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The relation between quasi valuation and valuation ring and filtered ring

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Abstract: In this paper we show the relation between filtered ring and quasi valuation and valuation ring . We show if R is a filtered ring then we can define a quasi valuation. And if R is some kind of filtered ring then we can define a valuation. Then we prove some properties and relations for R.

Keywords: Filtered ring, Quasi valuation ring, Valuation ring, Strongly filtered ring.

1. Introduction

In algebra valuation ring and filtered ring are two most important structure [5],[6],[7]. We know that filtered ring is also the most important structure since filtered ring is a base for graded ring especially associated graded ring and completion and some similar results, on the Andreadakis–Johnson filtration of the automorphism group of a free group [1], on the depth of the associated graded ring of a filtration [2],[3]. So, as these important structures, the relation between these structure is useful for finding some new structures, and if R is a discrete valuation ring then R has many properties that have many usage for example Decidability of the theory of modules over commutative valuation domains [7], Rees valuations and asymptotic primes of rational powers in Noetherian rings and lattices [6].

In this article we investigate the relation between filtered ring and valuation and quasi valuation ring. We prove that if we have filtered ring then we can find a quasi valuation on it. Continuously we show that if R be a strongly filtered then exist a valuation, Similarly we prove it for PID. At the end we explain some properties for them.

2. Preliminaries

Definition 2.1 A filtered ring R is a ring together with a family $\{R_n\}_{n\geq 0}$ of subgroups of R satisfying in the following conditions:

i. $R_0 = R$;

ii. $R_{n+1} \subseteq R_n$ for all $n \geq 0$;

iii. $R_n R_m \subseteq R_{n+m}$ for all $n, m \geq 0$.

Definition 2.2. Let R be a ring together with a family $\{R_n\}_{n\geq 0}$ of subgroups of R satisfying the following conditions:

i. $R_0 = R$;

ii. $R_{n+1} \subseteq R_n \; for \, all \, n \geq 0$;

iii. $R_n R_m = R_{n+m} \; for \, all \, n, m \geq 0$.

Then we say R has a strong filtration.

Definition 2.3. Let R be a ring and I an ideal of R. Then $R_n = I^n$ is called I-adic filtration.

Definition 2.4. A map $f : M \to N$ is called a homomorphism of filtered modules if: (i) f is R-module an homomorphism and (ii) $f(M_n) \subseteq N_n$ for all $n \geq 0$.

Definition 2.5. A subring R of a filed K is called a valuation ring of K if for every $\alpha \in K, \alpha \neq 0$, either $\alpha \in R$ or $\alpha^{-1} \in R$.

Definition 2.6. Let Δ be a totally ordered abelian group. A valuation v on R with values in Δ is a mapping $v: R^* \to \Delta$ satisfying:

i. $v(ab) = v(a) + v(b)$;

ii. $v(a + b) \geq Min\{v(a), v(b)\}$.

Definition 2.7. Let Δ be a totally ordered abelian group. A quasi valuation v on R with values in Δ is a mapping $v: R^* \to \Delta$ satisfying :

i. $v(ab) \geq v(a) + v(b)$;

ii. $v(a + b) \geq Min\{v(a), v(b)\}$.

Remark 2.1. R is said to be **vaulted ring**; $R_v = \{x \in R : v(x) \geq 0\}$ and $v^{-1}(\infty) = \{x \in R: v(x) = \infty\}$.

Definition 2.8. Let K be a filed. A discrete valuation on K is a valuation $v: K^* \to \mathbb{Z}$ which is surjective.

Theorem 2.1. If R is a UFD then R is a PID (see [2]).

Proposition 2.1. Any discrete valuation ring is a Euclidean domain(see[3]).

Remark 2.2. If R is a ring, we will denote by $Z(R)$ the set of **zero-divisors** of R and by $T(R)$ the **total ring of fractions** of R.

Definition 2.9. A ring R is said to be a **Manis valuation ring** (or simply a **Manis ring**) if there exist a valuation v on its total fractions $T(R)$, such that $R = R_v$.

Definition 2.10. A ring R is said to be a **Prüfer ring** if each overring of R is integrally closed in $T(R)$.

Definition 2.11. A Manis ring R_v is said to be v**-closed** if $R_v/v^{-1}(\infty)$ is a valuation domain (see Theorem 2 of [8]).

3. Quasi Valuation and Valuation derived from Filtered ring

Let R be a ring with unit and R a filtered ring with filtration $\{R_n\}_{n>0}$.

Lemma 3.1. Let R be a filtered ring with filtration $\{R_n\}_{n>0}$. Now we define $v: R \to \mathbb{Z}$ such that for every $\alpha \in R$ and $v(\alpha) = \min\{i | \alpha \in R_i \backslash R_{i+1}\}$.

Then we have $v(\alpha\beta) \geq v(\alpha) + v(\beta)$.

Proof. For any $\alpha, \beta \in R$ with $v(\alpha) = i$ and $v(\beta) = j$, $\alpha\beta \in R_i R_j \subseteq R_{i+j}$.

Now let $v(\alpha\beta) = k$ then we have $\alpha\beta \in R_k \backslash R_{k+1}$.

We show that $k \geq i + j$.

Let $k < i + j$ so we have $k + 1 \leq i + j$ hence $R_{K+1} \supset R_{i+j}$ then $\alpha\beta \in R_{i+j} \subseteq R_{k+1}$ it is contradiction. So $k \geq i + j$. Now we have $v(\alpha\beta) \geq v(\alpha) + v(\beta)$.

Lemma 3.2. Let R be a filtered ring with filtration $\{R_n\}_{n>0}$. Now we define $v: R \to \mathbb{Z}$ such that for every $\alpha \in R$ and $v(\alpha) = \min\{i |\ \alpha \in R_i \backslash R_{i+1}\}$.

Then $v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$

Proof. For any $\alpha, \beta \in R$ such that $v(\alpha) = i$ and $v(\beta) = j$ and $v(\alpha + \beta) = k$ so we have $\alpha + \beta \in R_k \backslash R_{k+1}$. Without losing the generality, let $i < j$ so $R_j \subset R_i$ hence $\beta \in R_i$. Now if $k < i$ then $k + 1 \leq i$ and $R_i \subset R_{k+1}$ so $\alpha + \beta \in R_i \subset R_{k+1}$ it is contradiction. Hence $k \geq i$ and so we have $v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$.

Theorem 3.1. Let R be a filtered ring. Then there exist a quasi valuation on R.

Proof. Let R be a filtered ring with filtration $\{R_n\}_{n>0}$. Now we define $v: R \to \mathbb{Z}$ such that for every $\alpha \in R$ and $v(\alpha) = \min\{i |\ \alpha \in R_i \backslash R_{i+1}\}$.

Then

i) By lemma (3.1) we have $v(\alpha\beta) \geq v(\alpha) + v(\beta)$.

ii) By lemma(3.2) we have $v(\alpha + \beta) \geq min\{v(\alpha), v(\beta)\}$. So by Definition 2.7 R is quasi valuation ring.

Proposition 3.1. Let R be a strongly filtered ring. Then there exists a valuation on R.

Proof. By theorem (3.1) we have $v(\alpha\beta) \geq v(\alpha) + v(\beta)$ and $v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$. Now we show $v(\alpha\beta) = v(\alpha) + v(\beta)$. Let $v(\alpha\beta) > v(\alpha) + v(\beta)$ so $k > i + j$ and it is contradiction. So $v(\alpha\beta) = v(\alpha) + v(\beta)$, then there is a valuation on R.

Corollary 3.1. Let R be a strongly filtered ring, then R is a Euclidean domain.

Proof. By proposition (3.1) R is a discrete valuation and so by proposition (2.1) R is a Euclidean domain.

Proposition 3.2. Let P is a prime ideal of R and $\{P^n\}_{n\geq 0}$ be P-adic filtration. Then there exists a valuation on R.

Proof. By theorem (3.1) we have $v(\alpha\beta) \geq v(\alpha) + v(\beta)$ and $v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$. Now we show $v(\alpha\beta) = v(\alpha) + v(\beta)$. Let $v(\alpha\beta) > v(\alpha) + v(\beta)$ so $k > i + j$ then $\alpha\beta \in P^k \subsetneq P^{i+j}$ and $k \geq i + j + 1$, since P is a prime ideal hence $\alpha \in P^{i+1}$ or $\beta \in P^{j+1}$ and it is contradiction. So $v(\alpha\beta) = v(\alpha) + v(\beta)$, then there is a valuation on R.

Proposition 3.3. Let R be a PID then there is a valuation on R.

Proof. By theorem (3.1) and proposition (3.2) there is a valuation on R.

Corollary 3.2. If R is an UFD then there exists a valuation on R, then R is a Euclidean domain.

Corollary 3.3. Let R be a ring and P is a prime ideal of R. If R has a P-adic filtration and $R = \cup_{i=0}^{+\infty} P^i$, then R is a Euclidean domain.

Proof. By proposition (3.2) R is a discrete valuation and so by proposition (2.1) R is a Euclidean domain.

Corollary 3.4. Let R be a PID then R is a Euclidean domain.

Proof. By proposition (3.3) and proposition (2.1) we have R is a Euclidean domain.

Corollary 3.5. Let R be a UFD then R is a Euclidean domain.

Corollary 3.6. Let R be a strongly filtered ring. Then R is Manis ring.

Corollary 3.7. Let P is a prime ideal of R and $\{P^n\}_{n\geq 0}$ be P-adic filtration. Then R is Manis ring.

Proposition 3.4. Let R_v be a Manis ring. If R_v is v-closed, then R_v is Prüfer.

Proof. See proposition 1 of [9]

Proposition 3.5. Let R be a strongly filtered ring. Then R is v-closed.

Proof. By proposition (3.1) and definition (2.9) we have R is Manis ring and $R = R_v$.

Now let $\alpha, \beta \in R$ and

$v(\alpha) = i$ and $v(\beta) = j$

Consequently if

$$\left(\alpha + v^{-1}(\infty)\right)\left(\beta + v^{-1}(\infty)\right) \in v^{-1}(\infty)$$

Then $i + j \geq \infty$ so $\alpha \in v^{-1}(\infty)$ or $\beta \in v^{-1}(\infty)$ .Hence by definition (2.11) R is v-closed.

Corollary 3.8. Let R be a strongly filtered ring. Then R is Prüfer.

Proof. By proposition (3.6) R is v-closed so by proposition (3.4) R is Prüfer.

Proposition 3.6. Let P is a prime ideal of R and $\{P^n\}_{n\geq 0}$ be P-adic filtration. Then R is v-closed.

Proof. By proposition (3.2) and definition (2.9) we have R is Manis ring and $R = R_v$.

Now let $\alpha, \beta \in R$ and

$v(\alpha) = i$ and $v(\beta) = j$

Consequently if

$$\left(\alpha + v^{-1}(\infty)\right)\left(\beta + v^{-1}(\infty)\right) \in v^{-1}(\infty)$$

Then $i + j \geq \infty$ so $\alpha \in v^{-1}(\infty)$ or $\beta \in v^{-1}(\infty)$ .Hence by definition (2.11) R is v-closed.

Corollary 3.9. Let P is a prime ideal of R and $\{P^n\}_{n\geq 0}$ be P-adic filtration. Then R is Prüfer.

Proof. By proposition (3.6) R is v-closed so by proposition (3.4) R is Prüfer.

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