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# Comparison between the new $(G'/G)$ expansion method and the extended homogeneous balance method

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Received: 11 June 2015, Revised: 16. June 2015, Accepted: 30 June 2015
Published online: 20 December 2015.

**Abstract:** In this paper, we compare the new $(G'/G)$ expansion method to the extended homogeneous balance method. Both of these methods are proposed for seeking the traveling wave solutions of nonlinear partial differential equations expressed by the hyperbolic functions, trigonometric functions and rational functions. Although both of them are subjected by some modifications using the Riccati equation and the reduced nonlinear ordinary differential equation, respectively, the new $(G'/G)$ expansion method is straightforward and concise, and taking special values for parameters and using some hyperbolic identities, all the solutions obtained by the extended homogeneous balance method coincide with the solutions obtained by the new $(G'/G)$ expansion method. Moreover, the new $(G'/G)$ expansion method gives the general form of solutions and is applied to nonlinear partial differential equations directly without using tedious calculation instead of the extended homogeneous balance method.

**Keywords:** The new $(G'/G)$ expansion method, the extended homogeneous balance method, travelling wave solution, the Riccati equation, the nonlinear partial differential equations.

## 1 Introduction

Nonlinear partial differential equations arise in a large number of physics, mathematics and engineering problems. In the soliton theory, the study of exact solutions to these nonlinear equations plays significiant role, as they provide much information about the physical models they describe. Various powerful methods have been employed to construct exact travelling wave solutions to nonlinear partial differential equations. These methods include the inverse scattering transform $[1]$, the Backlund transform $[2,3]$, the Darboux transform$[4]$, the Hirota bilinear method $[5]$, the tanh-function method $[6,7]$, the sine-cosine method $[8]$, the exp-function method $[9]$, the Riccati expansion method with constant coefficients $[10]$.

The $(G'/G)$ expansion method was first introduced by Wang and Zhang, has been widely used to obtain exact solutions of nonlinear partial differential equations. On the other hand, the homogeneous balance method proposed by Wang to find exact solutions of certain nonlinear PDEs. For more details about these methods, we refer the reader to $[11-25]$ and the references therein.

This paper is organized as follows. In section 2, we have presented the descriptions of the new $(G'/G)$ expansion method and the extended homogeneous balance method. In section 3 and 4, we have compared to these methods and applied it to Benjamin-Bona-Mahony-Burgers (BBMB) equation. Finally, we have given some conclusions in section 5.

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## 2 Outline of the methods

**(i) New $(G'/G)$ expansion method**

In this section, we describe the new $(G'/G)$ expansion method for finding travelling wave solutions of nonlinear partial differential equations. Suppose that a nonlinear partial differential equation(PDE), say in two independent variables $x$ and $t$, is given by

$$P(u, u_t, u_x, u_{xt}, u_{tt}, u_{xx}, ...) = 0 \tag{1}$$

where $u(x,t)$ is an unknown function, $P$ is a polynomial in $u = u(x,t)$ and its various partial derivatives, in which highest order derivatives and nonlinear terms are involved.

The summary of the new $(G'/G)$ expansion method, can be presented in the following six steps:

**Step 1.** To find the travelling wave solutions of Eq.(1) we introduce the wave variable

$$u(x,t) = u(\xi), \xi = x - Vt, \tag{2}$$

where the constant $V$ is generally termed the wave velocity. Substituting Eq.(2) into Eq.(1), we obtain the following ordinary differential equations(ODE) in $\xi$ (which illustrates a principal advantage of a travelling wave solution, i.e., a PDE is reduced to an ODE)

$$P(U, -VU', U', -VU'', V^2U'', U'', ...) = 0. \tag{3}$$

**Step 2.** If necessary we integrate Eq.(3) as many times as possible and set the constants of integration to be zero for simplicity.

**Step 3.** We suppose the solution of nonlinear partial differential equation can be expressed by a polynomial in $\phi$ as

$$u(\xi) = \sum_{i=0}^{m} a_i \phi(\xi)^i, \tag{4}$$

where $\phi = \phi(\xi)$ satisfy the Riccati differential equation

$$\phi'(\xi) = a\phi^2 + b\phi + c, \tag{5}$$

where $a, b$ and $c$ are real constants.

**Step 4.** The positive integer m can be accomplished by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq.(3) as follows: if we define the degree of $u(\xi)$ as $D[u(\xi)] = m$, then the degree of other expressions is defined by

$$D\left[\frac{d^q u}{d\xi^q}\right] = m + q,$$
$$D\left[u^r \left(\frac{d^q u}{d\xi^q}\right)^s\right] = mr + s(q+m).$$

Therefore, we can get the value of $m$ in Eq.(4).

**Step 5.** Finding the general solutions of Eq.(5) by using $(G'/G)$ expansion method as follows:

Let us consider the travelling wave solutions

$$\phi = \sum_{i=0}^{n} b_i \left(\frac{G'}{G}\right)^i.$$

Taking into account step 4 and balancing $\phi'$ with $\phi^2$ leads to $n = 1$. Substituting $\phi = b_0 + b_1\left(\frac{G'}{G}\right)$ into Eq.(5) where $G = G(\xi)$ satisfies the second-order linear ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0 \tag{6}$$

and $G' = \frac{dG}{d\xi}, G'' = \frac{d^2G}{d\xi^2}, \lambda, \mu$ are real constants it will be obtained

$$b_0 = \frac{-\lambda - b}{2a}, b_1 = -\frac{1}{a}$$

and

$$b^2 - 4ac = \lambda^2 - 4\mu.$$

Using the general solutions of Eq.(6) and the values $b_0, b_1,$ the general solution of Eq.(5) will be obtained as follows:

$$\phi = \begin{cases} -\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}\left(\frac{c_1 \sinh\left(\frac{\sqrt{b^2-4ac}}{2}\xi\right) + c_2 \cosh\left(\frac{\sqrt{b^2-4ac}}{2}\xi\right)}{c_1 \cosh\left(\frac{\sqrt{b^2-4ac}}{2}\xi\right) + c_2 \sinh\left(\frac{\sqrt{b^2-4ac}}{2}\xi\right)}\right), b^2 - 4ac > 0 \\ -\frac{b}{2a} - \frac{\sqrt{4ac-b^2}}{2a}\left(\frac{-c_1 \sin\left(\frac{\sqrt{4ac-b^2}}{2}\xi\right) + c_2 \cos\left(\frac{\sqrt{4ac-b^2}}{2}\xi\right)}{c_1 \cos\left(\frac{\sqrt{4ac-b^2}}{2}\xi\right) + c_2 \sin\left(\frac{\sqrt{4ac-b^2}}{2}\xi\right)}\right), b^2 - 4ac < 0 \\ -\frac{b}{2a} - \frac{1}{a}\left(\frac{c_1}{c_1\xi + c_2}\right), b^2 - 4ac = 0 \end{cases} \tag{7}$$

where $c_1$ and $c_2$ are arbitrary constants.

**Step 6.** Substituting Eq.(4) into Eq.(3), using Eq.(5) and collecting all terms with the same order of $\phi$ together, then setting each coefficient of this polynomial to zero yield a set of algebraic equations for $a_i$.Using the values obtained the explicit solutions of Eq.(1) is gained immediately depending of the sign of $b^2 - 4ac$ as in (7).

**(ii) The extended homogeneous balance method** In this section, we describe the extended homogeneous method for finding travelling wave solutions of nonlinear partial differential equations. The solution process has the same steps as in A), except step 5. There are different 5 cases to find the general solutions of Eq.(5) by using extended homogeneous balance method.

**Case 1.** In this case, it is assumed that $\phi = \sum_{i=0}^{n} k_i \tanh^i \xi$. Balancing $\phi'$ with $\phi^2$ leads to $n = 1$. Substituting

$\phi = k_0 + k_1 \tanh\xi$ into Eq.(5) results the following solution of Eq.(5):

$$\phi = -\frac{1}{2a}\left(b + 2\tanh\xi\right), \tag{8}$$

where $b^2 - 4ac = 4$.

**Case 2.** In this case, when a=1 and b=0, the Riccati equation has the following solutions:

$$\phi = \begin{cases} -\sqrt{-c}\tanh\left(\sqrt{-c}\xi\right), c < 0 \\ \sqrt{c}\tan\left(\sqrt{c}\xi\right), c > 0 \\ -\frac{1}{\xi}, c = 0 \end{cases} \tag{9}$$

**Case 3.** In this case, it is assumed that the Riccati equation has the following solution

$$\phi = A_0 + \sum_{i=0}^{n}\left(A_i f^i + B_i f^{i-1} g\right) \tag{10}$$

where

$$f = \frac{1}{\cosh\xi + r}, g = \frac{\sinh\xi}{\cosh\xi + r}$$

which satisfy

$$f'(\xi) = -f(\xi)g(\xi),$$

$$g'(\xi) = rf - r^2 f^2 + f^2.$$

Balancing $\phi'$with $\phi^2$ leads to

$$\phi = A_0 + A_1 f + B_1 g. \tag{11}$$

Substituting Eq.(11) into Eq.(5), collecting the coefficients of the terms with the same power of $f^i(\xi)g^j(\xi)$ and these coefficients to zero yield to the following set of algebraic equations

$$\begin{cases} aA_0^2 + aB_1^2 + bA_0 + c = 0, \\ 2aA_0A_1 - 2arB_1^2 - rB_1 + bA_1 = 0, \\ aA_1^2 + a\left(r^2 - 1\right)B_1^2 + \left(r^2 - 1\right)B_1 = 0, \\ 2aA_0B_1 + bB_1 = 0, \\ 2aA_1B_1 + A_1 = 0. \end{cases}$$

which have solutions

$$A_0 = -\frac{b}{2a}, A_1 = \mp\frac{\sqrt{r^2 - 1}}{2a}, B_1 = -\frac{1}{2a}, b^2 - 4ac = 1.$$

If we use these values in (11) we have

$$\phi = -\frac{1}{2a}\left(b + \frac{\sinh\xi \mp \sqrt{r^2 - 1}}{\cosh\xi + r}\right). \tag{12}$$

**Case 4.** In this case, it is assumed that the Riccati equation has the following solution:

$$\phi = e^{p_3\xi}\rho(z) + p_4(\xi) \tag{13}$$

where $z = e^{p_2\xi} + p_3$ and $p_1, p_2, p_3$ are constants to be determined. Substituting (13) into Eq.(5), it is obtained

$$p_1 = p_2, p_4 = \frac{p_1 - b}{2a}$$

and

$$b^2 - 4ac = p_1^2.$$

Using these values in (13) it is obtained

$$\phi = -\frac{b}{2a} - \frac{p_1}{2a}\left(\frac{e^{\frac{p_1\xi}{2}} - p_3 e^{-\frac{p_1\xi}{2}}}{e^{\frac{p_1\xi}{2}} + p_3 e^{-\frac{p_1\xi}{2}}}\right) \tag{14}$$

where $b^2 - 4ac = p_1^2$.

**Case 5.** In this case, it is assumed that the Riccati equation has the following solutions of the form:

$$\phi = A_0 + \sum_{i=1}^{n} \sinh^{i-1}(A_i \sinh w + B_i \cosh w) \tag{15}$$

where $\frac{dw}{d\xi} = \sinh w$ or $\frac{dw}{d\xi} = \cosh w$. Balancing $\phi'$ with $\phi^2$ leads to

$$\phi = A_0 + A_1 \sinh w + B_1 \cosh w. \tag{16}$$

When $\frac{dw}{d\xi} = \sinh w$, substituting (6) into (5) and setting the coefficients of $\sinh^i w \cosh^j w$ $(i = 0,1,2; j = 0,1)$ to zero, a set of algebraic equations is obtained as follows:

$$\begin{cases} aA_0^2 + aB_1^2 + bA_0 + c = 0, \\ 2aA_0A_1 + bA_1 = 0, \\ aA_0^2 + aB_1^2 = B_1, \\ 2aA_0B_1 + bA_1 = 0, \\ 2aA_0B_1 - A_1 = 0. \end{cases} \tag{17}$$

Solving the resulting system we find the following sets of solution

$$A_0 = -\frac{b}{2a}, A_1 = 0, B_1 = \frac{1}{a}, \text{where } b^2 - 4ac = 4, \tag{18}$$

$$A_0 = -\frac{b}{2a}, A_1 = \mp\frac{1}{2a}, B_1 = \frac{1}{2a}, \text{where } b^2 - 4ac = 1. \tag{19}$$

By using the hypothesis $\frac{dw}{d\xi} = \sinh w$, we obtain

$$\sinh w = -csch\xi, \cosh w = -\coth\xi. \tag{20}$$

By applying (20) in (18) and (19) with (16) we get

$$\phi = -\frac{b}{2a} - \frac{1}{a}\coth\xi, \text{where } b^2 - 4ac = 4, \tag{21}$$

$$\phi = -\frac{b}{2a} \pm \frac{1}{2a}csch\xi - \frac{1}{2a}\coth\xi, \text{where } b^2 - 4ac = 1. \tag{22}$$

## 3 Comparison new $(G'/G)$ expansion method to extended homogeneous balance method

It is well known that any expression containing exponents, trigonometric or hyperbolic functions can be rewritten in different forms. In the case of large expressions the equivalence of these forms is not obvious. Therefore, it is very easy to surmise that different forms of one expression are the different solutions.

In this section, we demonstrate that all these solutions obtained via homogeneous balance method coincide with the solutions obtained by new $(G'/G)$ expansion method. The Riccati equation (5) can be solved using the homogeneous balance method in 5 cases. However, most of these cases do not give any new solutions for the Riccati equation and they can be obtained easily from the solutions in (7).

Now we analyze these 5 cases and show that all the solutions obtained by the extended homogeneous balance method coincide with the solutions obtained by the new $(G'/G)$ expansion method.

**Case 1.** In this case, it is assumed that

$$\phi = \sum_{i=0}^{n} k_i \tanh^i \xi.$$

Balancing $\phi'$ with $\phi^2$ leads to $n = 1$. Substituting $\phi = k_0 + k_1 \tanh\xi$ into Eq.(5) results the following solution of Eq.(5):

$$\phi = -\frac{1}{2a}(b + 2\tanh\xi), \tag{23}$$

where $b^2 - 4ac = 4$. The fact that, when we take the values for $c_1 \neq 0$ and $c_2 = 0$ in (7), where $b^2 - 4ac = 4$, we will obtain $(3.1)$.

**Case 2.** In this case, when $a = 1$ and $b = 0$, the Riccati equation has the following solutions:

$$\phi = \begin{cases} -\sqrt{-c}\tanh\left(\sqrt{-c}\xi\right), & c < 0 \\ \sqrt{c}\tan\left(\sqrt{c}\xi\right) & , c > 0 \\ -\frac{1}{\xi} & , c = 0 \end{cases} \tag{24}$$

The fact that, when we take $a = 1, b = 0$, $c_1 \neq 0$ and $c_2 = 0$ in (7), we will obtain (24).

**Case 3.** In this case, it is assumed that the Riccati equation has the following solution

$$\phi = A_0 + \sum_{i=0}^{n} \left(A_i f^i + B_i f^{i-1} g\right) \tag{25}$$

where

$$f = \frac{1}{\cosh\xi + r}, g = \frac{\sinh\xi}{\cosh\xi + r},$$

which satisfy

$$f'(\xi) = -f(\xi)\,g(\xi),$$

$$g'(\xi) = rf - r^2 f^2 + f^2.$$

Balancing $\phi'$with $\phi^2$ leads to

$$\phi = A_0 + A_1 f + B_1 g. \tag{26}$$

Substituting Eq.(26) into Eq.(5), collecting the coefficients of the terms with the same power of $f^i(\xi)\,g^j(\xi)$ and setting it equal to zero we find the following set of algebraic equations:

$$\begin{cases} aA_0^2 + aB_1^2 + bA_0 + c = 0, \\ 2aA_0A_1 - 2arB_1^2 - rB_1 + bA_1 = 0, \\ aA_1^2 + a\left(r^2-1\right)B_1^2 + \left(r^2-1\right)B_1 = 0, \\ 2aA_0B_1 + bB_1 = 0, \\ 2aA_1B_1 + A_1 = 0. \end{cases}$$

Using Maple gives following solution:

$$A_0 = -\frac{b}{2a}, A_1 = \mp\frac{\sqrt{r^2-1}}{2a}, B_1 = -\frac{1}{2a},$$

where $b^2 - 4ac = 1$. If we set them in (26) we have

$$\phi = -\frac{1}{2a}\left(b + \frac{\sinh\xi \mp \sqrt{r^2-1}}{\cosh\xi + r}\right). \tag{27}$$

Although it seems a different solution, we can obtain it using (7) and some hyperbolic identities. Firstly, taking $c_1 = \cosh\left(\frac{\xi_0}{2}\right), c_2 = \sinh\left(\frac{\xi_0}{2}\right)$ in (26), where $b^2 - 4ac = 1$ and $r = \sqrt{1 + 4c_1^2 c_2^2}$ and using hyperbolic function identities we obtain

$$\phi = -\frac{1}{2a}\left(b + \tanh\left(\frac{\xi+\xi_0}{2}\right)\right) = -\frac{1}{2a}\left(b + \frac{\sinh(\xi) + \sinh(\xi_0)}{\cosh(\xi) + \cosh(\xi_0)}\right). \tag{28}$$

It can easily be seen (27) and (28) are equal.

**Case 4.** In this case, it is assumed that the Riccati equation has the following solution:

$$\phi = e^{p_3\xi}\rho(z) + p_4(\xi) \tag{29}$$

where $z = e^{p_2\xi} + p_3$ and $p_1, p_2, p_3$ are constants to be determined. Substituting (29) into (5), it is obtained

$$p_1 = p_2, p_4 = \frac{p_1 - b}{2a}$$

and

$$b^2 - 4ac = p_1^2.$$

Using these values in (29) it is obtained

$$\phi = -\frac{b}{2a} - \frac{p_1}{2a}\left(\frac{e^{\frac{p_1\xi}{2}} - p_3 e^{-\frac{p_1\xi}{2}}}{e^{\frac{p_1\xi}{2}} + p_3 e^{-\frac{p_1\xi}{2}}}\right). \tag{30}$$

The fact that, the solution in (30) is obtained easily from (7) for

$$c_1 = \frac{1+p_3}{2}, c_2 = \frac{1-p_3}{2}$$

where $b^2 - 4ac = p_1^2$.

**Case 5.** In this case, it is assumed that the Riccati equation has the following solutions of the form:

$$\phi = A_0 + \sum_{i=1}^{n} \sinh^{i-1}(A_i \sinh w + B_i \cosh w) \tag{31}$$

where $\frac{dw}{d\xi} = \sinh w$ or $\frac{dw}{d\xi} = \cosh w$. Balancing $\phi'$ with $\phi^2$ leads to

$$\phi = A_0 + A_1 \sinh w + B_1 \cosh w. \tag{32}$$

Assuming $\frac{dw}{d\xi} = \sinh w$ and substituting (32) into (5) also setting the coefficients of $\sinh^i w \cosh^j w$ $(i = 0,1,2; j = 0,1)$ to zero, a set of algebraic equationsis obtained as follows:

$$\begin{cases} aA_0^2 + aB_1^2 + bA_0 + c = 0, \\ 2aA_0A_1 + bA_1 = 0, \\ aA_0^2 + aB_1^2 = B_1, \\ 2aA_0B_1 + bA_1 = 0, \\ 2aA_0B_1 - A_1 = 0. \end{cases} \tag{33}$$

Using Maple gives following two solutions:

$$A_0 = -\frac{b}{2a}, A_1 = 0, B_1 = \frac{1}{a}, \text{where } b^2 - 4ac = 4, \tag{34}$$

$$A_0 = -\frac{b}{2a}, A_1 = \mp\frac{1}{2a}, B_1 = \frac{1}{2a}, \text{where } b^2 - 4ac = 1. \tag{35}$$

By using the hypothesis $\frac{dw}{d\xi} = \sinh w$, one gets $\sinh w = -csch\xi, \cosh w = -\coth\xi$. If we put these values in (34) and (35) to (32) we have

$$\phi = -\frac{b}{2a} - \frac{1}{a}\coth\xi, \text{where } b^2 - 4ac = 4, \tag{36}$$

$$\phi = -\frac{b}{2a} \pm \frac{1}{2a}csch\xi - \frac{1}{2a}\coth\xi, \text{where } b^2 - 4ac = 1. \tag{37}$$

The fact that, we can reach these two solutions by using (7). When $c_1 = 0$ and $c_2 \neq 0$, where $b^2 - 4ac = 4$, it is gained the solution (36). To obtain the solution (35), firstly, when $c_1 = 0$ and $c_2 \neq 0$, where $b^2 - 4ac = 1$ in (7), and by using some hyperbolic identities, it is seen that

$$\phi = -\frac{b}{2a} - \frac{1}{2a}csch\xi - \frac{1}{2a}\coth\xi, \text{where } b^2 - 4ac = 1. \tag{38}$$

and

$$\phi = -\frac{b}{2a} - \frac{1}{2a}\coth h\left(\frac{\xi}{2}\right) \tag{39}$$

are equal. In the same manner, when $c_1 \neq 0$ and $c_2 = 0$, where $b^2 - 4ac = 1$ in (7), and by using some hyperbolic identities, it is seen that

$$\phi = -\frac{b}{2a} + \frac{1}{2a}csch\xi - \frac{1}{2a}\coth\xi, \text{where } b^2 - 4ac = 1. \tag{40}$$

and

$$\phi = -\frac{b}{2a} - \frac{1}{2a}\tanh\left(\frac{\xi}{2}\right) \tag{41}$$

are equal, as well.

## 4 Application

The regularized long-wave equation, also known as Benjamin-Bona-Mahony (BBM) equation, in the form

$$u_t + u_x + uu_x - u_{xxt} = 0 \tag{42}$$

has been investigated, for the first time, by Benjamin et al. [26] as an alternative model to the Korteweg-de Vries equation for long waves and it plays an important role in the modeling of nonlinear dispersive systems. The BBM equation is applicable to the study of drift waves in plasma or the Ross by waves in rotating fluids. In this section, we will apply these two methods to solve these equations. We have noticed that they give the same solutions. However, the new $(G'/G)$ expansion method can be applied easily and directly to the problems.

Using the wave variable $\xi = x - Vt$ in Eq. (42), then integrating this equation and considering the integration constant to be zero, we obtain

$$(1 - V)U + \frac{U^2}{2} + VU'' = 0 \tag{43}$$

According to step 4 in the each methods, balancing $U^2$ and $U''$ gives $m = 2$. Therefore, the solutions of Eq.(43) can be written in the form

$$U(\xi) = a_0 + a_1\phi + a_2\phi^2, \tag{44}$$

where $a_0, a_1$ and $a_2$ are constants which are unknowns to be determined later and $\phi$ satisfy the Riccati equation (5). Substituting (44) into (43) and collecting all terms with the same power $\phi^i$ and setting each obtained coefficients to zero, yields the set of algebraic equations:

$$
\begin{aligned}
&\phi^4 : a_2^2 + 12Va^2a_2 = 0,\\
&\phi^3 : 2a_2a_1 + 20Vaba_2 + 4Va^2a_1 = 0,\\
&\phi^2 : 2a_2 + 16Vaca_2 + 8Vb^2a_2 - 2Va_2 + a_1^2 + 6Vaba_1 + 2a_0a_2 = 0,\\
&\phi^1 : 2a_1 + 2a_0a_1 + 2Vb^2a_1 - 2Va_1 + 12Vbca_2 + 4Vaca_1 = 0,\\
&\phi^0 : 2a_0 + 2Vbca_1 + a_0^2 - 2Va_0 + 4Vc^2a_2 = 0.
\end{aligned}
$$

Solving this algebraic equations by Maple, we find sets of solutions:

**Set 1.**

$$a_0 = \frac{-12ac}{1+4ac-b^2}, a_1 = \frac{-12ab}{1+4ac-b^2}, a_2 = \frac{-12a^2}{1+4ac-b^2}, V = \frac{1}{1+4ac-b^2};$$

**Set 2.**

$$a_0 = \frac{4ac+2b^2}{-1+4ac-b^2}, a_1 = \frac{12ab}{-1+4ac-b^2}, a_2 = \frac{12a^2}{-1+4ac-b^2}, V = \frac{1}{1-4ac+b^2}.$$

## *4.1 For the new $(G'/G)$ expansion method*

Using these coefficients and (7) in Eq.(44) when $b^2-4ac>0$, we obtain the hyperbolic solutions respectively:

$$u_1(x,t) = \frac{3\left(b^2-4ac\right)}{1+4ac-b^2}\left(1-\left(\frac{c_1\sinh\left(\frac{\sqrt{b^2-4ac}}{2}\xi\right)+c_2\cosh\left(\frac{\sqrt{b^2-4ac}}{2}\xi\right)}{c_1\cosh\left(\frac{\sqrt{b^2-4ac}}{2}\xi\right)+c_2\sinh\left(\frac{\sqrt{b^2-4ac}}{2}\xi\right)}\right)^2\right), \tag{45}$$

where $\xi = x - \frac{t}{1+4ac-b^2}$,

$$u_2(x,t) = \frac{b^2-4ac}{-1+4ac-b^2}\left(-1+3\left(\frac{c_1\sinh\left(\frac{\sqrt{b^2-4ac}}{2}\xi\right)+c_2\cosh\left(\frac{\sqrt{b^2-4ac}}{2}\xi\right)}{c_1\cosh\left(\frac{\sqrt{b^2-4ac}}{2}\xi\right)+c_2\sinh\left(\frac{\sqrt{b^2-4ac}}{2}\xi\right)}\right)^2\right), \tag{46}$$

where $\xi = x - \frac{t}{1-4ac+b^2}$.

When $b^2-4ac<0$, we obtain the trigometric solutions respectively:

$$u_3(x,t) = \frac{3\left(b^2-4ac\right)}{1+4ac-b^2}\left(1+\left(\frac{-c_1\sin\left(\frac{\sqrt{4ac-b^2}}{2}\xi\right)+c_2\cos\left(\frac{\sqrt{4ac-b^2}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{4ac-b^2}}{2}\xi\right)+c_2\sin\left(\frac{\sqrt{4ac-b^2}}{2}\xi\right)}\right)^2\right), \tag{47}$$

where $\xi = x - \frac{t}{1+4ac-b^2}$,

$$u_4(x,t) = \frac{4ac-b^2}{-1+4ac-b^2}\left(1+3\left(\frac{-c_1\sin\left(\frac{\sqrt{4ac-b^2}}{2}\xi\right)+c_2\cos\left(\frac{\sqrt{4ac-b^2}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{4ac-b^2}}{2}\xi\right)+c_2\sin\left(\frac{\sqrt{4ac-b^2}}{2}\xi\right)}\right)^2\right), \tag{48}$$

where $\xi = x - \frac{t}{1-4ac+b^2}$.

When $b^2-4ac=0$, we obtain the rational solution:

$$u_5(x,t) = -12\left(\frac{c_1}{c_1(x-t)+c_2}\right)^2 . \tag{49}$$

## 4.2 For the homogeneous balance method

**Case 1.** Using the values in Set 1 and Set 2 and (23) in (44) for the case 1, the solutions

$$u_6(x,t) = -4+4\tanh^2\left(x+\frac{t}{3}\right), \tag{50}$$

$$u_7(x,t) = \frac{4}{5}-\frac{12}{5}\tanh^2\left(x-\frac{t}{5}\right) \tag{51}$$

are obtained. In fact, these solutions are the special solutions for $c_1 \neq 0$ and $c_2 = 0$ in (45) and (46), where $b^2-4ac=4$. Particularly, taking some special values for $c_1, c_2$ and $b^2-4ac$, we can obtain a lot of solutions from (45) and (46).

**Case 2.** Using these values in Set 1 and Set 2 and (24) in Eq. (44) for the case 2, we obtain following solutions:

$$u_8(x,t) = \frac{-12c}{1+4c}\left(1-\tanh^2\left(\sqrt{-c}\left(x-\frac{t}{1+4c}\right)\right)\right), \tag{52}$$

$$u_9(x,t) = \frac{-4c}{-1+4c}\left(-1+3\tanh^2\left(\sqrt{-c}\left(x-\frac{t}{1-4c}\right)\right)\right), \tag{53}$$

$$u_{10}(x,t) = \frac{-12c}{1+4c}\left(1+\tan^2\left(\sqrt{c}\left(x-\frac{t}{1+4c}\right)\right)\right), \tag{54}$$

$$u_{11}(x,t) = \frac{4c}{-1+4c}\left(1+3\tan^2\left(\sqrt{c}\left(x-\frac{t}{1-4c}\right)\right)\right), \tag{55}$$

$$u_{12}(x,t) = -12\left(\frac{1}{x-t}\right)^2 \tag{56}$$

These solutions are the special solutions for $c_1 \neq 0$ and $c_2 = 0$ in (45), (46), (47), (48) and (49), where $a=1$ and $b=0$.

**Case 3.** Using the values in Set 2 and (27) in Eq. (44) for the case 3, the solution

$$u_{13}(x,t) = \frac{1}{2} - \frac{3}{2}\left(\frac{\sinh\xi \mp \sqrt{r^2-1}}{\cosh\xi + r}\right)^2, \tag{57}$$

is obtained. Setting $c_1 = \cosh\left(\frac{\xi_0}{2}\right), c_2 = \sinh\left(\frac{\xi_0}{2}\right)$ in (46), where $b^2 - 4ac = 1$ and $r = \sqrt{1+4c_1^2c_2^2}$and using hyperbolic function identities, we find same solution.

**Case 4.** Using these values in Set1 and Set 2 and (30) in (44) for the case 4, we find

$$u_{13}(x,t) = \frac{3p_1^2}{1-p_1^2}\left(1 - \left(\frac{e^{\frac{p_1\xi}{2}} - p_3 e^{-\frac{p_1\xi}{2}}}{e^{\frac{p_1\xi}{2}} + p_3 e^{-\frac{p_1\xi}{2}}}\right)^2\right) \tag{58}$$

where $\xi = x - \frac{t}{1-p_1^2}$, and

$$u_{14}(x,t) = \frac{p_1^2}{-1-p_1^2}\left(-1 + 3\left(\frac{e^{\frac{p_1\xi}{2}} - p_3 e^{-\frac{p_1\xi}{2}}}{e^{\frac{p_1\xi}{2}} + p_3 e^{-\frac{p_1\xi}{2}}}\right)^2\right), \tag{59}$$

where $\xi = x - \frac{t}{1+p_1^2}$. As the previous solutions, these solution are the special solutions for $c_1 = \frac{1+p_3}{2}, c_2 = \frac{1-p_3}{2}$ in (45) and (46), where $b^2 - 4ac = p_1^2$.

**Case 5**. Using these values in Set1 and Set 2 and (34) in (44) for the case 5

$$u_{15}(x,t) = -4 + 4\coth^2\left(x + \frac{t}{3}\right), \tag{60}$$

$$u_{16}(x,t) = \frac{4}{5} - \frac{12}{5}\coth^2\left(x - \frac{t}{5}\right) \tag{61}$$

are obtained. These solutions are the special solutions for $c_1 = 0$ and $c_2 \neq 0$ in (45) and (46), where $b^2 - 4ac = 4$. In the similar manner, using these values in Set 2 and (35) in (44) for the case 5

$$u_{17}(x,t) = \frac{1}{2} - \frac{3}{2}\coth^2\left(x - \frac{t}{2}\right) - \frac{3}{2}\operatorname{csc}h^2\left(x - \frac{t}{2}\right) \pm 3\operatorname{csc}h\left(x - \frac{t}{2}\right)\coth\left(x - \frac{t}{2}\right), \tag{62}$$

are otained. These solutions are the special solutions obtained by using some hyperbolic identities and, respectively for $c_1 = 0, c_2 \neq 0$ and $c_1 \neq 0, c_2 = 0$ in (46), where $b^2 - 4ac = 1$.

## 5 Conclusion

We have compared the new $(G'/G)$ expansion method to the extended homogeneous balance method and applied this comparison to well-known BBM equation. Although these two methods based on the solutions of the Riccati equation and have the same solution procedure, we conclude that the new $(G'/G)$ expansion method gives more general form of solutions for partial differential equations than the extended homogeneous balance method. As can be easily seen when looking at the overall in this work, the new $(G'/G)$ expansion method is easier to apply and more reliable according to the extended homogeneous balance method. Moreover, the new $(G'/G)$ expansion method is also a standard, direct and computerizable method, and taking special values for parameters and using some hyperbolic identities, all the solutions obtained by the extended homogeneous balance method coincide the solutions obtained by the new $(G'/G)$ expansion method without using tedious calculations. Moreover, in both methods, the solution procedure can be easily implemented in Maple.

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