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Real hypersurface of a complex space form

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Abstract: The purpose of the present paper is to give characterization of real hypersurface of a complex space form. We find conditions for these hypersurfaces to be ϕ-symmetric and to have η- parallel curvature tensor. Further we prove totally η-umbilical real hypersurfaces of complex space forms have ξ-parallel Ricci tensor and ξ-parallel structure Jacobi operator.

Keywords: Complex space form, real hypersurface, η-parallel, η umbilical, ξ-parallel, structure Jacobi operator.

1 Introduction

A complex $n(\geq 2)$ -dimensional K*ä*hlerian manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. The induced almost contact metric structure of a real hypersurface M of $M_n(c)$ is denoted by (ϕ, ξ, η, g).

In [1]and [2] Berndt has called real hypersurface in $M_n(c)$ with the principal vector ξ as Hopf real hypersurface. It can be easily seen that there does not exist any real hypersurface in $M_n(c)$, $c \neq 0$, which is locally symmetric that is $\nabla R = 0$. This motivates the introduction of the notion of η-parallel curvature tensor (Lee J.G etal [9]). The notion of η-parallel curvature tensor is defined by $g((\nabla_X R)(Y,Z)U,V) = 0$ for any X,Y,Z,U and V in a distribution orthogonal to ξ. This notion is weaker to the notion of η-parallel second fundamental tensor $g((\nabla_X A)Y,Z) = 0$ for any X,Y and Z.

2 preliminaries

Let $M_n(c)$ denote the complex space form of complex dimension n with constant holomorphic sectional curvature $4c$. Let M be a real $(2n-1)$-dimensional hypersurface immersed in $M_n(c)$ with parallel almost complex structure J and N be unit normal vector field on M. For any vector field X tangent to M, we define ϕ, η and ξ by

$$JX = \phi X + \eta(X)N, JN = -\xi, \tag{1}$$

where ϕX is the tangential part of JX, ϕ is a tensor field of a type (1, 1), η is a 1-form, and ξ is the unit vector field on M. Then they satisfy

$$\phi^2 X = -X + \eta(X)\xi,\ \phi\xi = 0,$$

$$\eta(\xi) = 1, g(X,\xi) = \eta(X), \eta(\phi X) = 0,$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

$$g(\phi X, Y) = -g(X, \phi Y), g(\phi X, X) = 0,$$

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$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y). \tag{2}$$

Thus (ϕ, ξ, η, g) is an almost contact metric structure on M. We denote by $\widetilde{\nabla}$ the operator of covariant differentiation in $M_n(c)$ and by ∇ the one in M determined by the induced metric. Then the Gauss and Weingarten formulae are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \tag{3}$$

$$\widetilde{\nabla}_X N = -AX, \tag{4}$$

for any vector fields X and Y tangent to M. We call A the shape operator of M. For the almost contact metric structure on M, we have

$$\nabla_X \xi = \phi AX, \tag{5}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{6}$$

We denote by R the Riemannian curvature tensor field of M. Then the equation of Gauss is given by

$$\begin{aligned} R(X,Y)Z =& \frac{c}{4}[g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & -2g(\phi X, Y)\phi Z] + g(AY, Z)AX - g(AX, Z)AY \end{aligned} \tag{7}$$

and the equation of Codazzi is

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\phi Y - \eta(X)\phi Y - 2g(\phi X, Y)\xi].$$

If the shape operator A of M is of the form $AX = aX + b\eta(X)\xi$ for some functions a and b, then M is said to be totally η-umbilical. We put $\alpha = g(A\xi, \xi)$. If ξ is a principal vector everywhere (i,e $A\xi = \alpha\xi$), we say that M is a Hopf hypersurface and M is called a hypersurface with recurrent shape operator if there exists a 1-form α such that A of M satisfies $(\nabla_X A)Z = \alpha(X)AZ$.

3 Real hypersurfaces of complex space forms with recurrent second fundamental form

Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. Taking covariant derivative of (7) with respect to X, we get

$$\begin{aligned} (\nabla_X R)(Y,Z)U =& \frac{c}{4}[\eta(Z)g(AX,U)\phi Y - \eta(U)g(AX,Z)\phi Y + \eta(Y)g(\phi Z,U)AX \\ & -g(AX,Y)g(\phi Z,U)\xi - \eta(Y)g(AX,U)\phi Z + \eta(U)g(AX,Y)\phi Z - \eta(Z)g(\phi Y,U)AX \\ & +g(AX,Z)g(\phi Y,U)\xi - 2\eta(Y)g(AX,Z)\phi U + 2\eta(Z)g(AX,Y)\phi U - 2\eta(U)g(\phi Y,Z)AX \\ & +g(AX,U)g(\phi Y,Z)\xi] + g((\nabla_X A)Z,U)AY + (\nabla_X A)Y g(AZ,U) - g((\nabla_X A)Y,U)AZ \\ & -g((\nabla_X A)Z,U)AZ. \end{aligned} \tag{8}$$

Contracting the above equation with respect to W, we get

$$\begin{aligned} g((\nabla_X R)(Y,Z)U,W) = \frac{c}{4}[\eta(Z)g(AX,U)g(\phi Y,W) - \eta(U)g(AX,Z)g(\phi Y,W)\\ +\eta(Y)g(AX,W)g(\phi Z,U) - \eta(W)g(AX,Y)g(\phi Z,U) - \eta(Y)g(AX,U)g(\phi Z,W)\\ +\eta(U)g(AX,Y)g(\phi Z,W) - \eta(Z)g(AX,W)g(\phi Y,U) + \eta(W)g(AX,Z)g(\phi Y,U)\\ -2\eta(Y)g(AX,Z)g(\phi U,W) + 2\eta(Z)g(AX,Y)g(\phi U,W) - 2\eta(U)g(AX,W)g(\phi Y,Z)\\ +2\eta(W)g(AX,U)g(\phi Y,Z)] + g((\nabla_X A)Z,U)g(AY,W) + g((\nabla_X A)Y,W)g(AZ,U)\\ -g((\nabla_X A)Y,U)g(AZ,W) - g((\nabla_X A)Z,U)g(AZ,W). \end{aligned} \tag{9}$$

If the shape operator A is recurrent, then we have

$$\begin{aligned} g((\nabla_X R)(Y,Z)U,W) = \frac{c}{4}[\eta(Z)g(AX,U)g(\phi Y,W) - \eta(U)g(AX,Z)g(\phi Y,W)\\ +\eta(Y)g(AX,W)g(\phi Z,U) - \eta(W)g(AX,Y)g(\phi Z,U) - \eta(Y)g(AX,U)g(\phi Z,W)\\ +\eta(U)g(AX,Y)g(\phi Z,W) - \eta(Z)g(AX,W)g(\phi Y,U) + \eta(W)g(AX,Z)g(\phi Y,U)\\ -2\eta(Y)g(AX,Z)g(\phi U,W) + 2\eta(Z)g(AX,Y)g(\phi U,W) - 2\eta(U)g(AX,W)g(\phi Y,Z)\\ +2\eta(W)g(AX,U)g(\phi Y,Z)] + 2\alpha(X)[g(AZ,U)g(AY,W) - g(AY,U)g(AZ,W)]. \end{aligned}$$

For all X,Y,Z,U,W orthogonal to ξ, the above equation reduces to

$$g((\nabla_X R)(Y,Z)U,W) = 2\alpha(X)[g(AZ,U)g(AY,W) - g(AY,U)g(AZ,W)].$$

Thus we have

Proposition 1. *A real hypersurface of a complex space form with recurrent second fundamental form has η -parallel curvature tensor if and only if*
$[g(AZ,U)g(AY,W) - g(AY,U)g(AZ,W)]$ *=0 holds.*

Applying ϕ^2 to (1), we get

$$\begin{aligned} \phi^2(\nabla_X R)(Y,Z)U = \frac{c}{4}[-\eta(Z)g(AX,U)\phi Y + \eta(U)g(AX,Z)\phi Y - \eta(Y)g(\phi Z,U)AX\\ +\eta(Y)\eta(AX)g(\phi Z,U)\xi + \eta(Y)g(AX,U)\phi Z - \eta(U)g(AX,Y)\phi Z + \eta(Z)g(\phi Y,U)AX\\ -\eta(Z)\eta(AX)g(\phi Y,U)\xi + 2\eta(Y)g(AX,Z)\phi U - 2\eta(Z)g(AX,Y)\phi U + 2\eta(U)g(\phi Y,Z)AX\\ -2\eta(U)\eta(AX)g(\phi Y,Z)\xi] - g((\nabla_X A)Z,U)AY + \eta(AY)g(\nabla_X A)Z,U)\xi - g(AZ,U)(\nabla_X A)Y\\ +\eta((\nabla_X A)Y)g(AZ,U)\xi + g((\nabla_X A)Y,U)AZ - \eta(AZ)g((\nabla_X A)Y,U)\xi + g(AY,U)(\nabla_X A)Z\\ -\eta((\nabla_X A)Z)g(AY,U)\xi. \end{aligned} \tag{10}$$

If the shape operator A is recurrent, then (10) takes the form

$$\begin{aligned} \phi^2(\nabla_X R)(Y,Z)U =& \frac{c}{4}[-\eta(Z)g(AX,U)\phi Y + \eta(U)g(AX,Z)\phi Y - \eta(Y)g(\phi Z,U)AX\\ &+\eta(Y)\eta(AX)g(\phi Z,U)\xi + \eta(Y)g(AX,U)\phi Z - \eta(U)g(AX,Y)\phi Z\\ &+\eta(Z)g(\phi Y,U)AX - \eta(Z)\eta(AX)g(\phi Y,U)\xi + 2\eta(Y)g(AX,Z)\phi U\\ &-2\eta(Z)g(AX,Y)\phi U + 2\eta(U)g(\phi Y,Z)AX - 2\eta(U)\eta(AX)g(\phi Y,Z)\xi]\\ &-2\alpha(X)[g(AY,U)AZ - g(AZ,U)AY]. \end{aligned} \tag{11}$$

For all X,Y,Z,U,W orthogonal to ξ, the above equation reduces to

$$\phi^2(\nabla_X R)(Y,Z)U = 2\alpha(X)[g(AY,U)AZ - g(AZ,U)AY]. \tag{12}$$

Thus we have

Proposition 2. *A real hypersurface of a complex space form with recurrent second fundamental form is ϕ -symmetric if and only if $[g(AZ,U)g(AY,W) - g(AY,U)g(AZ,W)]$ = 0 holds.*

The condition ϕ -symmetry on R is weaker to η -parallel condition on R. But these two conditions are equivalent as stated in the following . Combining proposition 1 and proposition 2, we can state the following:

Theorem 1. *In a real hypersurface M of a complex space form $M_n(c)$ with recurrent second fundamental form A satisfying the condition $[g(AZ,U)g(AY,W) - g(AY,U)g(AZ,W)]$ =0, the following are equivalent:*

(i) M has η -parallel curvature tensor.
(ii) M is ϕ -symmetric.

4 Totally η-umbilical real hypersurfaces of complex space forms

Let M be a totally η-umbilical hypersurface of the complex space form $M_n(c)$. Then using 10, we get

$$\begin{aligned}
g((\nabla_X R)(Y,Z)U,W) = & \frac{c}{4}[\eta(Z)g(AX,U)g(\phi Y,W) - \eta(U)g(AX,Z)g(\phi Y,W) \\
& +\eta(Y)g(AX,W)g(\phi Z,U) - \eta(W)g(AX,Y)g(\phi Z,U) \\
& -\eta(Y)g(AX,U)g(\phi Z,W) + \eta(U)g(AX,Y)g(\phi Z,W) \\
& -\eta(Z)g(AX,W)g(\phi Y,U) + \eta(W)g(AX,Z)g(\phi Y,U) \\
& -2\eta(Y)g(AX,Z)g(\phi U,W) + 2\eta(Z)g(AX,Y)g(\phi U,W) \\
& -2\eta(U)g(AX,W)g(\phi Y,Z) + 2\eta(W)g(AX,U)g(\phi Y,Z)] \\
& +(Xa)g(Z,U)g(aY,W) + (Xa)\eta(Y)\eta(W)g(Z,U) \\
& +(Xb)\eta(Z)\eta(U)g(aY,W) + b\eta(U)g(\phi AX,Z)g(aY,W) \\
& +b\eta(Z)g(\phi AX,U)g(aY,W) + (Xa)g(aZ,U)g(Y,W) \\
& +(Xb)\eta(Y)\eta(W)g(aZ,U) + b\eta(W)g(\phi AX,Y)g(aZ,U) \\
& +b\eta(Y)g(\phi AX,W)g(aZ,U) + b(Xa)\eta(Z)\eta(U)g(Y,W) \\
& -(Xa)g(Y,U)g(aZ,W) - (Xb)\eta(Y)\eta(U)g(aZ,W) \\
& -b\eta(U)g(\phi AX,Y)g(aZ,W) - b\eta(Y)g(\phi AX,U)g(aZ,W) \\
& -b\eta(Y)g(\phi AX,U)g(aZ,W) - b(Xa)\eta(Z)\eta(W)g(Z,U) \\
& -(Xa)g(Z,W)g(aY,U) - (Xb)\eta(Z)\eta(W)g(aY,U) \\
& -b\eta(W)g(\phi AX,Z)g(aY,U) - b\eta(Z)g(\phi AX,W)g(aY,U) \\
& -b(Xa)\eta(Y)\eta(U)g(Z,W).
\end{aligned} \tag{13}$$

Then for all X,Y,Z,U,W orthogonal to ξ, we have
$g((\nabla_X R)(Y,Z)U,W) = 0$. i.e. M has η -parallel curvature tensor .
Thus we have

Theorem 2. *A totally η-umbilical real hypersurface of a complex space form has η -parallel curvature tensor.*

Now from (1), for a totally η-umbilical real hypersurface, we have

$$\phi^2(\nabla_X R)(Y,Z)U = \frac{c}{4}[-\eta(Z)g(AX,U)\phi Y + \eta(U)g(AX,Z)\phi Y - \eta(Y)g(\phi Z,U)AX$$
$$+\eta(Y)\eta(AX)g(\phi Z,U)\xi + \eta(Y)g(AX,U)\phi Z - \eta(U)g(AX,Y)\phi Z + \eta(Z)g(\phi Y,U)AX$$
$$-\eta(Z)\eta(AX)g(\phi Y,U)\xi + 2\eta(Y)g(AX,Z)\phi U - 2\eta(Z)g(AX,Y)\phi U + 2\eta(U)g(\phi Y,Z)AX$$
$$-2\eta(U)\eta(AX)g(\phi Y,Z)\xi] - (Xa)g(Z,U)aY - (Xb)\eta(Z)\eta(U)aY - b\eta(U)g(\phi AX,Z)aY$$
$$-b(Xa)\eta(Y)g(Z,U)\xi + \eta(aY)(Xa)g(Z,U)\xi + (Xb)\eta(Z)\eta(U)\eta(aY)\xi - g(aZ,U)(Xa)Y$$
$$-b\eta(Y)g(aZ,U)\phi AX - b\eta(Z)\eta(U)(Xa)Y + (Xa)\eta(Y)g(aZ,U)\xi + (Xa)g(Y,U)aZ$$
$$+(Xb)\eta(U)\eta(Y)aZ + b\eta(U)g(\phi AX,Y)aZ + \eta(Y)g(\phi AX,U)aZ - b(Xa)\eta(aZ)g(Y,U)\xi$$
$$-(Xb)\eta(aZ)\eta(Y)\eta(U)\xi - b\eta(U)\eta(aZ)g(\phi AX,Y)\xi - b\eta(Y)\eta(aZ)g(\phi AX,U)\xi$$
$$+g(aY,U)(Xa)Z + \eta(Z)g(aY,U)\phi AX + b\eta(Y)\eta(U)(Xa)Z - \eta(Z)g(aY,U)(Xa)\xi.$$

For all X,Y,Z,U,W orthogonal to ξ, the above equation reduces to

$$\phi^2(\nabla_X R)(Y,Z)U) = -(Xa)g(Z,U)aY - g(AZ,U)(Xa)Y + (Xa)g(Y,U)aZ + g(AY,U)(Xa)Z].$$

It is well known that:

Remark 1. Any totally η-umbilical real hypersurface is a Hopf hypersurface.

Remark 2. In a Hopf hypersurface with ξ as a principal vector, principal curvature corresponding to ξ is a constant.

From the above remarks we have for constant a,

$$\phi^2(\nabla_X R)(Y,Z)U) = 0.$$

Thus we can state that

Theorem 3. *A totally η-umbilical real hypersurface of a complex space form is ϕ -symmetric.*

Now let us denote by Q the Ricci tensor of M in $M_n(c)$. Then from (7) which together with (5), we obtain

$$QX = \frac{c}{4}[(2n+1)X - 3\eta(x)\xi] + hAX - A^2X,$$

$$(\nabla_X Q)Y = -\frac{3c}{4}[g(\phi AX,Y)\xi + \eta(Y)\phi AX] + (Xh)AY + (hI - A)(\nabla_X A)Y) - (\nabla_X A)AY), \tag{14}$$

where I denote the identity map on the tangent space T_pM, $p \in M$.
If M is totally η-umbilical, then from (4) we obtain

$$(\nabla_\xi Q)Y = (\xi h)c\xi + (hI - A)((\xi a)Y + (\xi b)\eta(Y)\xi) - ((\xi a)AY + (\xi b)\eta(AY)\xi),$$

By remark 1 and remark 2, we have $(\nabla_\xi Q)Y = 0$, provided $a = -b$.

Theorem 4. *A totally η-umbilical real hypersurface of a complex space form has ξ -parallel Ricci tensor provided $a = -b$.*

From (5), we obtain

$$R_{\xi}Y = R(Y,\xi)\xi = \frac{c}{4}[Y - \eta(Y)\xi] + \alpha AY - \eta(AY)A\xi \tag{15}$$

for any vector field Y on M. Taking covariant derivative with respect to X of (15), we get

$$\begin{aligned}(\nabla_X R_{\xi})Y = &-\frac{c}{4}[(\nabla_X\eta)Y] - \frac{c}{4}[\eta(Y)(\nabla_X\xi + (X\alpha)AY \\ &+\alpha((\nabla_X A)Y) - ((\nabla_X\eta)AY)A\xi - (\eta(\nabla_X A)Y)A\xi \\ &-\eta(AY)\nabla_X A)\xi - \eta(AY)A(\nabla_X\xi)\end{aligned}$$

Contracting the above with respect to Z and using (4) and (5), we get

$$\begin{aligned}g((\nabla_X R_{\xi})Y,Z) = &-\frac{c}{4}[g(\nabla_X\xi,Y)\eta(Z) + \eta(Y)g(\nabla_X\xi,Z)] \\ &+X\alpha g(AY,Z) + \alpha g((\nabla_X A)Y,Z) + g(X,A\phi AY)g(A\xi,Z) \\ &-g((\nabla_X A)Y,\xi)g(A\xi,Z) - g(AY,\xi)g((\nabla_X A)\xi,Z) + g(AY,\xi)g(X,A\phi AZ).\end{aligned}$$

Since M is totally η-umbilical, the above equation reduces to

$$\begin{aligned}g((\nabla_X R_{\xi})Y,Z) = &-\frac{c}{4}[g(\phi AX,Y)\eta(Z) + \eta(Y)g(\phi AX,Z)] + (X\alpha)[ag(AY,Z) \\ &+b\eta(Y)\eta(Z)]\alpha[(Xa)g(Y,Z) + (Xb)\eta(Y)\eta(Z) + b(g(\phi AX,Y)\eta(Z) \\ &+\eta(Y)g(\phi AX,Z)] + \eta(AZ)g(X,A\phi AY) - [(Xa)\eta(Y) \\ &+(Xb)\eta(Y) + bg(\phi AX,Y)]\eta(AZ) - [(Xa)\eta(Z) \\ &+(Xb)\eta(Z) + bg(\phi AX,Z)]\eta(AY) + \eta(AY)g(X,A\phi AZ).\end{aligned}$$

For all X,Y,Z orthogonal to ξ, we obtain

$$g((\nabla_X R_{\xi})Y,Z) = (X\alpha)ag(AY,Z) + \alpha(Xa)g(Y,Z).$$

If a and b are constants, then we have

$$g((\nabla_X R_{\xi})Y,Z) = 0.$$

Theorem 5. *A totally η-umbilical real hypersurface of a complex space form has ξ -parallel structure Jacobi operator provided the associated scalars a and b are constants.*

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