New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2011115600

Essentially finitely indecomposable $QTAG$-modules

Alveera Mehdi[1], *Firdhousi Begam*[2] *and Fahad Sikander*[3]

[1,2]Department of Mathematics Aligarh Muslim University Aligarh-India
[3]College of Computing and Informatics, Saudi Electronic University, Jeddah, Kingdom of Saudi Arabia

Received: 23 March 2015, Revised: 26 March 2015, Accepted: 6 November 2015
Published online: 6 January 2016

Abstract: A right module M over an associative ring with unity is a $QTAG$-module if every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules. There are many fascinating results related to these modules and essentially indecomposable modules are extensively researched. Motivated by these modules we generalize them as essentially finitely indecomposable modules whose every direct decomposition $M = \bigoplus_{k\in I} M_k$ implies that there exists a positive integer n such that $H_n(M_i) = 0$ for all M_i's except for a finite number of M_i's. Here we investigate these modules and their relationship with HT-modules. The cases when the modules are not HT-modules are especially highlighted.

Keywords: QTAG-modules, Separable modules, essentially indecomposable modules, HT-modules.

1 Introduction

All the rings R considered here are associative with unity and right modules M are unital $QTAG$-modules. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R-module M with a unique decomposition series, $d(M)$ denotes its decomposition length. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup\left\{ d\left(\frac{yR}{xR}\right) \mid y \in M,\ x \in yR \text{ and } y \text{ uniform}\right\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M) = H_\omega(M)$ is the submodule of M generated by the elements of exponents at most k. M is h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ and it is h-reduced if it does not contain any h-divisible submodule. In other words it is free from the elements of infinite height.

A submodule $N \subset M$ is said to be high if it is a complement of M^1 *i.e* $M = N \oplus M^1$. A submodule N of M is h-pure in M if $H_k(N) = N \cup H_k(M)$ for every $k = 0, 1, 2, \ldots, \infty$. The sum of all simple submodules of M is called the *socle* of M and is denoted by $Soc(M)$.

A $QTAG$-module M is said to be separable if every finite set $\{x_1, x_2, \ldots, x_n\} \subset M$, can be embedded in a direct summand K of M, which is a direct sum of uniserial modules.

The set of modules $\{H_k(M)\},\ k = 0, 1, \ldots, \infty$, forms a base for the neighbourhood system of zero. This gives rise to a topology known as h-topology. The closure of a submodule $N \subset M$ is defined as $\overline{N} = \bigcap_{k=0}^{\infty} (N + H_k(M))$ and it is complete with respect to h-topology if $N = \overline{N}$ and N is h-dense in M if $\overline{N} = M$.

* Corresponding author e-mail: alveera_mehdi@rediffmail.com

The cardinality of the minimal generating set of M is denoted by $g(M)$ and fin $g(M)$ is defined as the infimum of $g(H_k(M))$ for $k = 0,1,2,\ldots,\infty$. For all ordinals α, $f_M(\alpha)$ is the α^{th}-$Ulm-Kaplansky$ invariant of M and it is equal to $g(Soc(H_\alpha(M))/Soc(H_{\alpha+1}(M)))$.

A submodule B of M is called a *basic submodule* of M, if the following conditions hold.

(i) B is a h- pure submodule of M.
(ii) B is a direct sum of uniserial modules.
(iii) M/B is a direct sum of uniform modules of infinite length *i.e.*, M/B is h-divisible.

M is a HT-module if every homomorphism from M to N is small whenever N is a direct sum of uniserial modules. Equivalently, M is a HT-module if and only if $N \supset Soc(H_k(M))$ for some $k < \omega$ whenever M/N is a direct sum of uniserial modules [6]. A $QTAG$-module M is $(\omega+n)$- *projective*, if there exists a submodule $N \subset H^n(M)$ such that M/N is a direct sum of uniserial modules [4]. Notations follows the standard work of Fuchs [1,2].

2 Main results

Several Mathematicians studied essentially indecomposable modules and here we generalize them as essentially finitely indecomposable modules. We start with the following:

Definition 1. *A QTAG-module M is essentially finitely indecomposable if* $M = \bigoplus_{k\in I} M_k$ *implies that there exists a positive integer n such that* $H_n(M_i) = 0$, *for all* M_i *except for a finite number of* M_i *for all decompositions.*

Proposition 1.*M is essentially finitely indecomposable if and only if for all decompositions* $M = N \oplus K$, *K a direct sum of uniserial modules, implies that K is bounded.*

Proof. If M is essentially finitely indecomposable and K is unbounded then for every $k < \omega$, there exists a uniserial summand of K of length greater than k which is a contradiction.

Conversely we may consider an unbounded K which can be written as the direct sum of infinite unbounded summands implying that M is not essentially finitely indecomposable.

Proposition 2. *Let M be a separable QTAG-module and* $\overline{M}$ *the closure of M with respect to h*-topology. *If* $g\left(\frac{\overline{M}}{M}\right) < 2^{\aleph_0}$, *then M is essentially finitely indecomposable.*

Proof. Suppose M is not essentially finitely indecomposable. Then M has an unbounded summand K which is a direct sum of uniserial modules and $M = N \oplus K$. Now N is an unbounded direct sum of uniserial modules and $g(N) \leq \aleph_0$. Therefore $g\left(\frac{\overline{N}}{N}\right) = 2^{\aleph_0}$ and $g\left(\frac{\overline{M}}{M}\right) \geq 2^{\aleph_0}$. It implies that M is essentially finitely indecomposable.

Remark. An unbounded closed module M [3] is essentially finitely indecomposable and for any $n < \omega$, there exist separable $QTAG$-modules with n unbounded summands but for $k > n$, M cannot be decomposed with k unbounded components. We can say if a $QTAG$-module M has no isomorphic submodules then the Ulm-Kaplansky invariants of M should be finite.

Now we investigate the situation with the help of this fact.

Lemma 1. *If M is a separable QTAG-module with no proper isomorphic submodules then M cannot contain an unbounded closed module.*

Proof. Let M be a $QTAG$-module with no isomorphic submodules then all of its Ulm-Kaplansky invariants must be finite. Let K be a closed submodule of M which is unbounded. Consider a basis X for the basic submodule B of M. Then there exists a subset Y of X and a positive integer n, for each $y \in Y$ such that $B = \bigoplus_{y \in Y} H_n(yR)$ is isomorphic to a basic submodule of M. Now the closure of B in K is closed, therefore it contains a submodule isomorphic to M. This contradiction leads to the result.

Proposition 3. *Let N be an unbounded direct sum of uniserial modules and $\bar{N}$ its closure. Let M be a h-pure submodule of $\bar{N}$, having no proper isomorphic submodules and $g\left(\frac{\overline{M}}{M}\right) = n$. If $M = \bigoplus_{i=1}^{m} M_i$ with each M_i unbounded then $m \leq n$.*

Proof. If $M = \bigoplus_{i=1}^{m} M_i$ then $\overline{M} = \bigoplus_{i=1}^{m} \overline{M_i}$. Now $g\left(\frac{\overline{M}}{M}\right) = n$ and each M_i is unbounded, then $g\left(\frac{\bigoplus_{i=1}^{m} \overline{M}_i}{\bigoplus_{i=1}^{m} M_i}\right) \geq m$ because by Lemma 1, M_i cannot be closed.

Remark. If $M = N \oplus K$ is a separable $QTAG$-module and M contains an unbounded closed submodule then either N or K contains an unbounded closed submodule.

To investigate further we define the following:

Definition 2. *Let M be a separable QTAG-module. A submodule N of M is M-essentially indecomposable (M-e.i.) if N contains no unbounded submodules which are summands of M.*

Lemma 2. *Let M be a separable QTAG-module with a submodule N which contains no unbounded summand of M. If N is h-pure and h-dense submodule of M then there exists an essentially finitely indecomposable module T with a high submodule N such that $T/H_\omega(T) \simeq M$.*

Proof. Consider the natural homomorphism $f : M \longrightarrow \frac{M}{N}$. We may define $g : \frac{M}{N} \longrightarrow \frac{M}{N}$ such that $g(\bar{x}) = 0$ if $d(\bar{x}R) = 1$ and $g(\bar{x}) = \bar{y}$ if $d(\bar{x}R) > 1$ and $d\left(\frac{\bar{x}R}{\bar{y}R}\right) = 1$. Now consider the set $T = \{(x, \bar{y}) \mid \phi(x) = g(\bar{y})\}$. If we define $\phi : T \longrightarrow M$ such that $\phi(x, \bar{y}) = x$, then $N' = \{(z, 0) \mid z \in N\}$ is a high submodule of T and $H_\omega(T) \simeq Soc\left(\frac{M}{N}\right) = Ker(g)$, therefore $Ker(\phi) = H_\omega(T)$. If T is not essentially finitely indecomposable then $T = K \oplus T'$ where K is an unbounded direct sum of uniserial modules. Let K' be a high submodule of T'. Then $K \oplus K'$ is a high submodule of T and $Soc(\phi(K + K')) = Soc(\phi(K)) \oplus Soc(\phi(K')) = Soc(N)$. If Q is a *h-pure* submodule of N supported by $Soc(\phi(K))$, then $M = Q \oplus \phi(T')$ and $Q \simeq K$. This contradicts the assumption that no unbounded submodule of N is a summand of M.

Lemma 3. *Let M be a separable QTAG-module such that every basic submodule of M contains an unbounded summand of M. Then any QTAG-module M' such that $\frac{M'}{H_\omega(M')} \simeq M$, is not essentially finitely indecomposable.*

Proof. Consider the $QTAG$-module N with the condition $\frac{N}{H_\omega(N)} \simeq M$ and the natural homomorphism $f : N \longrightarrow \frac{N}{H_\omega(N)}$. If B is a basic submodule of N, then B contains an unbounded submodule T of N such that $f(T)$ is a summand of $f(N)$. Now we may write $f(N) = f(T) \oplus T'$ and T can be extended to a basic submodule B' of N. Again T is a summand of B' and $B' = T \oplus T''$. Now $f(B')$ is a basic submodule of $f(N)$ and $Q = f(B') \cap T'$ is a basic submodule of T'. If K is the submodule of B' such that $f(K) = Q$, then $B' = T \oplus K$. By the properties of basic submodules, $\frac{N}{K \oplus H_\omega(N)} = \frac{T \oplus K \oplus H_\omega(N)}{K \oplus H_\omega(N)}$ is isomorphic to a *h-divisible* submodule D. Consider the natural homomorphism $\psi : N \longrightarrow \frac{N}{K \oplus H_\omega(N)}$ and $P = \psi^{-1}(D)$. Since $\frac{N}{K} = \frac{(T \oplus K)}{K} \oplus \frac{P}{K}$ and $K \cap T = 0$, we have $N = T \oplus P$ implying that N is not essentially finitely indecomposable.

Proposition 4. *Let M be a h-reduced QTAG-module M. If M is essentially finitely indecomposable then $\frac{M}{H_\omega(M)}$ is essentially finitely indecomposable if and only if for all separable QTAG-modules N, one basic submodule of N contains an unbounded summand of N.*

Proof. Suppose for every *h-reduced QTAG*-module M, if M is essentially finitely indecomposable then $M/H_\omega(M)$ is essentially finitely indecomposable. Now by Lemma 2, if one basic submodule of M is M-$e.i.$ then every basic submodule of M is M-$e.i.$
Conversely, if one basic submodule of a separable submodule is M-$e.i.$, then all basic submodules of it are M-$e.i.$ Now by Lemma 3, if M is essentially finitely indecomposable then $M/H_\omega(M)$ is essentially finitely indecomposable.

For a $QTAG$-module M, $fin\, g(M)$ is defined as $min(g(H_k(M))), k = 0,1,2,\ldots$. Here we shall use this concept to prove that a *h-reduced QTAG*-module M is essentially finitely indecomposable if and only if $M/H_\omega(M)$ is essentially finitely indecomposable.

Theorem 1. *Let M be a h-reduced QTAG-module with a summand K such that K is a direct sum of uniserial modules with $fin\, g(K) = \lambda$. Then for a basic submodule B of M, there exists a submodule N such that N is a summand of M and $fin\, g(N) = \lambda$.*

Proof. We may express $Soc(K) = \bigoplus_{i<\omega}(\bigoplus_{\alpha\in J(i)} x_\alpha R)$ where all $J(i)$'s are disjoint and $\alpha \in J(i)$ for all $i < \omega$ such that $H_M(x_\alpha) = i$. Now we may construct a sequence $n(i)$ such that $n(0) < n(1) < n(2) < \ldots$ and $0 < |J(i)| < |J(j)|$ for $i < j$ such that $|\bigcup_{i<\omega} J(i)| = \text{fin}\, g(K) = \lambda$.
Let $S = \bigoplus_{i<\omega}(\bigoplus_{\alpha\in J(n(i))} x_\alpha R)$ and T be a *h-pure* submodule of K such that $Soc(T) = S$. Since T is a summand of K, it is a summand of M and $M = M' \oplus T$. Again M/B is *h-divisible*, therefore every element of $\frac{Soc(T)}{Soc(B)}$ has infinite height in $\frac{M}{Soc(B)}$. For $x \in Soc(T)$ we define $A(x) = \{\alpha \mid \alpha \in \bigcup J(n(i)), x_\alpha R \neq 0, x = \bigoplus x_\alpha\}$ and $J = \bigcup J(n(i))$.

For $\beta = 0$, let $\alpha(0) \in J(n(0))$. Since $x_{\alpha(0)} + Soc(B)$ has infinite height in $M/Soc(B)$, let $b_0 \in Soc(B)$ such that $H_M(b_0 - x_{\alpha(0)}) > n(0)$. Then $b_0 = u + v_0$ for some $u \in M', v_0 \in S$. To apply transfinite induction we assume that $\alpha(\beta), b_\beta$ and v_β have been defined for all $\beta < \delta < \lambda$. Now put $\delta = \delta_0 + i$, where δ_0 is a limit ordinal and $i < \omega$. If $i = 0$, let $\alpha(\delta) \in J(n(l)) - \bigcup_{\beta<\delta} A(v_\beta)$ where l is the least integer such that $J(n(l)) > \delta$. If $i > 0$ let $\alpha(\delta) \in (n(l+1)) - \bigcup_{\beta<\delta} A(v_\beta)$, $\alpha(\delta - 1) \in J(n(l))$. Let $b_\delta \in Soc(B)$ such that $H_M(b_\delta - x_{\alpha(\delta)}) > n(i)$ where $\alpha(\delta) \in J(n(i))$. Then $b_\delta = u_\delta + v_\delta$, for some $u_\delta \in Soc(M')$ and $v_\delta \in Soc(T)$. Thus we get a sequence $\{v_\beta\}_{\beta<\lambda}$ of the elements of $Soc(T)$. Now the sum $\bigoplus_{\beta<\lambda} v_\beta R$ is direct, and there exists a cardinal $\mu < \lambda$ such that $\bigoplus_{\beta<\mu} v_\beta R$ supports a direct summand K of T with $\text{fin}\, g(K) = \lambda$. Thus $T = K \oplus K'$. Again $Soc(M) = Soc(M') \oplus Soc(K') \oplus (\bigoplus_{\beta<\mu} b_\beta R)$.
Let N be a *h-pure* submodule of B such that $Soc(N) = \bigoplus_{\beta<\mu} b_\beta R$. Then $M' \oplus K \oplus N$ is a *h-pure* submodule of M and $Soc(M' \oplus K \oplus N) = Soc(M)$. Thus $M = M' \oplus N \oplus K$ and N is a summand of M with $\text{fin}\, g(N) = \lambda$.

Corollary 1. *Let M be a h-reduced QTAG-module. If a basic submodule of M contains no unbounded summand of M, then no basic submodule of M have this property.*

The following theorem is a significant consequence of Theorem 1

Theorem 2. *A h-reduced QTAG-module is essentially finitely indecomposable if and only if $M/H_\omega(M)$ is essentially finitely indecomposable.*

With the help of Theorem 1, we shall prove another interesting result.

Proposition 5. *Let B be a basic submodule of a h-reduced QTAG-module M and let N be a h-pure submodule of M containing B. If N is essentially finitely indecomposable then M is essentially finitely indecomposable.*

Proof. Suppose M is not essentially finitely indecomposable. Then M has an unbounded summand which is a direct sum of uniserial modules. Then by Theorem 1, there exists an unbounded submodule K of B which is a summand of M. Now $N = K \oplus (N \cap M)$ implying that N is not essentially finitely indecomposable and we are done.

Consequently we may state.

Corollary 2. *Let M be a h-reduced QTAG-module. If one high submodule of M is essentially finitely indecomposable then M is essentially finitely indecomposable.*

Remark. It is important to note that if one high submodule of a *QTAG*-module is essentially finitely indecomposable then it is not necessary that all high submodules of M are essentially finitely indecomposable.

Now we shall prove that a *QTAG*-module M is a HT-module if and only if $M/H_\omega(M)$ is a HT-module. In order to prove this result we first need to prove the following.

Proposition 6. *A QTAG-module M is a HT-module if and only if $Soc(H_k(M)) \subset N$ for some $k < \omega$ and all submodules N such that M/N is a direct sum of uniserial modules.*

Proof. Let N be a submodule of a *QTAG*-module M such that M/N is a direct sum of uniserial modules and $M/H_\omega(M)$ is a HT-*module*. Therefore $H_\omega(M) \subset N$ and $\frac{M/H_\omega(M)}{N/H_\omega(M)} \simeq \frac{M}{N}$ is the direct sum of uniserial modules. If B' is any basic submodule of $M/H_\omega(M)$ then $M/H_\omega(M) = (N/H_\omega(M)) + B'$. Again if B is a basic submodule of M then $\frac{B+H_\omega(M)}{H_\omega(M)} \simeq B$ is a basic submodule of $M/H_\omega(M)$. Since $\frac{M}{H_\omega(M)} = \frac{N}{H_\omega(M)} + \frac{B+H_\omega(M)}{H_\omega(M)}$ we have $M = N + B$ as required.

For the converse suppose M is a HT-*module* and $\frac{N+H_\omega(M)}{H_\omega(M)} \subseteq \frac{M}{H_\omega(M)}$ for some submodule $N \subseteq M$ such that $\frac{M/H_\omega(M)}{(N+H_\omega(M))/H_\omega(M)} \simeq \left(\frac{M}{N+H_\omega(M)}\right)$ is a direct sum of uniserial modules. Now $N+H_\omega(M)$ is nice [5] in M. Since $Soc(H_k(M)) \subseteq N + H_\omega(M)$ for some $k \geq 0$, $Soc\left(\frac{H_k(M)}{H_\omega(M)}\right) = \frac{\bigcap_{n<\omega} Soc(H_k(M)) + H_n(M)}{H_\omega(M)}$.

Now by the above discussion

$$
\begin{aligned}
Soc\left(H_k\left(\frac{M}{H_\omega(M)}\right)\right) &= Soc\left(\frac{H_k(M)}{H_\omega(M)}\right) \\
&\simeq \frac{(\bigcap_{n<\omega} (N+H_\omega(M)+H_n(M)))}{H_\omega(M)} \\
&= \frac{N+H_\omega(M)+(\bigcap_{n<\omega} H_n(M))/H_\omega(M)}{(N+H_\omega(M))/H_\omega(M)}
\end{aligned}
$$

Again by the same arguments, $\frac{M}{H_\omega(M)}$ is a HT-*module*.

Following is the general consequence of the above.

Proposition 7. *A QTAG-module M is a HT*-module *if and only if* $\frac{M}{H_\omega(M)}$ *is a HT*-module.

Now we shall prove that there are essentially finitely indecomposable $QTAG$-modules which are not HT*-modules* and also prove some related results. We start by defining *h-compact* $QTAG$-modules.

Definition 3. *A QTAG-module M is is said to be h*-compact *if it is a direct summand of M' whenever M is a h*-pure *submodule of M'.*

Proposition 8. *Any* $(\omega+n)$-projective HT*-module is bounded.*

Proof. Let M be a $(\omega+n)$*-projective* $QTAG$-module [4] which is a HT*-module* also. Now there exists a submodule $N \subseteq M$ such that $N \subseteq H^n(M)$ and $\frac{M}{N}$ is a direct sum of uniserial modules, thus $\frac{M}{N}$ is also a HT*-module*. Since the direct sums of uniserial modules are bounded and vice-versa, $\frac{M}{N}$ is bounded and M retains the same property *i.e.* M is also bounded.

Proposition 9. *M is a HT*-module *such that $M/H_\omega(M)$ is* $(\omega+n)$-projective *if and only if M is h*-compact. *Therefore any* HT extension *of a direct sum of uniserial modules by a direct sum of uniserial modules is bounded.*

Proof. Since M is a HT*-module*, $M/H_\omega(M)$ is also a HT*-module* and being $(\omega+n)$*-projective* $M/H_\omega(M)$ is bounded. Therefore there exists $k \in \mathbb{Z}^+$ such that $H_k(M) = H_{k+1}(M)$. Now $H_k(M)$ is the maximal *h-divisible* submodule of M such that $H_k\left(\frac{M}{H_k(M)}\right) = 0$. Now M is the direct sum of a *h-divisible* module and a bounded module which is equivalent to the h-compactness of M.
For the next part consider the $QTAG$-modules $N \subseteq M$ such that N and $\frac{M}{N}$ are the direct sums of uniserial modules. Since M is a HT*-module*, the natural homomorphism $M \longrightarrow \frac{M}{N}$ is also small *i.e.* N contains a large submodule L of M. Therefore L is a direct sum of uniserial modules and since M is bounded it is a direct sum of uniserial modules, whence bounded.

Proposition 10. *Let N be a submodule of a QTAG-module M such that $\frac{M}{N}$ is bounded. Then M is a HT*-module *if and only if N is a HT*-module.

Proof. Since $\frac{M}{N}$ is bounded there exists some $k \in \mathbb{Z}^+$ such that $H_k(M) \subseteq N$. Suppose K is a submodule of N such that N/K is a direct sum of uniserial modules. Therefore $H_k\left(\frac{M+K}{K}\right)$ is also a direct sum of uniserial modules which holds for $\frac{M+K}{K} \simeq \frac{M}{M \cap K}$. Now there exists $n \in \mathbb{Z}^+$ such that $Soc(H_n(N)) \subseteq K$ and we are done.

For the converse suppose $\frac{M}{K}$ is a direct sum of uniserial modules. Now $\frac{N+K}{K} \simeq \frac{N}{N \cap K}$ is a submodule of $\frac{M}{K}$ which is also a direct sum of uniserial modules. Now there exists $n \in \mathbb{Z}^+$ such that $Soc(H_k(N)) \subseteq K$ implying that $Soc(H_{k+n}(M)) \subseteq K$ as required.

We conclude this article by proving a related result.

Proposition 11. *Let S be a fully invariant subsocle of M. If M/S is essentially finitely indecomposable then M is essentially finitely indecomposable.*

Proof. Let $M = N \oplus K$ where K is the direct sum of uniserial modules. Since S is fully invariant $\frac{M}{S} \simeq \frac{N}{N \cap S} \oplus \frac{K}{K \cap S}$. Now $K \cap S$ can be embedded in a summand of K which is contained in $H^1(K)$, thus $\frac{K}{K \cap S}$ is $(\omega+1)$*-projective*. It is an epimorphic image of the direct sum of uniserial modules $\frac{K}{S}$ where $K \cap S \subset H^1(K)$. Therefore $\frac{K}{K \cap S}$ is *U-decomposable* [5] such that $\frac{K}{K \cap S} = T \oplus Q$ where Q is the direct sum of uniserial modules and fin $g(Q) =$ fin $g\left(\frac{K}{K \cap S}\right)$. Now $\frac{M}{S} \simeq \frac{N}{N \cap S} \oplus T \oplus Q$. Since $\frac{M}{S}$ is essentially finitely indecomposable, Q is bounded and $\frac{K}{K \cap S}$ is also bounded implying that K is also bounded and the result follows.

References

[1] Fuchs L.,*Infinite Abelian Groups,*Vol. I, Academic Press, New York, 1970.

[2] Fuchs L.,*Infinite Abelian Groups,*Vol. II, Academic Press, New York, 1973.

[3] Mehdi A. and Khan M. Z., *On closed modules*, Kyungpook Math. J., **24**(1)(1984), 45–50.

[4] Mehdi A., Abbasi M. Y. and Mehdi F., *On* $(\omega + n)$*-projective modules*, Ganita Sandesh, **20**(1)(2006), 27–32.

[5] Mehdi A., Abbasi, M. Y. and Mehdi F., *U-decomposable* modules, J. Rajasthan Acad. Phy. Sc. **5**(1)(2006), 13-18.

[6] Mehdi A., Hasan A., Sikander F., *On HT-Modules*(To appear in GAMS)