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Socle-regular $QTAG$-modules

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Abstract: A right module M over an associative ring with unity is a $QTAG$-module if every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules. In this paper we focus our attention to the socles of fully invariant submodules and introduce a new class of modules, which we term socle-regular $QTAG$-modules. This class is shown to be large and strictly contains the class of fully transitive modules. Also, here we investigated some basic properties of such modules.

Keywords: $QTAG$ *-module, transitive modules, fully invariant submodules, socles.*

1. Introduction and preliminaries

The study of $QTAG$-modules was initiated by Singh [9]. Mehdi, Abbasi etc. worked a lot on this module [7]. They studied dfferent notions and structures on $QTAG$-modules and developed the theory of these modules by introducing several notions and some in-teresting properties of these modules and characterized different submodules of $QTAG$-modules. Yet there is much to explore.

Throughout this paper, all rings will be associative with unity and modules M are unital $QTAG$-modules. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R-module M with a unique composition series, $d(M)$ denotes its composition length. For a uniform element $x \in M, e(x) = d(xR)$ and $H_M(x) = \sup\left\{d\left(\frac{yR}{xR}\right) | y \in M, x \in yR \text{ and } y \text{ uniform}\right\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. M is h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ and it is h-reduced if it does not contain any h-divisible submodule. In other words it is free from the elements of infinite height. M is called separable if $M^1 = 0$.

For an ordinal σ, a submodule N of M is said to be σ-pure, if $H_\beta(M) \cap N = H_\beta(N)$ for all $\beta \leq \sigma$ and a submodule K of M is said to be isotype in M, if it is σ-pure for every ordinal σ. [3]

A $QTAG$-module M defines a well ordered sequence of submodules $M = M^0 \supset M^1 \supset M^2 \supset \cdots \supset M^T = 0$ for some ordinal τ. Here

$$M^1 \bigcap_{k\in\omega} H_k(M)\,, M^{\sigma+1} = (M^\sigma)^1 \text{ and } M^\sigma = \bigcap_{\rho<\sigma} M^\rho,$$

if σ is a limit ordinal. M^σ is said to be the σ^{th}*-Ulm submodule* of M. The σ^{th}*-Ulm invariant* of $M, f_M(\sigma)$ is the cardinality of $g\left(S_\alpha\left(H_\alpha(M)\right)/\mathrm{Soc}\left(H_{\sigma+1}(M)\right)\right)$. It is interesting to note that the results which hold for TAG-modules also hold good for $QTAG$-modules [9].

2. The class of socle-regular $QTAG$-modules

The classification of all fully invariant submodules of reduced $QTAG$-modules is a vast subject. We start these investigations by characterizing the socles of fully invariant modules. Here we deal with the socles of fully invariant submodules of reduced $QTAG$-modules.

To study fully invariant modules, the concept of U-sequences is extensively used. In a QTAG-module M, for $x \in M$, $U(x)$, the U-sequence for x is a monotonically increasing sequence of ordinals $\{\sigma_1\}, i \geq 0,\ \sigma_1 <$ length of M [8]. The symbol ∞ may be included in this U-sequence i.e. the sequence be ∞ from some point on but that if a gap occurs between σ_k and σ_{k+1}, the σ_k^{th}-Ulm invariant of M is non zero.

To study the socles of fully invariant submodules, we define the following:

Definition 2.1. *A h-reduced $QTAG$-module M is said to be socle-regular if for all fully invariant submodules N of M, there exists an ordinal σ such that $Soc(N) = Soc(H_\sigma(M))$. Hence σ depends on N.*

Definition 2.2. *For a submodule N of M, put $\sigma = min\{H(x) | x \in Soc(N)\}$ and denote $\sigma = inf(Soc(N))$. Here $Soc(N) \subseteq Soc(H_\sigma(M))$.*

Remark 2.1. *If K is submodule of M containing N, $inf(Soc(N))$ may be calculated with respect to N and M respectively. To differentiate we write $inf(Soc(N))_K$ and $inf(Soc(N))_M$ respectively, but if K is an isotype submodule of M, then $inf(Soc(N))_K = inf(Soc(N))_M$. However if K is not an isotype submodule of M, then $inf(Soc(N))_K \leq inf(Soc(N))_M$.*

To study these modules we need the following elementary facts:

Proposition 2.1. (i) If *N is a submodule of the h-reduced $QTAG$-module M such that $Soc(H_k(M)) \subseteq Soc(N)$ for some integer k, then $inf\ (Soc(N))$ is finite.*

(ii) If N is a fully invariant submodule of M and $inf(Soc(N)) = k,\ k < \omega$, then $Soc(N) = Soc(H_k(M))$.

Proof. (i) Let $\sigma = inf(Soc(N))$. Now $\sigma \leq \min\{H_M(x) | x \in Soc(H_k(M))\}$. If $\sigma \geq \omega$, then $Soc(H_k(M)) \subseteq H_\omega(M) = H_\omega(H_k(M))$. Thus $Soc(H_k(M)) \subseteq H_\omega(H_k(M))$. This means $H_k(M)$ is h-divisible (if not zero) which is not possible because M is h-reduced. Thus $inf(Soc(N)) < \omega$.

(ii) Since $inf(Soc(N)) = k, Soc(N) \subseteq Soc(H_k(M))$. Let x be a uniform element of $Soc(N)$ such that $H_M(x) = k$, then there exists $y \in M$ such that $d\left(\frac{yR}{xR}\right) = k$. Since every element of exponent one and finite height can be embedded in a direct summand, by [5] yR is a summand of M containing x. Therefore $M = yR \oplus M'$, for some M' of M. If z is an arbitrary uniform element of $Soc(H_k(M))$, then there exists $u \in M$ such that $d\left(\frac{uR}{zR}\right) = k$. Now $e(u) = k + 1$, we may define a homomorphism $f: M \to M$ such that $y \to u, f(M') = 0$ and $f(x) = z$. Since $Soc(N)$ is fully invariant in M, $z \in Soc(N)$ and $Soc(H_k(M)) \subseteq Soc(N)$, proving the result.

Remark 2.2. *For a fully invariant submodule N of a separable module M, $inf(Soc(N))$ is finite. Hence M is socle-regular.*

Let us recall the definition of fully transitive $QTAG$-modules [6]:

Definition 2.3. *A $QTAG$-module M is fully transitive if for every pair of uniform elements $x, y \in M$, $H_M(x_i) \leq H_M(y_i)$ for all $i \geq 0$ implies that there exists an endomorphism of M that maps x onto y. Here $d\left(\frac{x_iR}{xR}\right) = d\left(\frac{y_iR}{yR}\right) = i$.*

Remark 2.3. *We may extend this definition for all the elements if we consider U-sequences of the elements* [4], *consisting of ordinals and the symbol* ∞. *In other words if* $U(x) = (\alpha_1, \alpha_2, \dots)$ *and* $U(y) = (\beta_1, \beta_2, \dots)$ *such that* $\alpha_\rho < \beta_\rho$, *then there exists an endomorphism of* M *that maps* x *onto* y.

Definition 2.4. *Let* $\{\alpha_i\}$ *be a monotonically increasing sequence of ordinals defined for* $i \geq 0$. *If* λ *is the length of a module* M *and* $\alpha_i < \lambda$ *except that the sequence be* ∞ *from some point on,* $\{\alpha_i\}$ *is called a* U*-sequence relative to* M. *Whenever a gap occurs between* α_{n-1} *and* α_n, *the* α_n^{th}*-Ulm invariant of* M *is non-zero.*

Now we prove the following:

Theorem 2.1. *If* M *is a fully transitive* $QTAG$*-module, then* M *is socle-regular.*

Proof. The fully invariant submodule $N \subseteq M$ is generated by the elements x such that $U(x) \geq U$, where $U = \{\alpha_i\}$ is a U-sequence relative to M. If $x \in Soc(N)$, then $U(x) = (\beta, \infty, \dots)$ for some ordinal $\beta \geq \alpha_0$, therefore $x \in Soc\left(H_{\alpha_0}(M)\right)$ and $Soc(N) \subseteq Soc\left(H_{\alpha_0}(M)\right)$. On the other side if $z \in Soc\left(H_{\alpha_0}(M)\right)$ then $U(z) = (\beta, \infty, \dots)$ where $\beta \geq \alpha_0$. Now $U(z) \geq U$, therefore $z \in N$ and $Soc\left(H_{\alpha_0}(M)\right) \subseteq Soc(N)$ and $Soc(N) = Soc\left(H_{\alpha_0}(M)\right)$. Thus M is socle-regular.

To investigate the properties of socle-regular $QTAG$-modules we need the following lemmas:

Lemma 2.1. *Let* M *be* $QTAG$*-module such that* $M = \bigoplus_{i \in l} M_i$. *If* N *is a fully invariant submodule of* M, *then*

(i) $N = \bigoplus_{i \in l} (M_i \cap N)$;

(ii) *each* $M_i \cap N$ *is fully invariant in* M_i.

Proof. An endomorphism f of $M \oplus K$ may be expresed as the matrix $\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}$. Here f_2 is a homomorphism from M to K. Now $f(N \oplus 0) \subseteq f_1(N) \oplus f_2(N) \subseteq N \oplus f_2(N)$ because N is fully invariant in M. Since f_2 is a homomorphism from M to K and K is separable, f_2 maps $H_\omega(M)$ to zero. Also $N \subseteq H_\omega(M)$, $f_2(N) = 0$ therefore $f(N \oplus 0) \subseteq N \oplus 0$ and N is fully invariant in $M \oplus K$.

Theorem 2.2. *Let* $M = N \oplus K$ *be a* $QTAG$*-module with* K, *separable. Then* M *is socle-regular if and only if* N *is socle-regular.*

Proof. Suppose that N is socle-regular and L is fully invariant in M. By Lemma 2.1, $L = (L \cap N) \oplus (L \cap K)$ and $L \cap N$, $L \cap K$ are fully invariant in N and K respectively. If $L \cap K \neq 0$ then $inf\left(Soc(L \cap K)\right)_K$ is finite because K is separable. Now $Soc(L) = Soc(L \cap N) \oplus Soc(L \cap K)$ thus $inf\left(Soc(L)\right)_M \leq inf\left(Soc(L \cap K)\right)_M$. Being a direct summand, K is h-pure in M, therefore $inf\left(Soc(L \cap K)\right)_M = inf\left(Soc(L \cap K)\right)_K$. This implies $inf\left(Soc(L)\right)_M$ is also finite and by Proposition 2.1, $Soc(L) = Soc\left(H_k(M)\right)$, for some integer k.

If $L \cap K = 0$, then L is a fully invariant submodule of the socle-regular $QTAG$-module N. Therefore $Soc(L) = Soc\left(H_\alpha(N)\right)$ for some ordinal α. If $\alpha \geq \omega$, $H_\alpha M = H_\alpha(N)$ as K is separable and $Soc(L) = Soc\left(H_\alpha(M)\right)$ and if $\alpha < \omega$, $Soc(L) = Soc\left(H_k(N)\right)$ for some k and L is a fully invariant submodule of N. Now by Proposition 2.1 (i), $inf\left(Soc(L)\right)_N$ is finite. Being a direct summand N is h-pure in M, therefore $inf\left(Soc(L)\right)_M$ is also finite and by Proposition 2.1 (ii), $Soc(L) = Soc\left(H_k(M)\right)$ for some integer k.

Conversely suppose that M is socle-regular. If N is not socle-regular then there exists a fully invariant submodule L of M such that $Soc(L) \neq \left(H_\alpha(N)\right)$ for any ordinal α. If $inf\left(Soc(L)\right)$ is finite then by Proposition 2.1 (i), $Soc(L) = Soc\left(H_k(N)\right)$ for some finite k. This contradiction proves that $inf\left(Soc(L)\right)$ is infinite and $Soc(L) \subseteq H_\omega(N)$. Since N is fully invariant in N, $Soc(L)$ is also fully invariant in N. Now by Lemma 2.2, $Soc(L)$ is fully invariant in a socle-reguler module M. Therefore $Soc(L) = Soc\left(H_\alpha(M)\right)$, for some ordinal α. Since $Soc(L) \subseteq H_\omega(M)$, α must be infinite. Also

$H_\alpha(K) = 0, Soc(L) = Soc(H_\alpha(N)) \oplus Soc(H_\alpha(K)) = Soc(H_\alpha(N))$, which is a contradiction. Therefore N is socle-regular.

Theorem 2.3. *The $QTAG$-module M is socle-regular if and only if the direct sum of β copies of M, $\bigoplus_{\gamma<\beta} M_\gamma$ is socle-regular for any cordinal β.*

Proof. Let K be a fully invariant submodule of $\bigoplus_{\gamma<\beta} M_\gamma$, then by Lemma 2.1, $K = \bigoplus_{\gamma<\beta} (M_\gamma \cap K)$, where each M_γ is isomorphic to M. Now $Soc(K) = \bigoplus_{\gamma<\beta} Soc(M_\gamma \cap K)$ and each $M_\gamma \cap K$ is fully invariant in M_γ. Since M is socle-reguler, each $Soc(M_\gamma \cap K) = Soc\left(H_{\alpha_\gamma}(M_\gamma)\right)$ for ordinals α_γ's are not equal, the submodule $\bigoplus_{\gamma<\beta} Soc\left(H_{\alpha_\gamma}(M_\gamma)\right)$ is not fully invariant, therefore $Soc(K) = Soc\left(H_\alpha \bigoplus_{\gamma<\beta} M_\gamma\right)$ where $\alpha = \alpha_\gamma$ for all γ.

Conversely suppose $\bigoplus_{\gamma<\beta} M_\gamma$ is socle-reguler and N an arbitrary fully invariant sub-module of M. Now $\bigoplus_{\gamma<\beta} N_\gamma$ is fully invariant in $\bigoplus_{\gamma<\beta} M_\gamma$ which is socle-regular. Therefore we have $Soc\left(\bigoplus_{\gamma<\beta} N_\gamma\right) = Soc\left(H_\alpha\left(\bigoplus_{\gamma<\beta} M_\gamma\right)\right)$ for some ordinal α and $Soc(N) = Soc(H_\alpha(M))$ implying that M is socle-regular.

Proposition 2.2. *Let M be a socle-regular $QTAG$-module and L a fully invariant sub-module of M such that $H_\omega(L) = H_\omega(M)$. Then L is socle-regular.*

Proof. Let K be a fully invariant submodule of L. Then K is also fully invariant in M. Since M is socle-regular $Soc(K) = Soc(H_\alpha(M)) = Soc(H_\alpha(L))$ for all ordinals $\alpha \geq \omega$. Therefore $Soc(K) = Soc(H_\alpha(M)) = Soc(H_\alpha(L))$ if $\alpha \geq \omega$ and if α is finite, then $Soc(K) = Soc(H_k(M)) \supseteq Soc(H_k(L))$ and by Proposition 2.1 (i), $inf(Soc(K))_L$ is finite. Again by Proposition 2.1 (ii), $Soc(K) = Soc(H_j(L))$ for some j and L is socle-regular.

Remark 2.4. *For any large submodule L of M, $H_\omega(L) = H_\omega(M)$, therefore large sub-modules are socle-regular.*

For a $QTAG$-module M, the property of being socle-regular is shared with $H_\omega(M)$ under certain conditions.

Theorem 2.4. *Let M be a $QTAG$-module such that $M/H_\omega(M)$ is a direct sum of uniserial modules. Then M is socle-regular if and only if $H_\omega(M)$ is socle-regular.*

Proof. Let N be a fully invariant submodule of M. If $Soc(N) \nsubseteq Soc(H_\omega(M))$, then $inf(Soc(N))$ is finite and by Proposition 2.1, $Soc(N) = Soc(H_k(M))$, for some $k \in \mathbb{Z}^+$and if $Soc(N) \subseteq Soc(H_\omega(M))$, $Soc(N)$ is fully invariant in $H_\omega(M)$. Since $H_\omega(M)$ is socle-regular, $Soc(N) = Soc\left(H_\alpha(H_\omega(M))\right)$ for some ordinal α and $Soc(N) = Soc(H_{\omega+\alpha}(M))$ and M is socle-regular. Necessity is trivial.

Theorem 2.5. *Let $M = N \oplus K$ be a socle-regular module such that every homomorphism from N to K is small, then N is socle-regular.*

Proof. Let L be a fully invariant submodule of N. If $inf(Soc(L))$ is finite then by Proposition 2.1, $Soc(L) = Soc(H_k(N))$ for some $k \in \mathbb{Z}^+$, otherwise $Soc(L) \subseteq Soc(H_\omega(N))$. Since any endomorphism f of M may be expressed as the matrix $\begin{pmatrix} f_1 & f_2 \\ g_1 & g_2 \end{pmatrix}$ where $f_2 \in$ Hom (N, K) i.e. f_2 is small. Now $f(Soc(L \oplus 0)) \subseteq f_1(Soc(L)) \oplus f_2(Soc(L))$ and $Soc(L) \subseteq H_\omega(N)$ imply that $f_2(Soc(L)) = 0$ as f_2 is small. Therefore $Soc(L) \oplus 0$ is fully invariant in M and $Soc(L) \oplus 0 = Soc(H_\lambda(M))$ for some ordinal λ. Thus $Soc(L) = Soc(H_\lambda(N))$ and N is socle-regular.

We end this paper with the following open problem:

Problem. Are all the $QTAG$-modules of length $\omega + 1$ regular?

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