alpha)^2 \right]B \right] c_i^4 \left( 4 - c_i^2 \right)^2 \]
\[ - 2 \left( 4(1 + \alpha)^2 (1 + 3\alpha) - \left[ 2(1 + \alpha)^2 (1 + 3\alpha) - (1 + \alpha)(1 + 2\alpha)^2 \right] c_i^4 \right) (4 - c_i^2) \]
\[ + 4(1 + \alpha)(1 + 2\alpha)^2 c_i (4 - c_i^2) \left( 1 - |x|^2 \right) z \]

Assume that \( c_i = c \) and \( c \in [0, 2] \), using triangular inequality and \(|z| \leq 1\), we have
\[ \left| a_j - a_i \right| = \frac{(A-B)^2}{C(\alpha)} f(\delta), \text{ where } \delta = |\alpha| \leq 1 \] 

\[ F(\delta) = 2(4-c^2) \left( 8(1+\alpha)^2(1+3\alpha) - 2(1+\alpha)^2(1+3\alpha)^2 - 4(1+\alpha)(1+2\alpha)^2 c(4-c^2) \right) \delta^2 
+ \left[ (1+2\alpha)(1+5\alpha) A + \left( 8(1+\alpha)^2(1+3\alpha) - 4(1+\alpha)(1+2\alpha)^2 \right) B \right] c(4-c^2) \delta 
+ \left[ (1+2\alpha)(1+5\alpha) AB + \left( 8(1+\alpha)^2(1+3\alpha) - 2(1+\alpha)(1+2\alpha)^2 - (1+2\alpha)(1+5\alpha) \right) B \right] c^4 
+ 4(1+\alpha)(1+2\alpha)^2 c(4-c^2) \]

is an increasing function. Therefore \( \text{Max} F(\delta) = F(1) \).

Consequently

\[ \left| a_j - a_i \right| \leq \frac{(A-B)^2}{C(\alpha)} G(c), \quad (3.9) \]

where

\[ G(c) = F(1). \]

So \( G(c) = S(\alpha)c^3 + T(\alpha)c^2 + 64(1+\alpha)(1+3\alpha) \)

where

\[ S(\alpha) = \left[ (1+2\alpha)(1+5\alpha) AB + \left( 8(1+\alpha)^2(1+3\alpha) - 2(1+\alpha)(1+2\alpha)^2 \right) \right. 
+ \left. 2 \left( 1+2\alpha \right)^2(1+3\alpha)(1+2\alpha) \right] \]

and

\[ T(\alpha) = \left[ 4 \left( 1+2\alpha \right) \left( 1+5\alpha \right) A + \left( 8(1+\alpha)^2(1+3\alpha) - 4(1+\alpha)(1+2\alpha)^2 \right) \right. 
- \left. 8 \left( 1+2\alpha \right)^2(1+3\alpha)(1+2\alpha) \right] \]

Now \( G'(c) = 4S(\alpha)c^3 + 2T(\alpha)c \) and \( G^*(c) = 12S(\alpha)c^2 + 2T(\alpha) \).

\( G'(c) = 0 \) gives

\[ c \left[ 2S(\alpha)c^2 + T(\alpha) \right] = 0. \]

\( G^*(c) \) is negative at \( c = 0 \).

So \( \text{Max} G(c) = G(1) \).
Hence from (3.9), we obtain (3.1).

The result is sharp for \(c_1 = 0, \ c_2 = 2 \quad \text{and} \quad c_3 = 0.\)

For \(A = 1 \quad \text{and} \quad B = -1\) in Theorem 3.1, we obtain the following result:

**Corollary 3.1.1.** If \(f(z) \in S^*_n(\alpha)\), then

\[
|a_2a_4 - a^2| \leq \frac{1}{(1+2\alpha)^2}.
\]

For \(\alpha = 0, \ A = 1 \quad \text{and} \quad B = -1\), Theorem 3.1 gives the following result due to Janteng et al.[5].

**Corollary 3.1.2.** If \(f(z) \in S^*_n\), then

\[
|a_2a_4 - a^2| \leq 1.
\]

For \(\alpha = 1, \ A = 1 \quad \text{and} \quad B = -1\), Theorem 3.1 gives the following result due to Janteng et al.[5].

**Corollary 3.1.3.** If \(f(z) \in K_n\), then

\[
|a_2a_4 - a^2| \leq \frac{1}{9}.
\]

Putting \(\alpha = 0\) in Theorem 3.1, we obtain the following result:

**Corollary 3.1.4.** If \(f(z) \in S^*_n(A, B)\), then

\[
|a_2a_4 - a^2| \leq \frac{(A-B)^2}{4}.
\]

Putting \(\alpha = 1\) in Theorem 3.1, we obtain the following result:

**Corollary 3.1.5.** If \(f(z) \in K_n(A, B)\), then

\[
|a_2a_4 - a^2| \leq \frac{(A-B)^2}{36}.
\]

**References**


