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The Generalized Order-*k* Jacobsthal Lengths of The Some Centro-Polyhedral Groups

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Abstract: In [7], the authors defined the generalized order-*k* Jacobsthal orbit $J_A^k(G)$ of a finitely generated group $G = \langle A \rangle$. In this study, we obtain the generalized order-*k* Jacobsthal lengths of the centro-polyhedral groups $\langle 2, -n, 2 \rangle, \langle -2, n, 2 \rangle$ and $\langle 2, n, -2 \rangle$.

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1 Introduction and Preliminaries

It is known that the Jacobsthal sequence $\{J_n\}$ is defined recursively by the equation

$$J_n = J_{n-1} + 2J_{n-2} \tag{1}$$

for $n \geq 2$, where $J_0 = 0$ and $J_1 = 1$.

In [11], Koken and Bozkurt showed that the Jacobsthal numbers are also generated by a matrix

$$F = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \; F^n = \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix}$$

Kalman [9] mentioned that these sequences are special cases of a sequence which is defined recursively as a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1},$$

where $c_0, c_1, \cdots, c_{k-1}$ are real constants. In [9], Kalman derived a number of closed-form formulas for the generalized sequence by companion matrix method as follows:

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Then by an inductive argument he obtained that

$$A_k^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

In [15], Yilmaz and Bozkurt defined the k sequences of the generalized order-*k* Jacobsthal numbers as follows:

for $n > 0$ and $1 \leq i \leq k$

$$J_n^i = J_{n-1}^i + 2J_{n-2}^i + \cdots + J_{n-k}^i \,, \tag{2}$$

with initial conditions

$$J_n^i = \begin{cases} 1 & \text{if } n = 1 - i, \\ & \qquad \text{for } 1 - k \leq n \leq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where J_n^i is the nth term of the ith sequence. If $k = 2$ and $i = 1$ the generalized order-k Jacobsthal sequence is reduced to the conventional Jacobsthal sequence.

In [15], Yilmaz and Bozkurt showed that

$$\begin{bmatrix} J_{n+1}^i \\ J_n^i \\ J_{n-1}^i \\ \vdots \\ J_{n-k+2}^i \end{bmatrix} = C. \begin{bmatrix} J_n^i \\ J_{n-1}^i \\ J_{n-2}^i \\ \vdots \\ J_{n-k+1}^i \end{bmatrix} \tag{3}$$

where C is called the generalized order-k Jacobsthal matrix and C is a k-square matrix as following:

$$C = \begin{bmatrix} 1 & 2 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{4}$$

Also, it was obtained that $B_n = C \cdot B_{n-1}$ where

$$B_n = \begin{bmatrix} J_n^1 & J_n^2 & \cdots & J_n^k \\ J_{n-1}^1 & J_{n-1}^2 & \cdots & J_{n-1}^k \\ \vdots & \vdots & & \vdots \\ J_{n-k+1}^1 & J_{n-k+1}^2 & \cdots & J_{n-k+1}^k \end{bmatrix} \tag{5}$$

Lemma 1. (Yilmaz and Bozkurt [15]). Let C and B_n be as in (4) and (5), respectively. Then, for all integers $n \geq 0$

$$B_n = C^n.$$

Reducing the generalized order-k Jacobsthal sequence $(k \geq 2)$ by a modulus m, we can get the repeating sequences, denoted by

$$\{J_n^{k,m}\} = \{J_{1-k}^{k,m}, J_{2-k}^{k,m}, \cdots, J_0^{k,m}, J_1^{k,m}, \cdots, J_i^{k,m}, \cdots\}$$

where $J_i^{k,m} \equiv J_i^k (\text{mod } m)$. It has the same recurrence relation as in (2) [8].

Theorem 1. (Deveci et al [7]). The sequence $\{J_n^{k,m}\}$ $(k \geq 2)$ is periodic.

The notation $hJ^{k,m}$ denotes the smallest period of $\{J_n^{k,m}\}$ $(k \geq 2)$ [8].

Theorem 2. (Deveci et.al [7]). If p is a prime such that $p \neq 2$, then $hJ^{k,p^\alpha} = |\langle C \rangle_{p^\alpha}|$.

Definition 1. (Deveci and Sağlam [8]). Let $hJ_{(a_1,a_2,\cdots,a_k)}^{k,m}$ denote the smallest period of the integer-valued recurrence relation $u_n = u_{n-1} + 2u_{n-2} + \cdots + u_{n-k}$, $u_1 = a_1, u_2 = a_2, \cdots, u_k = a_k$ when each entry is reduced modulo m.

Theorem 3. (Deveci and Sağlam [8]). For $a_1, a_2, \cdots, a_k, x_1, x_2, \cdots, x_k \in \mathbb{Z}$, p is a prime such that $p \neq 2$, $\gcd(a_1, a_2, \cdots, a_k, \mathrm{p}) = 1$ and $\gcd(x_1, x_2, \cdots, x_k, p) = 1$,

$$hJ^{k,p}_{(a_1,a_2,\cdots,a_k)} = hJ^{k,p}_{(x_1,x_2,\cdots,,x_k)}.$$

A sequence of group elements is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called the period of the sequence. For example, the sequence $a, b, c, d, e, b, c, d, e, b, c, d, e, \cdots$ is periodic after the initial element a and has period 4. A sequence of group elements is simply periodic with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, d, e, f, a, b, c, d, e, f, a, b, c, d, e, f, \cdots$ is simply periodic with period 6.

Definition 2. (Deveci et.al [7]). The generalized order-*k* Jacobsthal orbit $J^k_A(G)$ for a finitely generated group $G = \langle A \rangle$, where $A = \{a_1, a_2, \cdots, a_k\}$ as following: the generalized order-*k* Jacobsthal orbit $J^k_A(G)$ with respect to the generating set A to be the sequence $\{x_i\}$ of the elements of G such that

$$x_i = \alpha_{i+1} \text{ for } 0 \leq i \leq k-1, x_{i+k} = \begin{cases} (x_i)^2(x_{i+1}), & k = 2 \\ (x_i)\cdots(x_{i+k-2})^2(x_{i+k-1}), & k \geq 3 \end{cases} \quad \text{for } i \geq 0$$

The length of the period of the generalized order-*k* Jacobsthal orbit $J^k_A(G)$ is denoted by $LJ^k_A(G)$ and is called the generalized order-*k* Jacobsthal length of G.

Many references may be given for some special linear recurrence sequences in groups and related issues; see for example, [1,3,5,10,12-14]. Campbell and Campbell calculated the Fibonacci lengths of certain centro-polyhedral groups [2]. Deveci et.al obtained the periods of *k*-nacci sequences in centro-polyhedral groups and related groups [6]. In this paper, we obtain the lengths $LJ^3_{(x,y,z)}(\langle 2, -n, 2\rangle), LJ^3_{(x,y,z)}(\langle -2, n, 2\rangle)$ and $J^3_{(x,y,z)}(\langle 2, n, -2\rangle)$.

2 Main Results and Proofs

Definition 3. The *polyhedral group* (l, m, n), for $l, m, n > 1$ is defined by the presentation

$$\langle x, y, z : x^l = y^m = z^n = xyz = 1 \rangle.$$

The *polyhedral group* (l, m, n) is finite if, and only if, the number

$$\mu = lmn\left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1\right) = mn + nl + lm - lmn$$

is positive. Its order is $2lmn/\mu$.

For more information on these groups see [4, p.67-68].

Definition 4. The *centro-polyhedral group* $\langle l, m, n \rangle$, for $l, m, n \in \mathbb{Z}$ is defined by the presentation

$$\langle x, y, z : x^l = y^m = z^n = xyz \rangle.$$

For more information on these groups see [2,4].

Theorem 4. $LJ^3_{(x,y,z)}(\langle 2, -n, 2\rangle) = 7$.

Proof. These group have orders $4n$. We first note that in the group defined by this presentation z^2 is central and $|x| = 4, |z| = 4$ and $|y| = 2n$ then $y^{-n} = y^n$. The orbit $J^3_{(x,y,z)}(\langle 2, -n, 2\rangle)$ becomes:

$$x, y, z, zx, e, yx, x^3, x, y, z, \cdots,$$

which has period 7. That is $J^3_{(x,y,z)}(\langle 2, -n, 2\rangle) = 7$.

Theorem 5. $LJ^3_{(x,y,z)}(\langle -2,n,2\rangle)$= $LJ^3_{(x,y,z)}(\langle 2,n,-2\rangle) = hJ^{3,4(n-1)}$.

Proof. These groups have orders $4n(n-1)$. Let us consider the group given by the presentation $\langle -2,n,2\rangle$. We first note in the group defined this presentation both x^{-2} and z^2 are central, $|x| = |z| = 4(n-1)$,$|y| = 2n(n-1)$ and $x^{-3} = yz$.

Let us consider the recurrence relations defined by the following:

$$u_{n+3} = u_n + 2u_{n+1} + u_{n+2} \text{ for } n \geq 3 \text{ where } u_0 = 1, u_1 = 0 \text{ and } u_2 = 0;$$

$$v_{n+3} = v_n + 2v_{n+1} + v_{n+2} \text{ for } n \geq 3 \text{ where } v_0 = 0, v_1 = 1 \text{ and } v_2 = 0;$$

$$w_{n+3} = w_n + 2w_{n+1} + w_{n+2} \text{ for } n \geq 3 \text{ where } w_0 = 0, w_1 = 0 \text{ and } w_2 = 1;$$

Then a routine induction shows that the number of x's,y's and z's in nth entry of the *Jacobsthal* sequence is given by u_n , v_n and w_n, respectively.

Here the start of the orbit $J^3_{(x,y,z)}(\langle -2,n,2\rangle)$ is

$$x_0 = x, x_1 = y, x_2 = z, x_3 = xy^2z, x_4 = y^3xz^3, x_5 = x^3y^7z^6,$$

$$x_6 = x^6z^{13}y^{15}, x_7 = x^{13}y^{32}z^{28}, x_8 = x^{28}y^{69}z^{60}, \cdots.$$

We can see that the sequence will separate into some natural layers and each layer will be of such as

$$x_n = \begin{cases} x^{u_n}z^{w_n}y^{v_n}, & n \equiv 0 \text{ mod } 6, \\ x^{u_n}y^{v_n}z^{w_n}, & n \equiv 1 \; mod \; 6 \\ x^{u_n}y^{v_n}z^{w_n}, & n \equiv 2 \; mod \; 6 \\ x^{u_n}y^{v_n}z^{w_n}, & n \equiv 3 \; mod \; 6 \\ y^{v_n}x^{u_n}z^{w_n}, & n \equiv 4 \; mod \; 6 \\ x^{u_n}y^{v_n}z^{w_n}, & n \equiv 0 \; mod \; 6 \end{cases}$$

Now the proof is finished if we note that the sequence will repeat when $x_{hJ^{3,4(n-1)}} = x$, $x_{hJ^{3,4(n-1)}+1} = y$ and $x_{hJ^{3,4(n-1)}+2} = z$. Since the sequence can be said to form layers lenth seven then the period is $7.\mu$,, $(\mu \in \mathbb{N})$that is $P \equiv 0$ mod 7, $P + 1 \equiv 1$mod 7 and $P + 2 \equiv 2$ mod 7. Where we denote $LJ^3_{(x,y,z)}(\langle -2,n,2\rangle)$ by P. Examining this statement in more detail gives

$$x_P = x^{u_P}z^{w_P}y^{v_P},$$

$$x_{P+1} = x^{u_{P+1}}y^{v_{P+1}}z^{w_{P+1}},$$

$$x_{P+2} = x^{u_{P+2}}y^{v_{P+2}}z^{w_{P+2}}$$

Using $P \equiv 0$ mod 7, $P + 1 \equiv 1$ mod 7 and $P + 2 \equiv 2$ mod 7 we obtain

$$u_P \equiv u_0 = 1, u_{P+1} \equiv u_1 = 0,, u_{P+2} \equiv u_2 = 0$$

$$v_P \equiv v_0 = 0, v_{P+1} \equiv v_1 = 1, v_{P+2} \equiv v_2 = 0$$

and

$$w_P \equiv w_0 = 0, w_{P+1} \equiv w_1 = 0, w_{P+2} \equiv w_2 = 1 .$$

So, from the above equalities we have

$$x_P = x, x_{P+1} = y, x_{P+2} = z .$$

Then from Theorem 3 it is clear that the smallest non-trivial integer satisfying the above conditions occurs when the period is $hJ^{2,2^{n-1}}$. That is $LJ^3_{(x,y,z)}(\langle -2,n,2\rangle) = hJ^{3,4(n-1)}$.

The proof for the orbit $J^3_{(x,y,z)}(\langle 2,n,-2\rangle)$ is similar to the above and is omitted.

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