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The period of fibonacci sequences over the finite field of order p^2

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Abstract: In this paper , we obtain the period of Fibonacci sequence in the finite fields of order p^2 by using equality recursively defined by $F_{n+1} = A_1F_n + A_0F_{n-1}$, for $n > 0$, where $F_0 = 0, F_1 = 1$ and A_1, A_0 are generators elements of these fields of order p^2 .

Keywords: Fibonacci sequence, period, finite fields.

1 Introduction

Generalized Fibonacci sequence have been intensively studied for many years and have become into an interesting topic in Applied Mathematics. Fibonacci sequences and their related higher-order (tribonacci, k-nacci) sequences are generally viewed as sequences of integers. The notation of Wall number was first proposed by D. D. Wall [7] in 1960. In [7], he gave some theorems and properties concerning Wall number of the Fibonacci sequences. K. Lu and J. Wang [5] contributed to the study of the Wall number for the k-step Fibonacci sequences. D. J. De Carli [2] gave a generalized Fibonacci sequences over an arbitrary ring in 1970. Special cases of Fibonacci sequences over an arbitrary ring have been considered by R. G. Bauschman [1], A. F. Horadam [4] and N. N. Vorobyov [6] where this ring was taken to be the set integers. O. Wyler [8] also worked with such a sequence over a particular commutative ring with identity. Classification of finite rings of order $p^2$with p a prime have been studied by B. Fine [3].

A sequence of ring elements is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called the period of the sequence. For example, the sequence $a,b,c,d,e,b,c,d,e,b,c,d,e,...$ is periodic after the initial element a and has period 4. A sequence of ring elements is simply periodic with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a,b,c,d,e,f,a,b,c,d,e,f,a,b,c,d,e,f,...$ is simply periodic with period 6.

Definition 1. *Let f_n^k denote the n th member of the k-step Fibonacci sequence defined as*

$$f_n^{(k)} = \sum_{j=1}^{k} f_{n-j}^{(k)} \quad for \quad n > k \tag{1}$$

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with boundary conditions $f_i^{(k)} = 0$ *for* $1 \leq i < k$ *and* $f_k^{(k)} = 1$*.Reducing this sequence modulo* m *, we can get a repeating sequence, denoted by* $f(k,m) = (f_1^{(k,m)}, f_2^{(k,m)}, ..., f_n^{(k,m)}, ...)$ *where* $f_i^{(k,m)} = f_i^{(k)}(mod m)$*. Then we have that*

$$f(k,m) = (f_1^{(k,m)}, f_2^{(k,m)}, ..., f_k^{(k,m)}) = (0,0,...,0,1)$$

and it has the same recurrence relation as in (1), [5].

Theorem 1. $f(k,m)$ *is a periodic sequence [5].*

Theorem 2. *For any prime* p*, up to isomorphism, the finite* 2*-generator field of order* p^2 *is given by the following presentations [3]:*

$$GF(p^2) = \begin{cases} \langle a,b : pa = pb = 0, a^2 = a, b^2 = ja, ab = b, ba = b \rangle, \\ \quad \text{where } j \text{ is not a square in } \mathbb{Z}_p, \text{ if } p \neq 2 \\ \langle a,b : 2a = 2b = 0, a^2 = a, b^2 = a+b, ab = b, ba = b \rangle, \\ \quad \text{if } p = 2 \end{cases}$$

Definition 2. *Let* R *be a ring with identity* I*. The sequence* $\{M_n\}$ *of elements of* R *recursively is defined by*

$$M_{n+2} = A_1 M_{n+1} + A_0 M_n \quad \text{for} \quad n \geq 0, \tag{2}$$

where M_o, M_1, A_0 *and* A_1 *are arbitrary elements of* R *[2].*

Definition 3. *A special case of equality (2) is denoted by* $\{F_n\}$ *and defined by*

$$F_{n+2} = A_1 F_{n+1} + A_0 F_n \quad \text{for} \quad n \geq 0,$$

where $F_o = 0, F_1 = I$*, and* A_0, A_1 *are arbitrary elements of* R *[2].*

We next denote the identity of the $GF(p^2)$ by 1.

Theorem 3. *If* $F_{n+2} = A_1 F_{n+1} + A_0 F_n$*, then* $F_{n+2} = F_{n+1} A_1 + F_n A_0$ *[2].*

Theorem 4. *Let*

$$GF(p^2) = \begin{cases} \langle a,b : pa = pb = 0, a^2 = a, b^2 = ja, ab = b, ba = b \rangle, \\ \quad \text{where } j \text{ is not a square in } \mathbb{Z}_p, \text{ if } p \neq 2 \\ \langle a,b : 2a = 2b = 0, a^2 = a, b^2 = a+b, ab = b, ba = b \rangle, \\ \quad \text{if } p = 2 \end{cases}$$

(i) *If* $j = p-1$,

$$0, a, b, 0, b, ja, 0, ja, jb, 0, jb, a, 0, a, b, ...$$

Fibonacci sequences is simple periodic and period is 12.

(ii) *If* $j = p-2$,

$$0, a, b, (j+1)a, 0, (j+1)a, (j+1)b, a, 0, a, b, ...$$

Fibonacci sequences is simple periodic and period is 8.

(iii) *If* $j = p-3$,

$$0, a, b, , (j+1)a, (j+2)b, a, 0, a, b, ...$$

Fibonacci sequences is simple periodic and period is 6.

(iv) *If $j = p-4$,*

$$0,a,b,(j+1)a,(j+2)b,\underbrace{(4k+1)a,(2k+1)b,(j-(4k-1))a,(j-(2k-2))b,}_{k=1}$$
$$\underbrace{(4k+1)a,(2k+1)b,(j-(4k-1))a,(j-(2k-2))b}_{k=2},\ldots,0,1,b,\ldots$$

Fibonacci sequences is simple periodic and period is $4p$.

Proof. Let us consider the $Definition 1.5$. For $F_{n+2} = A_1F_{n+1} + A_0F_n$ where $F_0 = 0, F_1 = 1$ and $A_1 = b, A_0 = a, n \geq 0$.

(i) Suppose that a single period of $mod(p)$ is partitioned into smaller finite subsequences $A_0, A_1, A_2, \ldots$ as shown below $a = 1$:

$$\underbrace{0,a,b}_{A_0},\underbrace{0,b,ja}_{A_1},\underbrace{0,ja,jb}_{A_2},\underbrace{0,jb,a}_{A_3},0,a,b,\ldots$$

If it is use $ja = b^2, jab = b^2b \Longrightarrow jb = b^3$

$$\underbrace{0,a,b}_{A_0},\underbrace{0,b,b^2}_{A_1},\underbrace{0,b^2,b^3}_{A_2},\underbrace{0,b^3,a}_{A_3},0,a,b,\ldots$$

Each subsequence A_i has $\alpha = 3$ terms and it contains exactly one zero. Every subsequence A_i for $i \geq 1$ is a multiply of A_0, more precisely, the following congruences hold modulo p

$$\begin{aligned}
A_1 &= bA_0\\
A_2 &= b^2A_0\\
A_3 &= b^3A_0\\
&.\\
&.\\
&.\\
A_{n-1} &= b^{n-1}A_0\\
A_n &= b^nA_0
\end{aligned}$$

Now, the last term in A_{n-1} is b^n , the last term in A_0 is b and the last term in A_3 is $b^4 = a = 1$,i.e order of b is 4. If the number of subsequences A_i is $\beta = 4$, clearly it follows that Fibonacci sequence is simple periodic and period is $\alpha.\beta = 3.4 = 12$.

(ii) Suppose that a single period of $mod(p)$ is partitioned into smaller finite subsequences $A_0, A_1, A_2, \ldots$ as shown below $a = 1$:

$$0,a,b,(j+1)a,0,(j+1)a,(j+1)b,a,0,a,b,\ldots$$

If it is use $ja = b^2$, $jab = b^2b \Longrightarrow jb = b^3, jb^2 = 4a = b^4, 4ab = 4b = b^5, \ldots, (j+1)a = b^{p-1}$, $(j+1)b = b^p, \ldots$ Then,

$$\underbrace{0,a,b,b^{p-1}}_{A_0},\underbrace{0,b^{p-1},b^p,b^{2p-2} = a}_{A_1},0,a,b,\ldots$$

Each subsequence A_i has $\alpha = 4$ term and it contains exactly one zero. Every subsequence A_1 is a multiply of A_0, more precisely, the following congruences hold modulo p

$$A_1 \equiv b^{p-1} A_0.$$

Now , the last term in A_o is b^{p-1} and the last term in A_1 is $b^{2p-2} = a = 1$, i.e., order of b is $2p-2$. If number of subsequences A_i is $\beta = 2$. Clearly, it follows that period is $\alpha.\beta$. So, Fibonacci sequence is simple periodic and period is $4.2 = 8$.

(iii) If $j = p-3$,

$$\underbrace{0, a, b, (j+1)a, (j+2)b, a}_{A_0}, 0, a, b, \ldots$$

It is clear that only subsequence A_0 has $\alpha = 6$ term and it contains exactly one zero. Thus, Fibonacci sequence is simple periodic and period is $1.6 = 6$.

(iv) Suppose that a single period of $mod(p)$ is partitioned into smaller finite subsequences $A_0, A_1, A_2, \ldots$ as shown below $a = 1$:

$$0, a, b, (j+1)a, (j+2)b, \underbrace{(4k+1)a, (2k+1)b, (j-(4k-1))a, (j-(2k-2))b}_{k=1},$$
$$\underbrace{(4k+1)a, (2k+1)b, (j-(4k-1))a, (j-(2k-2))b}_{k=2}, \ldots, 0, a, b, \ldots$$

and

$$\underbrace{0, a, b, (j+1)a, (j+2)b, \underbrace{(4k+1)a, (2k+1)b, (j-(4k-1))a, (j-(2k-2))b}_{k=1},}_{A_0}$$
$$\underbrace{\underbrace{(4k+1)a, (2k+1)b, (j-(4k-1))a, (j-(2k-2))b}_{k=2}, \ldots, \underbrace{(4k+1)a, (2k+1)b}_{k=r},}_{A_0}$$
$$\underbrace{\underbrace{(j-(4k-1))a, (j-(2k-2))b}_{k=r}, \underbrace{(4k+1)a, (2k+1)b}_{k=r+1}(4k+1)a, (2k+1)b, \ldots, \underbrace{(4k+1)a, (2k+1)b}_{k=s},}_{A_1}$$
$$\underbrace{\underbrace{(j-(4k-1))a, (j-(2k-2))b}_{k=s}, \underbrace{(4k+1)a, (2k+1)b}_{k=s+1}(4k+1)a, (2k+1)b, \ldots, \underbrace{(4k+1)a, (2k+1)b}_{k=t},}_{A_2}$$
$$\underbrace{\underbrace{(j-(4k-1))a, (j-(2k-2))b}_{k=t}, \underbrace{(4k+1)a, (2k+1)b}_{k=t+1}(4k+1)a, (2k+1)b, \ldots, \underbrace{(4k+1)a, (2k+1)b}_{k=u},}_{A_3}$$
$$0, a, \ldots$$

Each subsequences A_i has p term and it contains exactly one zero. If $j = 4k-1, F_{pn} = 0, F_{pn+1} = (j-(2k-2))b, 1 \leq n \leq 3$, then

$$F_{4p} = 0, F_{4p+1} = a, F_{4p+2} = b, \ldots$$

Thus, Fibonacci sequence is simple periodic and period is $4p$.

Example 1. (i) For $p=11$, the presentation of $GF(11^2)$

$$GF(11^2)=\langle a,b:11a=11b=0,a^2=a,b^2=ja,ab=b,ba=b\rangle$$

j is not square an element in the $\mathbb{Z}_{11}$. If $p=11$, the non-square elements of $\mathbb{Z}_{11}$ can be calculated as follows.

$$1^2=1,2^2=4,3^2=9,4^2=5,5^2=3,6^2=3,7^2=5,8^2=9,9^2=4$$

where $\mathbb{Z}_{11}=\{\overline{0},\overline{1},\overline{2},\overline{3},\overline{4},\overline{5},\overline{6},\overline{7},\overline{8},\overline{9},\overline{10}\}$. After this point, the numbers repeat. For example $10^2=(-1)^2=1^2=1$.

Therefore the set of elements that are not square in the $\mathbb{Z}_{11}$ is $\{\overline{2},\overline{6},\overline{7},\overline{8},\overline{10}\}$ and the set of elements that are square in the $\mathbb{Z}_{11}$ is $\{\overline{0},\overline{1},\overline{3},\overline{4},\overline{5},\overline{9}\}$. From $Theorem$ 1.7. i., $j=11-1=10$

$$GF(11^2)=\langle a,b:11a=11b=0,a^2=a,b^2=10a,ab=b,ba=b\rangle$$

Let us consider the $Definition$ 1.5. For $F_{n+2}=A_1F_{n+1}+A_0F_n$ where $F_0=0,F_1=1$ and $A_1=b,A_0=a$, $n\geq 0$.

$$\begin{aligned}&0,1,=a,b,b^2+a^2=10a+a=11a=0,ba=b,b^2=10a,\\&10ab+ba=11b=0,10a^2=10a,10ab=10b,10b^2+10a^2=110a=0,\\&10ba=10b,10b^2=100a=a,ab+10ba=11b=0,a^2=a,ab=b,\dots\end{aligned}$$

From relations in the $GF(11^2)$, we have follows

$$b^2=10a,b^3=10ab=10b,b^4=10b^2=100a=a$$

Then the sequence is

$$\underbrace{0,a,b}_{A_0},\underbrace{0,b,b^2}_{A_1},\underbrace{0,b^2,b^3}_{A_2},\underbrace{0,b^3,a}_{A_3},0,a,b,\dots$$

Each subsequence has 3 terms and the number of subsequences 4 . Thus, Fibonacci sequence is simple periodic and period is $3.4=12$.

(ii) For $p=13$, the presentation of $GF(13^2)$

$$GF(13^2)=\langle a,b:13a=13b=0,a^2=a,b^2=ja,ab=b,ba=b\rangle$$

j is not square an element in the $\mathbb{Z}_{13}$. Set of elements that are not square in the $\mathbb{Z}_{13}$ is $\{\overline{2},\overline{5},\overline{6},\overline{7},\overline{8},\overline{11}\}$ and the set of elements that are square in the $\mathbb{Z}_{13}$ is $\{\overline{0},\overline{1},\overline{3},\overline{4},\overline{9},\overline{10},\overline{12}\}$. From $Theorem$ 1.7. (ii), $j=13-2=11$

$$GF(13^2)=\langle a,b:13a=13b=0,a^2=a,b^2=11a,ab=b,ba=b\rangle$$

Let us consider the *Definition* 1.5. For $F_{n+2} = A_1F_{n+1} + A_0F_n$ where $F_0 = 0, F_1 = 1$ and $A_1 = b, A_0 = a$, $n \geq 0$.

$$\begin{aligned}
&0,1,a,b,b^2+a^2 = 11a+a = 12a, 12ab+ba = 13b = 0, 12a^2 = 12a,\\
&12ab = 12b, 12b^2+12a^2 = 132a+12a = 144a = a, ab+12ba = 0,\\
&a^2 = a, ab = b, \ldots
\end{aligned}$$

From relations in the $GF(13^2)$, we have follows

$$\begin{aligned}
&b^2 = 11a, b^3 = 11ab = 11b, b^4 = 11b^2 = 121a = 4a, b^5 = 4ab = 4b,\\
&b^6 = 5a, b^7 = 5b, b^8 = 3a, b^9 = 3b, b^{10} = 7a, b^{11} = 7b,\\
&b^{17} = 9b, b^{18} = 99a = 8a, b^{19} = 8b, b^{20} = 88a = 10a, b^{21} = 10b,\\
&b^{22} = 110a = 6a, b^{23} = 6ab = 6b, b^{24} = 66a = a, \ldots
\end{aligned}$$

That is $b^{2p-2} = b^{26-2} = b^{24} = a$. Then the sequence is

$$\underbrace{0,a,b,12a}_{A_0},\underbrace{0,12a,12b,a}_{A_1},0,a,b,\ldots$$

$$\underbrace{0,a,b,b^{12}}_{A_0},\underbrace{0,b^{12},b^{13},b^{24}}_{A_1},0,a,b\ldots$$

Each subsequence has 4 terms and the number of subsequences 2. Thus, Fibonacci sequence is simple periodic and period is $2.4 = 8$.

(iii) For $p = 17$, the presentation of $GF(17^2)$

$$GF(17^2) = \langle a,b : 17a = 17b = 0, a^2 = a, b^2 = ja, ab = b, ba = b \rangle$$

j is not square an element in the $\mathbb{Z}_{17}$. Set of elements that are not square in the $\mathbb{Z}_{17}$ is $\{\overline{3},\overline{5},\overline{6},\overline{7},\overline{10},\overline{11},\overline{12},\overline{14}\}$ and the set of elements that are square in the $\mathbb{Z}_{17}$ is $\{\overline{0},\overline{1},\overline{2},\overline{4},\overline{8},\overline{9},\overline{13},\overline{15},\overline{16}\}$. From *Theorem* 1.7. iii., $j = 17-3 = 14$

$$GF(17^2) = \langle a,b : 17a = 17b = 0, a^2 = a, b^2 = 14a, ab = b, ba = b \rangle$$

Let us consider the *Definition* 1.5. For $F_{n+2} = A_1F_{n+1} + A_0F_n$ where $F_0 = 0, F_1 = 1$ and $A_1 = b, A_0 = a$, $n \geq 0$.

$$\begin{aligned}
&0,1,=a,b,b^2+a^2 = 14a+a = 15a, 15ba+ab = 16b,\\
&16b^2+15a^2 = 224a+15a = 239a = a, ba+16ab = b+16b = 17b = 0,\\
&b0+aa = a^2 = a, ba+a0 = b, bb+aa = b^2+a^2 = 14a+a = 15a, \ldots
\end{aligned}$$

Then the sequence is

$$\underbrace{0,a,b,15a,16b,a}_{A_0},0,a,b,15a,\ldots$$

Subsequence has 6 terms and there is a subsequence. Thus, Fibonacci sequence is simple periodic and period is 6 .

(iv) For $p=19$, the presentation of $GF(19^2)$

$$GF(19^2)=\langle a,b:19a=19b=0,a^2=a,b^2=ja,ab=b,ba=b\rangle$$

j is not square an element in the $\mathbb{Z}_{19}$. Set of elements that are not square in the $\mathbb{Z}_{19}$ is $\{\overline{2},\overline{3},\overline{7},\overline{8},\overline{10},\overline{12},\overline{13},\overline{14},\overline{15},\overline{18}\}$ and the set of elements that are square in the $\mathbb{Z}_{19}$ is $\{\overline{0},\overline{1},\overline{4},\overline{5},\overline{6},\overline{9},\overline{11},\overline{16},\overline{17}\}$. From *Theorem* 1.7. (iii), $j=19-4=15$

$$GF(19^2)=\langle a,b:19a=19b=0,a^2=a,b^2=15a,ab=b,ba=b\rangle$$

Let us consider the *Definition* 1.5. For $F_{n+2}=A_1F_{n+1}+A_0F_n$ where $F_0=0,F_1=1$ and $A_1=b,A_0=a$, $n\geq 0$. It is use relations to $GF(19^2)$,

$$\underbrace{0,a,b,16a,17b,5a,3b,12a,15b,9a,5b,8a,13b,13a,7b,4a,11b,17a,9b}_{A_0}$$

$$\underbrace{0,9b,2a,11b,15a,7b,6a,13b,11a,5b,10a,15b,7a,3b,14a,17b,3a,b,18a}_{A_1}$$

$$\underbrace{0,18a,18b,3a,2b,14a,16b,7a,4b,10a,14b,11a,6b,6a,12b,15a,8b,2a,10b}_{A_2}$$

$$\underbrace{0,10b,17a,8b,4a,12b,13a,6b,8a,14b,9a,4b,12a,16b,5a,2b,16a,18b,a}_{A_3}$$

$$0,a,\ldots$$

For $k=4,\quad j=4.4-1,F_{19.1}=0,\quad F_{19.1+1}=F_{20}=(15-(2.4-2))b=9b,\quad 1\leq n\leq 3,\quad F_{4.19}=0,\quad F_{4.19+1}=a,\ F_{4.19+2}=b.$

It is clear that subsequence $A_i, 0\leq i\leq 3$, has $p=19$ term and it contains exactly one zero. Thus, Fibonacci sequence is simple periodic and period is $4p=4.19=76$.

2 Conclusion

For any prime p, up to isomorphism, it can be seen that the period of the Fibonacci sequence $GF(p^2)$ of field of order p^2 is determined by j in the presentation of $GF(p^2)$. Consider $p=11$:

(i) From Example i., we have that the period of the Fibonacci sequence is 12 for $j=p-1, p=11$.

(ii) We not use Theorem 1.7., ii. for $p=11$ because $j=p-2=11-2=9\notin\{\overline{2},\overline{6},\overline{7},\overline{8},\overline{10}\}$ for Theorem 1.7., ii. where j is not square an element in the $\mathbb{Z}_{11}$, $j\in\{\overline{2},\overline{6},\overline{7},\overline{8},\overline{10}\}$.

(iii) Let us consider the Theorem 1.7., iii. for $j=p-3=11-3=8\in\{\overline{2},\overline{6},\overline{7},\overline{8},\overline{10}\}$

$$GF(11^2)=\langle ab:11a=11b=0,a^2=a,b^2=8a,ab=b,ba=b\rangle$$

From *Definition* 1.5. For $F_{n+2}=A_1F_{n+1}+A_0F_n$ where $F_0=0,F_1=1$ and $A_1=b,A_0=a$, $n\geq 0$.

$$0,1=a,b,b^2+a^2=8a+a=9a,9ba+ab=10b,$$
$$10b^2+9a^2=80a+9a=a,ba+10ab=b+10b=11b=0,$$
$$b0+aa=a^2=a,ba+a0=b,bb+aa=b^2+a^2=8a+a=9a,\ldots$$

Then the sequence is

$$\underbrace{0,a,b,9a,10b,a}_{A_0},0,a,b,9a,\ldots$$

Subsequence has 6 terms and there is a subsequence. Thus, Fibonacci sequence is simple periodic and period is 6 .

(iv) If it is use Theorem 1.7., iv. for $j=p-4=11-4=7$, it can be seen clearly that the period of the Fibonacci sequence is 44 .

Consequently, the period of the Fibonacci sequence is 12 for $j=p-1$, the period of the Fibonacci sequence is 6 for $j=p-3$ and the period of the Fibonacci sequence is 44 for ,$j=p-4,p=11$.

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