Generalization of \( ([e], [e] \lor [c]) \)-Ideals of BE-algebras

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Abstract: In this paper, using \( N \)-structure, the notion of an \( N \)-ideal in a BE-algebra is introduced. To obtain a more general form of an \( N \)-ideal, a point \( N \)-structure which is \((k\text{-conditionally})\) employed in an \( N \)-structure is proposed. Using these notions, the concept of an \( ([e], [e] \lor [c_k]) \)-ideal is introduced and related properties are investigated. The notion \( ([e], [e] \lor [c_k]) \)-ideal is a generalization of \( ([e], [c]) \)-ideal. We derive some characterizations of \( ([e], [e] \lor [c_k]) \)-ideals of BE-algebras.

Keywords: BE-algebra, (Transitive, self distributive) BE-algebra, Ideal, \( N \)-ideal, \( ([e], [e] \lor [c_k]) \)-ideal.

1 Introduction

A (crisp) set \( A \) in a universe \( X \) can be defined in the form of its characteristic function \( \mu_A : X \rightarrow \{0, 1\} \) yielding the value 1 for elements belonging to the set \( A \) and the value 0 for elements excluded from the set \( A \).

So far most of the generalization of the crisp set have been conducted on the unit interval \([0, 1]\) and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets spread positive information that fit the crisp point \( \{1\} \) into the interval \([0, 1]\).

Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply a mathematical tool.

To attain such an object, Lee et al.\([12]\) introduced a new function which is called a negative-valued function, and constructed \( N \)-structures. They applied \( N \)-structures to BCK/BCI-algebras, and discussed \( N \)-ideals in BCK/BCI-algebras. In 1966, Iseki and Imai \([7]\) and Iseki \([8]\) introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. As a generalization of a BCK-algebra, Kim and Kim \([3]\) introduced the notion of a BE-algebra and filter of a BE-algebra, and investigated several properties. So and Ahn \([6]\) introduced the notion of ideals in BE-algebras. They considered several descriptions of ideals in BE-algebras. They defined the upper set in BE-algebra and derived some relation between ideal theory and upper set in BE-algebras. Kim and Lee generalized the concept of So and Ahn, to defined extended upper set in BE-algebras \([4]\). They provided the relations between filters and extended upper set in BE-algebras. Ahn et al. introduced the concepts of self-distributive and transitive BE-algebras and also discussed some properties of filters in

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commutative BE-algebras [1]. They discussed some properties of characterizations of generalized upper set \( A_\alpha(u,v) \) to the structure of ideals and filters in BE-algebras. The Congruence’s and BE-relations on BE-algebra was defined by Yon et al. [2]. Recently, Saeid et al. studied filters in BE-algebras [10]. They defined positive filters, normal filters and implicative filters of BE-algebras. Also gave some relation among these types of filters in BE-algebras. They introduced some interesting results on these filters. Also gave some characterization of BE-algebras by these filters.

Kang, and Jun [9], introduced the notion of an \( N \)-ideal of BE-algebra. In [9], a point \( N \)-structure which is (Conditionally) employed in an \( N \)-structure is proposed. The concept of \([e],[e \lor e]\)-ideals and discussed the related properties. The present author introduce the new notion called \( k \)-conditionally employed of point \( N \)-structure. We use this concept to \( N \)-ideal of BE-algebra to introduce a generalization of \([e],[e \lor e]\)-ideals of BE-algebras.

In this paper, we introduce the notion of \( k \)-conditionally employed to \( N \)-structure. By applying of employed and \( k \)-conditionally employed to \( N \)-structure with point \( N \)-structure to introduce the concept of an \([e],[e \lor e]\)-ideal in a BE-algebra and give some related properties. The aim of this paper is to introduce a new generalization of the concept of \([e],[e \lor e]\)-ideal. We give some characterization of \([e],[e \lor e]\)-ideals of BE-algebras by the level sets. We further discuss some interesting results of \([e],[e \lor e]\)-ideals of BE-algebra.

### 2 Preliminaries

**Definition 1.** [3] Let \( K(\tau) \) be a class of type \( \tau = (2,0) \). A system \((X,*,1) \in K(\tau)\) define a **BE-Algebra** if the following axioms hold:

\[
\begin{align*}
(V_1) & \quad (\forall x \in X) \ (x*x = 1), \\
(V_2) & \quad (\forall x \in X) \ (x*1 = 1), \\
(V_3) & \quad (\forall x \in X) \ (1*x = x), \\
(V_4) & \quad (\forall x,y,z \in X) \ (x*(y*z) = y*(x*z)) .
\end{align*}
\]

**Definition 2.** [6] A relation "\( \leq \)" on a BE-algebra \( X \) is defined by \((\forall x,y \in X) \ (x \leq y \iff x*y = 1)\).

**Definition 3.** [6] A BE-algebra \( X \) is called **Self-distributive** if \( x*(y+z) = (x*y)*(x*z) \) for all \( x,y,z \in X \).

**Definition 4.** [4] A BE-algebra \((X,*,1)\) is said to be **Transitive** if it satisfies: \((\forall x,y,z \in X) \ (y*z \leq (x*y)*(x*z))\).

**Definition 5.** [6] Let \( I \) a non-empty subset of an BE-algebra \( X \) then \( I \) is called an **Ideal** of \( X \) if:

\[
\begin{align*}
(1) & \quad (\forall x \in X,s \in I) \ (x*s \in I), \\
(2) & \quad (\forall x \in X,s,q \in I) \ ((x*(q*x))*x \in I).
\end{align*}
\]

**Lemma 1.** [6] A non-empty subset \( I \) of \( X \) is an ideal of \( X \) if and only if it satisfies:

\[
\begin{align*}
(1) & \quad 1 \in I, \\
(2) & \quad (\forall x,z \in X) \ (\forall y \in I) \ (x*(y*z) \in I \Rightarrow x*z \in I).
\end{align*}
\]

### 3 \( N \)-ideals of a BE-algebra

**Definition 6.** [9] An element of \( \tau(X,[-1,0]) \) is called a **Negative-valued function** from \( X \) to \([-1,0]\) (briefly, \( N \)-function on \( X \)).
Definition 7. [9] An ordered pair \((X, f)\) of \(X\) and an \(N\)-function \(f\) on \(X\) is called an \(N\)-structure.

Definition 8. [9] For any \(N\)-structure \((X, f)\) the non-empty set

\[ C(f; t) := \{ x \in X \mid f(x) \leq t \} \]

is called a closed \((f, t)\)-cut of \((X, f)\), where \(t \in [-1, 0]\).

Definition 9. [9] By an \(N\)-ideal of \(X\) we mean an \(N\)-structure \((X, f)\) which satisfies the following condition: \((\forall t \in [-1, 0]) \ (C(f; t) \in J(X) \cup \{\emptyset\})\). where \(J(X)\) is a set of all ideal of \(X\).

Example 1. Let \(X = \{1, \alpha, h, m, 0\}\) be a set with a multiplication table given by:

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Then \((X; +, 1)\) is a BE-algebra. Consider an \(N\)-structure \((X, f)\) in which \(t\) is defined by:

\[ f(y) = \begin{cases} -0.7 & \text{if } y \in \{1, \alpha, h\} \\ -0.2 & \text{if } y \in \{m, 0\} \end{cases} \]

Then

\[ C(f; t) = \begin{cases} \{1, \alpha, h\} & \text{if } t \in [-0.7, 0] \\ \emptyset & \text{if } t \in [-1, -0.7] \end{cases} \]

Note that \(\{1, \alpha, h\}\) is an ideals of BE-algebra \(X\), and hence \((X, f)\) is an \(N\)-ideal of \(X\).

Lemma 2. Each \(N\)-ideal \((X, f)\) of BE-algebra \(X\) satisfies the condition \((\forall x \in X) \ (f(1) \leq f(x))\).

Proof. Since in BE-algebra we have \(x * x = 1\), thus we have \(f(1) = f(x * x) \leq f(x)\) for all \(x \in X\).

Proposition 1. Each \(N\)-ideal \(f\) of BE-algebra \(X\) satisfies the condition \((\forall x, y \in X) \ (f((x * y) * y) \leq f(x))\).

Proof. Straightforward.

Proposition 2. Each \(N\)-ideal \(f\) of BE-algebra \(X\) satisfies the condition \((\forall x, y \in X) \ (f(y) \leq \max\{f(x), f(x * y)\})\).

Proof. It can be easily proved.

Corollary 1. If \(x \leq y\), then each \(N\)-ideal \(f\) of BE-algebra \(X\) satisfies the condition; \(f(y) \leq f(x)\).

Proof. Suppose \(x \leq y\) for all \(x, y \in X\). Then \(x * y = 1\), so

\[ f(y) = f(1 * y) = f((x * y) * y) \]

By proposition 1, \(f((x * y) * y) \leq f(x)\), hence \(f(y) \leq f(x)\).
4 \([\{e\}, \{e\} \lor \{c_k\}]\)-Ideals

In [9], Jun et. al introduced the concept of N-ideals in BE-algebras and generalized N-ideals in BE-algebra by using the concept of employed and conditionally employed. In this section, we give further generalization of conditionally employed which is called \(k\)-conditionally employed. We use this concept to N-ideals of BE-algebras to introduce the notion of \([\{e\}, \{e\} \lor \{c_k\}]\)-ideals of BE-algebras. The concept of \([\{e\}, \{e\} \lor \{c_k\}]\)-ideals of BE-algebras is a generalization of N-ideals and \([\{e\}, \{e\} \lor \{c\}]\)-ideals of BE-algebras.

Let \(f\) be an \(N\)-structure of \(X\) in which \(f\) is given by:

\[
 f(y) = \begin{cases} 
 0 & \text{if } y \neq x \\
 t & \text{if } y = x 
\end{cases}
\]

where \(t \in [-1, 0]\). In this case, \(f\) is represented by \(\frac{t}{\alpha}\). \((X, \frac{t}{\alpha})\) is called Point \(N\)-structure [9]. A Point \(N\)-structure \((X, \frac{t}{\alpha})\) is called Employed in an \(N\)-structure \((X, f)\) of BE-algebra \(X\) if \(f(x) \leq t\) for all \(x \in X\), and \(t \in [-1, 0]\). It is represented as \((X, \frac{t}{\alpha})[e](X, f)\) or \(\frac{t}{\alpha}[e]f\). A point \(N\)-structure \((X, \frac{t}{\alpha})\) is called \((k\)-Conditionally Employed\) with an \(N\)-structure \((X, f)\) if \(f(x) + t + k + 1 < 0\) for all \(x \in X\), \(t \in [-1, 0]\) and \(k \in [0, 1]\). It is denoted by \((X, \frac{t}{\alpha})[c_k](X, f)\) or \(\frac{t}{\alpha}[c_k]f\). To say that \((X, \frac{t}{\alpha})[e](X, f)\) (or briefly, \(\frac{t}{\alpha}[e]f\)) we mean \((X, \frac{t}{\alpha})[e](X, f)\) or \((X, \frac{t}{\alpha})[c_k](X, f)\) (or briefly, \(\frac{t}{\alpha}[e]\) or \(\frac{t}{\alpha}[c_k]f\)).

To say that \(\frac{t}{\alpha}f\) we mean \(\frac{t}{\alpha}f\) does not hold for \(\alpha \in \{e\}, \{c_k\}, \{e\} \lor \{c_k\}\).

**Definition 10.** An \(N\)-structure \((X, f)\) is called \([\{e\}, \{e\} \lor \{c_k\}]\)-ideal of \(X\) if it satisfied:

1. \(\frac{t}{\alpha}[e]f \Rightarrow \frac{t+r}{\alpha+y}([e] \lor [c_k])f\),
2. \(\frac{t}{\alpha}[e]f, \frac{t}{\alpha}f \Rightarrow \frac{1}{\max(\{x+y\})}([e] \lor [c_k])f\) for all \(x, y, z \in X\), where \(t, r \in [-1, 0]\) and \(k \in [0, 1]\).

**Example 2.** Let \(X = \{1, \gamma, 0, m, \omega\}\) be a set with a multiplication table given by:

\[
\begin{array}{c|ccccc}
* & 1 & \gamma & 0 & m & \omega \\
\hline
1 & 1 & \gamma & 0 & m & \omega \\
\gamma & 1 & 1 & \gamma & m & m \\
0 & 1 & 1 & 1 & m & m \\
m & 1 & \gamma & 0 & 1 & \gamma \\
\omega & 1 & 1 & \gamma & 1 & 1 \\
\end{array}
\]

Let \((X, f)\) be an \(N\)-structure. Then \(f\) is defined an \(N\)-structure \((X, f)\), as:

\[
f = \begin{pmatrix}
 1 & \gamma & 0 & m & \omega \\
-0.9 & -0.8 & -0.7 & -0.9 & -0.8 
\end{pmatrix}
\]

For \(k \in (0.4, 1)\) Hence, \(f\) is an \([\{e\}, \{e\} \lor \{c_k\}]\)-ideal of \(X\).

**Theorem 1.** For any \(N\)-structure \((X, f)\), the following are equivalent:

1. \((X, f)\) is a \([\{e\}, \{e\} \lor \{c_k\}]\)-ideal of \(X\).
2. \((X, f)\) satisfies the following inequalities:
2.1. \((\forall x,y \in X)\ (f(x \ast y) \leq \max\{f(y), \frac{-k-1}{2}\}\). 
2.2. \((\forall x,y,z \in X)\ (f((x \ast (y \ast z)) \ast z) \leq \max\{f(x), f(y), \frac{-k-1}{2}\}\) where \(k \in (-1,0]\).

**Proof.** Let \((X,f)\) be a \(\{[e],[e] \vee [c_k]\}\)-ideal of \(X\). Suppose that \(f(x \ast y) > \max\{f(y), \frac{-k-1}{2}\}\) for all \(x,y \in X\). If we take \(t_i := \max\{f(y), \frac{-k-1}{2}\}\), \(t_i \in [\frac{-k-1}{2},0]\), \(\frac{1}{t_i}[e]\) and \(\frac{1}{t_i}[f]\). Also, \(f(x \ast y) + t_i + k + 1 > 2t_i + k + 1 \geq 0\), and so \(\frac{1}{t_i}[e] \leq \frac{1}{t_i}[f]\). This is a contradiction. Thus, \(f(x \ast y) \leq \max\{f(y), \frac{-k-1}{2}\}\) for all \(x,y \in X\). Also, suppose that \(f((x \ast (y \ast z)) \ast z) > \max\{f(x), f(y), \frac{-k-1}{2}\}\) for some \(x,y,z \in X\). Then, \(t := \max\{f(x), f(y), \frac{-k-1}{2}\}\). Then, \(t \geq \frac{-k-1}{2}, \frac{1}{t}[e] f\) and \(\frac{1}{t}[f]\), but \(\frac{1}{t}[e] f\) is a contradiction, and hence \(f((x \ast (y \ast z)) \ast z) \leq \max\{f(x), f(y), \frac{-k-1}{2}\}\) for all \(x,y,z \in X\).

Conversely, suppose that \((X,f)\) satisfies (2.1) and (2.2). Let \(\frac{1}{t}[e] f\) for all \(y \in X\) and \(t \in [-1,0]\). Then, \(f(y) \leq t\). We shall prove that \(\frac{1}{t}[e] \leq \frac{1}{t}[f]\). Since from (2.1) we have \(f(x \ast y) \leq \max\{f(y), \frac{-k-1}{2}\}\). Then, if \(f(y) > \frac{-k-1}{2}\), then \(f(x \ast y) \leq \max\{f(y), \frac{-k-1}{2}\} = f(y) \leq t\). Thus, \(\frac{1}{t}[e] f\) which is a contradiction. If \(f(y) \leq \frac{-k-1}{2}\), which implies that \(f(x \ast y) + t + k + 1 < 2 f(x \ast y) + t + k + 1 \leq 2 \max\{f(y), \frac{-k-1}{2}\} + k + 1 = 0\), i.e., \(\frac{1}{t}[e] f\). Thus \(\frac{1}{t}[e] f\) and \(\frac{1}{t}[f]\) for all \(x,y,z \in X\) and \(t \in [-1,0]\). Then \(f(x) \leq t\) and \(f(y) \leq r\). Suppose that \(\frac{1}{t}[e] f\), i.e., \(f((x \ast (y \ast z)) \ast z) > \max\{t,r\}\). If \(\max\{f(x), f(y)\} > \frac{-k-1}{2}\), then

\[
f((x \ast (y \ast z)) \ast z) \leq \max\{f(x), f(y), \frac{-k-1}{2}\} = \max\{f(x), f(y)\} \leq \max\{t,r\}.
\]

This is impossible, and so \(\max\{f(x), f(y)\} \leq \frac{-k-1}{2}\). It follows that \(f((x \ast (y \ast z)) \ast z) + \max\{t,r\} + k + 1 < 2 f((x \ast (y \ast z)) \ast z) + k + 1 \leq 2 \max\{f(x), f(y), \frac{-k-1}{2}\} + k + 1 = 0\). Hence \(\frac{1}{t}[e] f\) and \(\frac{1}{t}[f]\) is a \(\{[e],[e] \vee [c_k]\}\)-ideal of \(X\).

If \(k = 0\), then the following holds.

**Corollary 2.** For any \(N\)-structure \((X,f)\), the following are equivalent:
1. \((X,f)\) is a \(\{[e],[e] \vee [c_k]\}\)-ideal of \(X\).
2. \((X,f)\) satisfies the following inequalities:

2.1. \((\forall x,y \in X)\ (f(x \ast y) \leq \max\{f(y), -0.5\}\). 
2.2. \((\forall x,y,z \in X)\ (f((x \ast (y \ast z)) \ast z) \leq \max\{f(x), f(y), \frac{-k-1}{2}\}\) where \(k \in (-1,0]\).

**Theorem 2.** Every \(\{[e],[e] \vee [c_k]\}\)-ideal \((X,f)\) of an BE-algebra \(X\) satisfies the following inequalities:

1. \((\forall x \in X)\ (f(1) \leq \max\{f(x), \frac{-k-1}{2}\}\) ,
2. \((\forall x,y \in X)\ (f((x \ast y) \ast y) \leq \max\{f(x), \frac{-k-1}{2}\}\) where \(k \in (-1,0]\).

**Proof.** (1): By using (V1) and theorem 1(2.1), we have

\[
f(1) = f(x \ast x) \leq \max\{f(x), \frac{-k-1}{2}\}
\]

for all \(x \in X\).

(2): By using (V3), we have \(f((x \ast y) \ast y) = f((x \ast (1 \ast y)) \ast y)\) for all \(x,y \in X\).

Then by using theorem 1(2.2), we get

\[
f((x \ast (1 \ast y)) \ast y) \leq \max\{f(x), f(1), \frac{-k-1}{2}\} = \max\{f(x), \frac{-k-1}{2}\},
\]

because by (1) \(f(1) \leq \max\{f(x), \frac{-k-1}{2}\}\) for all \(x,y \in X\).

Hence, \(f((x \ast y) \ast y) \leq \max\{f(x), \frac{-k-1}{2}\}\) for all \(x,y \in X\).

If \(k = 0\), then the following holds.
Lemma 3. Hence, \( (\forall x \in X)(f(1) \leq \max \{f(x), -0.5\} ) \).

Lemma 4. Hence, \( (\forall x, y \in X)(f((x * y) * y) \leq \max \{f(x), -0.5\} ) \).

Corollary 3. Every \( ([e], [e] \lor [c]) \)-ideal of a BE-algebra \( X \) satisfies the following inequalities:

1. \( (\forall x \in X)(f(1) \leq \max \{f(x), -0.5\} ) \),
2. \( (\forall x, y \in X)(f((x * y) * y) \leq \max \{f(x), -0.5\} ) \).

Corollary 4. Each \( ([e], [e] \lor [c_k]) \)-ideal \( X, f \) satisfies the following implication;

\( (\forall x, y \in X)(x \leq y \Rightarrow f(y) \leq \max \{f(x), \frac{-k-1}{2}\} ) \), where \( k \in (-1, 0) \).

Proof. Let \( x \leq y \) for all \( x, y \in X \). Then \( x * y = 1 \), and so

\[
f(y) = f(1 * y) = f((x * y) * y) \leq \max \{f(x), \frac{-k-1}{2}\}
\]

Hence, \( f(y) \leq \max \{f(x), \frac{-k-1}{2}\} \).

If \( k = 0 \), then the following holds.

Lemma 3. Each \( ([e], [e] \lor [c]) \)-ideal \( X, f \) satisfies the following implication;

\( (\forall x, y \in X)(x \leq y \Rightarrow f(y) \leq \max \{f(x), -0.5\} ) \).

Proposition 3. Let \( (X, f) \) be an \( N \)-structure such that

1. \( (\forall x \in X)(f(1) \leq \max \{f(x), \frac{-k-1}{2}\} ) \),
2. \( (\forall x, y, z \in X)(f((x * z) \leq \max \{f(x * (y * z)), f(y), \frac{-k-1}{2}\} ) \).

Then the following implication is valid.

\( (\forall x, y \in X)(x \leq y \Rightarrow f(y) \leq \max \{f(x), \frac{-k-1}{2}\} ) \), where \( k \in (-1, 0) \).

Proof. Suppose \( x \leq y \) for all \( x, y \in X \). Then \( x * y = 1 \), and by using (1) we get

\[
f(y) = f(1 * y) \leq \max \{f(1 * (x+y)), f(x), \frac{-k-1}{2}\}
\]

\[
= \max \{f(1 * 1), f(x), \frac{-k-1}{2}\}
\]

\[
= \max \{f(1), f(x), \frac{-k-1}{2}\}
\]

\[
= \max \{f(x), \frac{-k-1}{2}\}
\]

Hence, \( f(y) \leq \max \{f(x), \frac{-k-1}{2}\} \).

If \( k = 0 \), then the following holds.

Lemma 4. Let \( (X, f) \) be an \( N \)-structure such that

1. \( (\forall x \in X)(f(1) \leq \max \{f(x), -0.5\} ) \),
2. \( (\forall x, y, z \in X)(f((x * z) \leq \max \{f(x * (y * z)), f(y), -0.5\} ) \).

Then the following implication is valid.

\( (\forall x, y \in X)(x \leq y \Rightarrow f(y) \leq \max \{f(x), -0.5\} ) \).

Theorem 3. Let \( (X, f) \) be an \( N \)-structure of transitive BE-algebra \( X \). Then \( (X, f) \) is an \( ([e], [e] \lor [c_k]) \)-ideal of \( X \) if and only if it satisfies the following inequalities:

1. \( (\forall x \in X)(f(1) \leq \max \{f(x), \frac{-k-1}{2}\} ) \),
2. \( (\forall x, y, z \in X)(f((x * z) \leq \max \{f(x * (y * z)), f(y), \frac{-k-1}{2}\} ) \), where \( k \in (-1, 0) \).
Proof. Suppose that \((X, f)\) is an \(([e], [e] \lor [e])\)-ideal of \(X\). From theorem 2(1), it is easily seen that
\[
f(1) \leq \max \{ f(x), \frac{-k-1}{2} \}.
\]
Since \(X\) is transitive,
\[
((y \lor z) \lor z) \lor ((x \lor (y \lor z)) \lor (x \lor z)) = 1 \quad (G)
\]
for all \(x, y, z \in X\). By using \((V_3)\) and \((G)\)
\[
f(x \lor z) = f(1 \lor (x \lor z)) = f(((y \lor z) \lor z) \lor ((x \lor (y \lor z)) \lor (x \lor z)) \lor (x \lor z))
\]
By using theorem 1(2.2), 2(2), we have
\[
f(((y \lor z) \lor z) \lor ((x \lor (y \lor z)) \lor (x \lor z)) \lor (x \lor z)) \leq \max \{ f((y \lor z) \lor z), f(x \lor (y \lor z)), \frac{-k-1}{2} \} = \max \{ f(x \lor (y \lor z)), f((y \lor z) \lor z), \frac{-k-1}{2} \} \leq \max \{ f(x \lor (y \lor z)), f(y), \frac{-k-1}{2} \}.
\]
Hence \(f(x \lor z) \leq \max \{ f(x \lor (y \lor z)), f(y), \frac{-k-1}{2} \}\) for all \(x, y, z \in X\).
Conversely suppose that \((X, f)\) satisfies (1) and (2). By using (2), \((V_1)\), \((V_2)\) and (1)
\[
f(x \lor y) \leq \max \{ f(x \lor (y \lor y)), f(y), \frac{-k-1}{2} \} = \max \{ f(x), f(y), \frac{-k-1}{2} \} = \max \{ f(x), \frac{-k-1}{2} \} = \max \{ f(y), \frac{-k-1}{2} \}
\]
Also by using (2) and (1) we get
\[
f((x \lor y) \lor y) \leq \max \{ f((x \lor y) \lor (x \lor y)), f(x), \frac{-k-1}{2} \} = \max \{ f(1), f(x), \frac{-k-1}{2} \}
\]
for all \(x, y \in X\). Now, since \((y \lor z) \lor z \leq (x \lor (y \lor z)) \lor (x \lor z)\) for all \(x, y, z \in X\), it follows that from proposition 3, we have
\[
f((x \lor (y \lor z)) \lor (x \lor z)) \leq \max \{ f((y \lor z) \lor z), \frac{-k-1}{2} \}
\]
So, from (2), we have
\[
f((x \lor (y \lor z)) \lor z) \leq \max \{ f((x \lor (y \lor z)) \lor (x \lor z)), f(x), \frac{-k-1}{2} \} \leq \max \{ f((y \lor z) \lor z), f(x), \frac{-k-1}{2} \} \leq \max \{ f(x), f(y), \frac{-k-1}{2} \}
\]
for all \(x, y, z \in X\). Using theorem 1, we conclude that \((X, f)\) is a \([e], [e] \lor [c]\)-ideal of \(X\).

If \((k = 0)\), then the following holds.

**Corollary 5.** Let \((X, f)\) be an \(N\)-structure of transitive BE-algebra \(X\). Then \((X, f)\) is an \([e], [e] \lor [c]\)-ideal of \(X\) if and only if it satisfies the following inequalities:

1. \((\forall x \in X) (f(1) \leq \max \{f(x), -0.5\})\).
2. \((\forall x, y, z \in X) (f(x \ast z) \leq \max \{f(x \ast (y \ast z)), f(y), -0.5\})\).

**Theorem 4.** Let \(X\) be a transitive BE-algebra. If \((X, f)\) is a \([e], [e] \lor [c]\)-ideal of \(X\) such that \(f(1) > -\frac{k-1}{2}\), then \((X, f)\) is an \(N\)-ideal of \(X\), where \(k \in (-1, 0]\).

**Proof.** Suppose that \((X, f)\) is a \([e], [e] \lor [c]\)-ideal of \(X\) such that \(-\frac{k-1}{2} < f(1)\). Then \(-\frac{k-1}{2} < f(x)\) and so \(-\frac{k-1}{2} < f(1) \leq f(x)\) for all \(x \in X\) by theorem 3(1)

\[
f(1) \leq \max \{f(x), -\frac{k-1}{2}\}
\]

for all \(x \in X\). It follows that from theorem 3(2),

\[
f(x \ast z) \leq \max \{f(x \ast (y \ast z)), f(y), -\frac{k-1}{2}\} = \max \{f(x \ast (y \ast z)), f(y)\}
\]

for all \(x, y, z \in X\). Hence \((X, f)\) is an \(N\)-ideal of \(X\).

If \((k = 0)\), then the following holds.

**Corollary 6.** Let \(X\) be a transitive BE-algebra. If \((X, f)\) is a \([e], [e] \lor [c]\)-ideal of \(X\) such that \(f(1) > -0.5\), then \((X, f)\) is an \(N\)-ideal of \(X\).

**Theorem 5.** If \((X, f)\) is a \([e], [e] \lor [c]\)-ideal of a transitive BE-algebra \(X\). Show that

\[
(\forall t \in [-1, -\frac{k-1}{2}]) (Q(f; t) \in J(X) \cup \{\emptyset\})
\]

where \(Q(f; t) := \{x \in X \mid \frac{f}{t}[c_k] f\}, J(X)\) is a set of all ideal of \(X\) and \(k \in (-0.5, 0]\).

**Proof.** Suppose that \(Q(f; t) \neq \emptyset\) for all \(t \in [-1, -\frac{k-1}{2}]\). Then there exists \(x \in Q(f; t)\), and so \(\frac{f}{t}[c_k] f\), i.e., \(f(x) + t + k + 1 < 0\).

Using theorem 3(1), we have

\[
f(1) \leq \max \{f(x), -\frac{k-1}{2}\}
\]

\[
= \begin{cases} \frac{-k-1}{2} & \text{if } f(x) \leq \frac{-k-1}{2} \\ f(x) & \text{if } f(x) > \frac{-k-1}{2} \\ -1-t-k & \end{cases}
\]

which indicates that \(1 \in Q(f; t)\). Let \(x \ast (y \ast z) \in Q(f; t)\) for all \(x, y, z \in X\) here \(y \in Q(f; t)\). Then \(\frac{f(x \ast (y \ast z))}{t}[c_k] f\) and \(\frac{f}{t}[c_k] f\), i.e., \(f(x \ast (y \ast z)) + t + k + 1 < 0\) and \(f(y) + t + k + 1 < 0\). Using theorem 3(2), we get

\[
f(x \ast z) \leq \max \{f(x \ast (y \ast z)), f(y), -\frac{k-1}{2}\}
\]
Thus, if \( \max\{f(x \ast (y \ast z), f(y)\} > \frac{-k-1}{2} \), then
\[
f(x \ast z) \leq \max\{f(x \ast (y \ast z)), f(y)\} < -1 - t - k
\]
If \( \max\{f(x \ast (y \ast z)), f(y)\} \leq \frac{-k-1}{2} \), then \( f(x \ast z) \leq \frac{-k-1}{2} < -1 - t - k \). This show that \( \frac{-k}{2} + [e]f \) i.e., \( x \ast z \in Q(f; t) \). By using lemma 1, we have \( Q(f; t) \) is an ideal of \( X \).

If \( (k = 0) \), then the following holds.

**Corollary 7.** If \( (X, f) \) is a \( ([e], [e] \lor [c]) \)-ideal of a transitive BE-algebra \( X \). Show that
\[
(\forall t \in [-1, -0.5]) \ (Q(f; t) \in J(X) \cup \{0\})
\]
where \( Q(f; t) := \{x \in X \mid \frac{t}{2} \in [e]f\} \), and \( J(X) \) is a set of all ideal of \( X \).

**Theorem 6.** Let \( X \) be a transitive BE-algebra. Then the followings are equivalent:

1. An \( N \)-structure \( (X, f) \) is a \( ([e], [e] \lor [c]) \)-ideal of \( X \)
2. \( (\forall t \in [-1, 0]) \ (\forall f \in J(X) \cup \{0\}) \)

where \( [f]_t := C(f; t) \cup \{x \in X \mid f(x) + t + k + 1 \leq 0\}, J(X) \) is a set of all ideal of \( X \), and \( k \in (-1, 0] \).

**Proof.** (1) \( \Rightarrow \) (2): Suppose that (1) satisfies. Let \( [f]_t \neq \emptyset \), here \( t \in [-1, 0) \). Then there exists \( x \in [f]_t \), and so \( f(x) \leq t \) or \( f(x) + t + k + 1 \leq 0 \) for all \( x \in X \) and \( t \in [-1, 0) \). If \( f(x) \leq t \), then
\[
f(1) \leq \max\{f(x), \frac{-k-1}{2}\} \leq \max\{t, \frac{-k-1}{2}\}
\]
\[
= \begin{cases} 
  t & \text{if } t > \frac{-k-1}{2} \\
  \frac{-k-1}{2} & \text{if } t \leq \frac{-k-1}{2}
\end{cases}
\]
By theorem 3(1). Hence \( 1 \in [f]_t \). If \( f(x) + t + k + 1 \leq 0 \), then
\[
f(1) \leq \max\{f(x), \frac{-k-1}{2}\} \leq \max\{-1 - t - k, \frac{-k-1}{2}\}
\]
\[
= \begin{cases} 
  -1 - t - k & \text{if } t < \frac{-k-1}{2} \\
  \frac{-k-1}{2} & \text{if } t \geq \frac{-k-1}{2}
\end{cases}
\]
And so \( 1 \in [f]_t \). Let \( x, y, z \in X \) be such that \( y \in [f]_t \) and \( x \ast (y \ast z) \in [f]_t \). Then \( f(y) \leq t \) or \( f(y) + t + k + 1 \leq 0 \), and \( f(x \ast (y \ast z)) \leq t \) or \( f(x \ast (y \ast z)) + t + k + 1 \leq 0 \). Thus we let the four cases:

- \( a_1 \): \( f(y) \leq t \) and \( f(x \ast (y \ast z)) \leq t \),
- \( a_2 \): \( f(y) \leq t \) and \( f(x \ast (y \ast z)) + t + k + 1 \leq 0 \),
- \( a_3 \): \( f(y) + t + k + 1 \leq 0 \) and \( f(x \ast (y \ast z)) \leq t \),
- \( a_4 \): \( f(y) + t + k + 1 \leq 0 \) and \( f(x \ast (y \ast z)) + t + k + 1 \leq 0 \).
For case (a1), theorem 3(2), implies that
\[ f(x \ast z) \leq \max\{f(x \ast (y \ast z)), f(y), \frac{-k-1}{2}\} \leq \max\{t, \frac{-k-1}{2}\} \]
\[ = \begin{cases} \frac{-k-1}{2} & \text{if } t < \frac{-k-1}{2} \\ t & \text{if } t \geq \frac{-k-1}{2} \end{cases} \]
so that \( x \ast z \in C(f; t) \) or \( f(x \ast z) + t + k \leq \frac{-k-1}{2} + \frac{-k-1}{2} + k = -1 \). Thus \( x \ast z \in [f]_x \). For case (a2), we have
\[ f(x \ast z) \leq \max\{f(x \ast (y \ast z)), f(y), \frac{-k-1}{2}\} \leq \max\{-1-t-k, t, \frac{-k-1}{2}\} \]
\[ = \begin{cases} -1-t-k & \text{if } t < \frac{-k-1}{2} \\ \frac{-k-1}{2} & \text{if } t \geq \frac{-k-1}{2} \end{cases} \]
Thus \( x \ast z \in [f]_x \).
For case (a3), the prove is same to case (a2). For case (a4) we have,
\[ f(x \ast z) \leq \max\{f(x \ast (y \ast z)), f(y), \frac{-k-1}{2}\} \leq \max\{-1-t-k, \frac{-k-1}{2}\} \]
\[ = \begin{cases} -1-t-k & \text{if } t < \frac{-k-1}{2} \\ \frac{-k-1}{2} & \text{if } t \geq \frac{-k-1}{2} \end{cases} \]
So that, \( x \ast z \in [f]_x \) By using lemma 1, \([f]_x \) is an ideal of \( X \).

(2) \( \Rightarrow \) (1): Suppose that (2) hold. If \( f(1) \geq \max\{f(y), \frac{-k-1}{2}\} \) for all \( y \in X \), then \( f(1) \geq t_y \geq \max\{f(y), \frac{-k-1}{2}\} \) for some \( t_y \in [-\frac{k-1}{2}, 0] \). It follows that \( x \in C(f; t_y) \subseteq [f]_x \), but \( 1 \notin C(f; t_y) \). Also, \( f(1) + t_y + k + 1 > 2t_y + k + 1 \geq 0 \). Hence \( 1 \notin [f]_x \), which contradicts the supposition. So, \( f(1) \leq \max\{f(y), \frac{-k-1}{2}\} \) for all \( y \in X \). Suppose that for some \( x, z \in X \), we have \( f(x \ast z) > \max\{f(x \ast (y \ast z)), f(y), \frac{-k-1}{2}\} \) (D).

Taking \( t := \max\{f(x \ast (y \ast z)), f(y), \frac{-k-1}{2}\} \) implies that \( t \in [-\frac{k-1}{2}, 0] \), \( x \in C(f; t) \subseteq [f]_x \), and \( x \ast (x \ast z) \in C(f; t) \subseteq [f]_x \). Since \([f]_x \) is an ideal of \( X \), we have \( x \ast z \in [f]_x \), and so \( f(x \ast z) \leq t \) or \( f(x \ast z) + t + k + 1 \leq 0 \). The inequality (D) induces \( x \ast z \notin C(f; t) \), and \( f(x \ast z) + t + k + 1 > 2t + k + 1 \geq 0 \). Thus \( x \ast z \notin [f]_x \). It contradicts the supposition. Hence \( f(x \ast z) \leq \max\{f(x \ast (y \ast z)), f(y), \frac{-k-1}{2}\} \) for all \( x, y \in X \). Using theorem 3, we have, \((X, f)\) is a \([e, [e] \vee [c_k]]\)-ideal of \( X \).

If \((k = 0)\), then the following holds.

**Corollary 8.** Let \( X \) be a transitive BE-algebra. Then the followings are equivalent:
1. An \( N \)-structure \((X, f)\) is a \([e, [e] \vee [c_k]]\)-ideal of \( X \)
2. \((\forall t \in [-1, 0]) ([f]_x \in J(X) \cup \{\emptyset\}) \)
where \([f]_x := C(f; t) \cup \{x \in X \mid f(x) + t + 1 \leq 0\}\), and \( J(X) \) is a set of all ideal of \( X \)

### 5 Conclusion:

BE-algebra is a type of logical algebra like BCK/BCI/BCH-algebras. A BE-algebra is another generalization of BCK/BCI/BCH-algebras. In this paper, we have investigated the concept of \([e, [e] \vee [c_k]]\)-ideal of a BE-algebra by using \( (k\text{-conditionally}) \) employed of \( N \)-structure with point \( N \)-structure. We also characterized transitive and distributive BE-algebra by \([e, [e] \vee [c_k]]\)-ideal. We also discussed their related properties and provide characterizations of \([e, [e] \vee [c_k]]\)-ideals.
References