New Trends in Mathematical Sciences

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On exponential Pompeiu's type inequalities for double integrals with applications to ostrowski's inequality

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Abstract: The main of this paper is to derive some new inequalities of Ostrowski type using Pompeiu's mean value theorem for double integrals involving functions of two independent variables via being used exponential function.

Keywords: Ostrowski's inequality, Pompeiu's inequalities, absolutely continuous function, double integrals.

1 Introduction

In 1938, the classical integral inequality established by Ostrowski [10] as follows:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f^{'}:(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty=\sup\limits_{t\in(a,b)}|f'(t)|<\infty$. Then, the inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\left\|f'\right\|_\infty \tag{1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

Inequality (1) has wide applications in numerical analysis and in the theory of some special means; estimating error bounds for some special means, some mid-point, trapezoid and Simpson rules and quadrature rules, etc. Hence inequality (1) has attracted considerable attention and interest from mathematicans and researchers. Due to this, over the years, the interested reader is also refered to ([2]-[5], [8],[14]-[18]) for integral inequalities in several independent variables.. In addition, the current approach of obtaining the bounds, for a particular quadrature rule, have depended on the use of Peano kernel. The general approach in the past has involved the assumption of bounded derivatives of degree greater than one.

In 1946, Pompeiu [12] derived a variant of Lagrange's mean value theorem, now known as *Pompeiu's mean value theorem*.

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Theorem 2. *For every real valued function f differentiable on an interval $[a,b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a,b]$, there exist a point ξ between x_1 and x_2 such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi).$$

In [13], E.C. Popa using a mean value theorem obtained following theorem.

Theorem 3. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Assume that $\alpha \notin [a,b]$. Then for any $x \in [a,b]$, we have the inequality*

$$\left| \left(\frac{a+b}{2} - \alpha \right) f(x) + \frac{\alpha - x}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \left\| f - l_\alpha f' \right\|_\infty .$$

In [11], the authors have proved the following Ostrowski type inequality:

Theorem 4. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $0 < a < b$. Then for $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have the inequality*

$$\left| \frac{a+b}{2} \cdot \frac{f(x)}{x} + \frac{1}{b-a} \int_a^b f(t)dt \right| \leq PU(x,p) \left\| f - l f' \right\|_p ,$$

for $x \in [a,b]$, where

$$PU(x,p) := (b-a)^{\frac{1}{p}-1} \left[\left(\frac{a^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - a^{1+q} x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} + \left(\frac{b^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - b^{1+q} x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \right].$$

In the cases $(p,q) = (1,\infty), (\infty,1)$ and $(2,2)$ the quantity $PU(x,p)$ has to be taken as the limit as $p \longrightarrow 1, \infty$ and 2, respectively.

In [7], Dragomir has proved the Ostrowski type inequalities for complex valued absolutely continuous functions as follows:

Theorem 5. *Let $f : [a,b] \longrightarrow \mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ and $\alpha \in \mathbb{C}$ with $Re(\alpha) > 0$. Then for any $x \in [a,b]$ we have*

$$\left| \frac{\exp(\alpha b) - \exp(\alpha a)}{\alpha} f(x) - \exp(\alpha x) \int_a^b f(t)dt \right| \leq \begin{cases} |Re(\alpha)| \left\| f' - \alpha f \right\|_\infty B_1(a,b,x,\alpha), \; \text{if } f' - \alpha f \in L_\infty[a,b], \\ q^{\frac{1}{q}} |Re(\alpha)|^{\frac{1}{q}} (b-a)^{\frac{1}{p}} \qquad \text{if } f' - \alpha f \in L_p[a,b], \\ \times \left\| f' - \alpha f \right\|_p \left| B_q(a,b,x,\alpha) \right|^{\frac{1}{q}}, \qquad p > 1, \; \frac{1}{p} + \frac{1}{q} = 1, \\ \left\| f' - \alpha f \right\|_1 B_\infty(a,b,x,\alpha), \qquad \text{if } f' - \alpha f \in L_1[a,b], \end{cases}$$

where

$$B_q(a,b,x,\alpha) := \left[2\exp(xqRe(\alpha)) \left(x - \frac{a+b}{2} \right) + \frac{1}{qRe(\alpha)} \left(\frac{\exp(bqRe(\alpha)) + \exp(aqRe(\alpha))}{2} - \exp(xqRe(\alpha)) \right) \right]$$

for $q \geq 1$ and

$$B_\infty(a,b,x,\alpha) := \exp(xRe(\alpha))(x-a) + \frac{\exp(bRe(\alpha)) - \exp(xRe(\alpha))}{Re(\alpha)}.$$

The interested reader is also referred to ([1]-[9], [11], [13], [7], [4]-[22]) for integral inequalities by using Pompeiu's mean value theorem. The main aim of this paper is to establish some Pompeiu's type inequlity for complex valued absolutely continuous functions with double integrals involving functions of two independent variables.

2 Main Result

To prove our theorems, we need the following lemma:

Lemma 1. *$f:\Delta=[a,b]\times[c,d]\to\mathbb{C}$ be an absolutely continuous function such that the partial derivative of order 2 exists for all $(t,s)\in\Delta$ and $\alpha\in\mathbb{C}$ with $Re(\alpha)>0$. Then for any $(t,s),(x,y)\in\Delta$ with $x\neq y\neq t\neq s$, we have*

$$\frac{f(x,y)}{\exp(\alpha x)\exp(\alpha y)}-\frac{f(x,s)}{\exp(\alpha x)\exp(\alpha s)}-\frac{f(t,y)}{\exp(\alpha t)\exp(\alpha y)}+\frac{f(t,s)}{\exp(\alpha t)\exp(\alpha s)} \tag{2}$$

$$=\int_t^x\int_s^y\left[\frac{f_{uv}(u,v)-\alpha f_u(u,v)-\alpha f_v(u,v)+\alpha^2 f(u,v)}{\exp(\alpha u)\exp(\alpha v)}\right]dvdu$$

or, equaivalently

$$\exp(\alpha t)\exp(\alpha s)f(x,y)-\exp(\alpha y)\exp(\alpha t)f(x,s)-\exp(\alpha x)\exp(\alpha s)f(t,y)+\exp(\alpha x)\exp(\alpha y)f(t,s) \tag{3}$$
$$=\exp(\alpha x)\exp(\alpha y)\exp(\alpha t)\exp(\alpha s)\times\int_t^x\int_s^y\left[\frac{f_{uv}(u,v)-\alpha f_u(u,v)-\alpha f_v(u,v)+\alpha^2 f(u,v)}{\exp(\alpha u)\exp(\alpha v)}\right]dvdu.$$

Proof. Since f is continuously differantiable function, $\frac{f(u,v)}{\exp(\alpha u)\exp(\alpha v)}$ is an absolutely continuous function on Δ. Then we get

$$\int_t^x\int_s^y\frac{\partial^2}{\partial u\partial v}\left[\frac{f(u,v)}{\exp(\alpha u)\exp(\alpha v)}\right]dvdu=\int_t^x\frac{\partial}{\partial u}\left(\frac{f(u,y)}{\exp(\alpha u)\exp(\alpha y)}-\frac{f(u,s)}{\exp(\alpha u)\exp(\alpha s)}\right)du \tag{4}$$
$$=\frac{f(x,y)}{\exp(\alpha x)\exp(\alpha y)}-\frac{f(x,s)}{\exp(\alpha x)\exp(\alpha s)}-\frac{f(t,y)}{\exp(\alpha t)\exp(\alpha y)}+\frac{f(t,s)}{\exp(\alpha t)\exp(\alpha s)},$$

for all $(t,s),(x,y)\in\Delta$ with $x\neq y\neq t\neq s$. On the other hand,

$$\frac{\partial^2}{\partial u\partial v}\left[\frac{f(u,v)}{\exp(\alpha u)\exp(\alpha v)}\right]=\frac{f_{uv}(u,v)-\alpha f_u(u,v)-\alpha f_v(u,v)+\alpha^2 f(u,v)}{\exp(\alpha u)\exp(\alpha v)} \tag{5}$$
$$\tag{6}$$

for all $(t,s),(x,y)\in\Delta$ with $x\neq y\neq t\neq s$. Thus, from (4) and (5), it follows that

$$\int_t^x\int_s^y\frac{\partial^2}{\partial u\partial v}\left[\frac{f(u,v)}{u^r v^r}\right]dvdu=\int_t^x\int_s^y\left[\frac{f_{uv}(u,v)-\alpha f_u(u,v)-\alpha f_v(u,v)+\alpha^2 f(u,v)}{\exp(\alpha u)\exp(\alpha v)}\right]dvdu$$
$$=\frac{f(x,y)}{\exp(\alpha x)\exp(\alpha y)}-\frac{f(x,s)}{\exp(\alpha x)\exp(\alpha s)}-\frac{f(t,y)}{\exp(\alpha t)\exp(\alpha y)}+\frac{f(t,s)}{\exp(\alpha t)\exp(\alpha s)}$$

which this completes the proof.

Theorem 6. *$f:\Delta\to\mathbb{C}$ be an absolutely continuous function such that the partial derivative of order 2 exists for all $(t,s)\in\Delta$ and $\alpha\in\mathbb{C}$ with $Re(\alpha)>0$. Then for any $(t,s),(x,y)\in\Delta$ with $x\neq y\neq t\neq s$, we have*

$$
\begin{aligned}
&|\exp(\alpha t)\exp(\alpha s)f(x,y)-\exp(\alpha t)\exp(\alpha y)f(x,s)-\exp(\alpha x)\exp(\alpha s)f(t,y)+\exp(\alpha x)\exp(\alpha y)f(t,s)| \\
&\leq \begin{cases} Re^2(\alpha)\,\|F\|_\infty\,|\exp(xRe(\alpha))-\exp(tRe(\alpha))|\times|\exp(yRe(\alpha))-\exp(sRe(\alpha))|, & if\ F\in L_\infty(\Delta), \\ q^{\frac{2}{q}}Re^{\frac{2}{q}}(\alpha)\,\|F\|_p\times|\exp(xqRe(\alpha))-\exp(tqRe(\alpha))|^{\frac{1}{q}}\times|\exp(yqRe(\alpha))-\exp(sqRe(\alpha))|^{\frac{1}{q}}, & if\ F\in L_p(\Delta),\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \|F\|_1\max\{\exp(tRe(\alpha)),\exp(xRe(\alpha))\}\times\max\{\exp(sRe(\alpha)),\exp(yRe(\alpha))\}, & if\ F\in L_1(\Delta) \end{cases}
\end{aligned} \tag{7}
$$

where $F=f_{uv}-\alpha f_u-\alpha f_v+\alpha^2 f$.

Proof. From Lemma 1 and by using modulus, we have

$$
\begin{aligned}
&\left|\frac{f(x,y)}{\exp(\alpha x)\exp(\alpha y)}-\frac{f(x,s)}{\exp(\alpha x)\exp(\alpha s)}-\frac{f(t,y)}{\exp(\alpha t)\exp(\alpha y)}+\frac{f(t,s)}{\exp(\alpha t)\exp(\alpha s)}\right| \\
&=\left|\int_t^x\int_s^y\left[\frac{f_{uv}(u,v)-\alpha f_u(u,v)-\alpha f_v(u,v)+\alpha^2 f(u,v)}{\exp(\alpha u)\exp(\alpha v)}\right]dvdu\right| \\
&\leq\left|\int_t^x\int_s^y\frac{|f_{uv}(u,v)-\alpha f_u(u,v)-\alpha f_v(u,v)+\alpha^2 f(u,v)|}{|\exp(\alpha u)\exp(\alpha v)|}dvdu\right|:=|I|
\end{aligned} \tag{8}
$$

Firstly, we will consider the case $p=\infty$ and $q=1$. Then, we have

$$
|I|\leq\sup_{(u,v)\in[t,x]\times[s,y]}\left|f_{uv}(u,v)-\alpha f_u(u,v)-\alpha f_v(u,v)+\alpha^2 f(u,v)\right|\times\left|\int_t^x\int_s^y\frac{1}{|\exp(\alpha u)\exp(\alpha v)|}dvdu\right|. \tag{9}
$$

Since $\alpha=Re(\alpha)+iIm(\alpha)$ and $(u,v)\in\Delta$, we have

$$
|\exp(\alpha u)\exp(\alpha v)|=\exp(uRe(\alpha))\exp(vRe(\alpha)). \tag{10}
$$

Using the identity (10) we get

$$
\begin{aligned}
\int_t^x\int_s^y\frac{1}{|\exp(\alpha u)\exp(\alpha v)|}dvdu&=\int_t^x\int_s^y\frac{1}{\exp(uRe(\alpha))\exp(vRe(\alpha))}dvdu \\
&=\left(\int_t^x\frac{1}{\exp(uRe(\alpha))}du\right)\left(\int_s^y\frac{1}{\exp(vRe(\alpha))}dv\right) \\
&=Re^2(\alpha)\left[\frac{1}{\exp(tRe(\alpha))}-\frac{1}{\exp(xRe(\alpha))}\right] \\
&\quad\left[\frac{1}{\exp(sRe(\alpha))}-\frac{1}{\exp(yRe(\alpha))}\right].
\end{aligned} \tag{11}
$$

Utilizing the inequalities (9) and (11), we obtain

$$
|I|\leq Re^2(\alpha)\left\|f_{uv}-\alpha f_u-\alpha f_v+\alpha^2 f\right\|_\infty\times\left|\frac{1}{\exp(tRe(\alpha))}-\frac{1}{\exp(xRe(\alpha))}\right|\left|\frac{1}{\exp(sRe(\alpha))}-\frac{1}{\exp(yRe(\alpha))}\right|.
$$

Now, consider the case $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, by using Hölder's inequality we have

$$|I| \le \left|\int_t^x\int_s^y \left|f_{uv}(u,v)-\alpha f_u(u,v)-\alpha f_v(u,v)+\alpha^2 f(u,v)\right|^p dvdu\right|^{\frac{1}{p}} \times \left|\int_t^x\int_s^y \frac{1}{|\exp(\alpha u)\exp(\alpha v)|^q}dvdu\right|^{\frac{1}{q}}$$
$$= q^{\frac{2}{q}} Re^{\frac{2}{q}}(\alpha)\left\|f_{uv}-\alpha f_u-\alpha f_v+\alpha^2 f\right\|_p \times \left|\frac{1}{\exp(tqRe(\alpha))}-\frac{1}{\exp(xqRe(\alpha))}\right|^{\frac{1}{q}} \left|\frac{1}{\exp(sqRe(\alpha))}-\frac{1}{\exp(yqRe(\alpha))}\right|^{\frac{1}{q}}.$$

Finally, we consider the case $p=1$ and $q=\infty$. Then, we get

$$|I| \le \left|\int_t^x\int_s^y \left|f_{uv}(u,v)-\alpha f_u(u,v)-\alpha f_v(u,v)+\alpha^2 f(u,v)\right| dvdu\right| \times \sup_{(u,v)\in[t,x]\times[s,y]}\left(\frac{1}{\exp(\alpha u)\exp(\alpha v)}\right)$$
$$= \left\|f_{uv}-\alpha f_u-\alpha f_v+\alpha^2 f\right\|_1 \times \frac{1}{\min\{\exp(tRe(\alpha)),\exp(xRe(\alpha))\}}\frac{1}{\min\{\exp(sRe(\alpha)),\exp(yRe(\alpha))\}}.$$

This completes the proof.

Now, we examine some particular case of Theorem 6.

Corollary 1. *$f:\Delta\to\mathbb{C}$ be an absolutely continuous function such that the partial derivative of order 2 exists for all $(t,s)\in\Delta$. Then for any $(t,s),(x,y)\in\Delta$ with $x\neq y\neq t\neq s$, we have*

$$|\exp(t)\exp(s)f(x,y)-\exp(t)\exp(y)f(x,s)-\exp(x)\exp(s)f(t,y)+\exp(x)\exp(y)f(t,s)|$$
$$\le \begin{cases} \|F_1\|_\infty |\exp(x)-\exp(t)|\,|\exp(y)-\exp(s)|, & \text{if } F_1\in L_\infty(\Delta), \\ q^{\frac{2}{q}}\|F_1\|_p |\exp(xq)-\exp(tq)|^{\frac{1}{q}}\,|\exp(yq)-\exp(sq)|^{\frac{1}{q}}, & \text{if } F_1\in L_p(\Delta),\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \|F_1\|_1 \max\{\exp(t),\exp(x)\}\max\{\exp(s),\exp(y)\}, & \text{if } F_1\in L_1(\Delta), \end{cases}$$

where $F_1=f_{uv}-f_u-f_v+f$.

Remark. If we take $Re(\alpha)=0$, then the inequality (7) becomes for any $(t,s),(x,y)\in\Delta$

$$\begin{aligned}&\Bigg|\frac{f(x,y)}{\exp(iIm(\alpha)x)\exp(iIm(\alpha)y)}-\frac{f(x,s)}{\exp(iIm(\alpha)x)\exp(iIm(\alpha)s)}\\&-\frac{f(t,y)}{\exp(iIm(\alpha)t)\exp(iIm(\alpha)y)}+\frac{f(t,s)}{\exp(iIm(\alpha)t)\exp(iIm(\alpha)s)}\Bigg|\\&\le \begin{cases} \|F_2\|_\infty |x-t|\,|y-s|, & \text{if } F_2\in L_\infty(\Delta), \\ \|F_2\|_p |x-t|^{\frac{1}{q}}\,|y-s|^{\frac{1}{q}}, & \text{if } F_2\in L_p(\Delta),\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \|F_2\|_1, & \text{if } F_2\in L_1(\Delta), \end{cases}\end{aligned} \tag{12}$$

where $F_2=f_{uv}-iIm(\alpha)f_u-iIm(\alpha)f_v-Im^2(\alpha)f$. In particular, we have for any $(t,s),(x,y)\in\Delta$.

$$\left|\frac{f(x,y)}{\exp(ix)\exp(iy)}-\frac{f(x,s)}{\exp(ix)\exp(is)}-\frac{f(t,y)}{\exp(it)\exp(iy)}+\frac{f(t,s)}{\exp(it)\exp(is)}\right|$$
$$\le \begin{cases} \|F_3\|_\infty |x-t|\,|y-s|, & \text{if } F_3\in L_\infty(\Delta), \\ \|F_3\|_p |x-t|^{\frac{1}{q}}\,|y-s|^{\frac{1}{q}}, & \text{if } F_3\in L_p(\Delta),\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \|F_3\|_1, & \text{if } F_3\in L_1(\Delta), \end{cases}$$

or, equivalently

$$
\begin{aligned}
&\left|\exp(it)\exp(is)f(x,y)-\exp(it)\exp(iy)f(x,s)-\exp(ix)\exp(is)f(t,y)+\exp(ix)\exp(iy)f(t,s)\right| \\
&\leq \begin{cases} \|F_3\|_\infty |x-t|\,|y-s|, & \text{if } F_3\in L_\infty(\Delta), \\ \|F_3\|_p |x-t|^{\frac{1}{q}}\,|y-s|^{\frac{1}{q}}, & \text{if } F_3\in L_p(\Delta), \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \|F_3\|_1, & \text{if } F_3\in L_1(\Delta), \end{cases}
\end{aligned}
$$

where $F_3=f_{uv}-if_u-if_v-f$.

Theorem 7. *$f:\Delta\to\mathbb{C}$ be an absolutely continuous function such that the partial derivative of order 2 exists for all $(t,s)\in\Delta$ and $\alpha\in\mathbb{C}$ with $Re(\alpha)>0$. Then for any $(t,s),(x,y)\in\Delta$ with $x\neq y\neq t\neq s$, we have*

$$
\begin{aligned}
&\left|\frac{(\exp(\alpha b)-\exp(\alpha a))(\exp(\alpha d)-\exp(\alpha c))}{\alpha^2}f(x,y)-\frac{(\exp(\alpha b)-\exp(\alpha a))}{\alpha}\exp(\alpha y)\int_c^d f(x,s)ds\right. \\
&\left.-\frac{(\exp(\alpha d)-\exp(\alpha c))}{\alpha}\exp(\alpha x)\int_a^b f(t,y)dt+\exp(\alpha x)\exp(\alpha y)\int_a^b\int_c^d f(t,s)dsdt\right| \\
&\leq \begin{cases} Re^2(\alpha)\|F\|_\infty B_1(a,b,x,\alpha)B_1(c,d,y,\alpha), & \text{if } F\in L_\infty(\Delta), \\ q^{\frac{2}{q}}Re^{\frac{2}{q}}(\alpha)(b-a)^{\frac{1}{p}}(d-c)^{\frac{1}{p}} & \text{if } F\in L_p(\Delta), \\ \times\|F\|_p\left|B_q(a,b,x,\alpha)\right|^{\frac{1}{q}}\left|B_q(c,d,y,\alpha)\right|^{\frac{1}{q}}, & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \|F\|_1 B_\infty(a,b,x,\alpha)B_\infty(c,d,y,\alpha), & \text{if } F\in L_1(\Delta), \end{cases}
\end{aligned}
\tag{13}
$$

where $F=f_{uv}-\alpha f_u-\alpha f_v+\alpha^2 f$,

$$
B_q(a,b,x,\alpha):=2\left[\exp(xqRe(\alpha))\left(x-\frac{a+b}{2}\right)+\frac{1}{qRe(\alpha)}\left(\frac{\exp(bqRe(\alpha))+\exp(aqRe(\alpha))}{2}-\exp(xqRe(\alpha))\right)\right]
$$

for $q\geq 1$ and

$$
B_\infty(a,b,x,\alpha):=\exp(xRe(\alpha))(x-a)+\frac{\exp(bRe(\alpha))-\exp(xRe(\alpha))}{Re(\alpha)}.
$$

Proof. By using the first inequality in (7), it follows that

$$
\begin{aligned}
(|A)|&:=(|\frac{((\exp(\alpha b)-\exp(\alpha a)))((\exp(\alpha d)-\exp(\alpha c)))}{\alpha^2}f(x,y))-\frac{((\exp(\alpha b)-\exp(\alpha a)))}{\alpha}\exp(\alpha y)\int_c^d f(x,s)ds \\
&-\frac{((\exp(\alpha d)-\exp(\alpha c)))}{\alpha}\exp(\alpha x)\int_a^b f(t,y)dt+\exp(\alpha x)\exp(\alpha y)\int_a^b\int_c^d f(t,s)dsdt)| \\
&\leq\int_a^b\int_c^d(|\exp(\alpha t)\exp(\alpha s)f(x,y)-\exp(\alpha t)\exp(\alpha y)f(x,s)-\exp(\alpha x)\exp(\alpha s)f(t,y)+\exp(\alpha x)\exp(\alpha y)f(t,s))|dsdt \\
&\leq Re^2(\alpha)(\|f_{uv}-\alpha f_u-\alpha f_v+\alpha^2 f)\|_\infty\times\int_a^b\int_c^d(|\exp(xRe(\alpha))-\exp(tRe(\alpha)))|(|\exp(yRe(\alpha))-\exp(sRe(\alpha)))|dsdt.
\end{aligned}
\tag{14}
$$

Because of $Re(\alpha)>0$ in condition of theorem, we have

$$\begin{aligned}
&\int\limits_a^b |\exp(xRe(\alpha))-\exp(tRe(\alpha))|\,dt \qquad (15)\\
=&\int\limits_a^x (\exp(xRe(\alpha))-\exp(tRe(\alpha)))\,dt+\int\limits_x^b (\exp(tRe(\alpha))-\exp(xRe(\alpha)))\,dt\\
=&2\left[\exp(xRe(\alpha))\left(x-\frac{a+b}{2}\right)+\frac{1}{Re(\alpha)}\left(\frac{\exp(bRe(\alpha))+\exp(aRe(\alpha))}{2}-\exp(xRe(\alpha))\right)\right]\\
=&B_1(a,b,x,\alpha)
\end{aligned}$$

and

$$\begin{aligned}
\int\limits_c^d |\exp(yRe(\alpha))-\exp(sRe(\alpha))|\,ds=&\int\limits_c^y (\exp(yRe(\alpha))-\exp(sRe(\alpha)))\,ds+\int\limits_y^d (\exp(sRe(\alpha))-\exp(yRe(\alpha)))\,ds \qquad (16)\\
=&B_1(c,d,y,\alpha).
\end{aligned}$$

If we substitute the identities (15) and (16) in (14), then we obtain the first inequality in (13). By using the second inequality in (7), it follows that

$$\begin{aligned}
|A|\le&\int\limits_a^b\int\limits_c^d |\exp(\alpha t)\exp(\alpha s)f(x,y)-\exp(\alpha t)\exp(\alpha y)f(x,s)-\exp(\alpha x)\exp(\alpha s)f(t,y)+\exp(\alpha x)\exp(\alpha y)f(t,s)|\,dsdt\\
\le&q^{\frac{2}{q}}Re^{\frac{2}{q}}(\alpha)\left\|f_{uv}-\alpha f_u-\alpha f_v+\alpha^2 f\right\|_p\\
&\times\int\limits_a^b\int\limits_c^d |\exp(xqRe(\alpha))-\exp(tqRe(\alpha))|^{\frac{1}{q}}\,|\exp(yqRe(\alpha))-\exp(sqRe(\alpha))|^{\frac{1}{q}}\,dsdt.
\end{aligned}$$

By Hölder's integral inequality we also have

$$\begin{aligned}
&\int\limits_a^b\int\limits_c^d |\exp(xqRe(\alpha))-\exp(tqRe(\alpha))|^{\frac{1}{q}}\,|\exp(yqRe(\alpha))-\exp(sqRe(\alpha))|^{\frac{1}{q}}\,dsdt \qquad (17)\\
=&\left(\int\limits_a^b\int\limits_c^d dsdt\right)^{\frac{1}{p}}\times\left[\int\limits_a^b\int\limits_c^d\left(|\exp(xqRe(\alpha))-\exp(tqRe(\alpha))|^{\frac{1}{q}}\,|\exp(yqRe(\alpha))-\exp(sqRe(\alpha))|^{\frac{1}{q}}\right)^q dsdt\right]^{\frac{1}{q}}\\
=&(b-a)^{\frac{1}{p}}(d-c)^{\frac{1}{p}}\times\left[\int\limits_a^b\int\limits_c^d |\exp(xqRe(\alpha))-\exp(tqRe(\alpha))|\,|\exp(yqRe(\alpha))-\exp(sqRe(\alpha))|\,dsdt\right]^{\frac{1}{q}}.
\end{aligned}$$

Because of $Re(\alpha)>0$ in condition of theorem, we have

$$\int\limits_a^b |\exp(xqRe(\alpha))-\exp(tqRe(\alpha))|\,dt=B_q(a,b,x,\alpha) \qquad (18)$$

and

$$\int\limits_c^d |\exp(yqRe(\alpha))-\exp(sqRe(\alpha))|\,ds=B_q(c,d,y,\alpha). \qquad (19)$$

If we substitute the identities (19) and (18) in (17), then we obtain the second inequality in (13).

Finally, using the third inequality of (7) we have

$$|A| \leq \int_a^b \int_c^d |\exp(\alpha t)\exp(\alpha s) f(x,y) - \exp(\alpha t)\exp(\alpha y) f(x,s) - \exp(\alpha x)\exp(\alpha s) f(t,y) + \exp(\alpha x)\exp(\alpha y) f(t,s)|\, ds dt$$

$$\leq \left\| f_{uv} - \alpha f_u - \alpha f_v + \alpha^2 f \right\|_1 \times \int_a^b \int_c^d \max\left\{\exp(tRe(\alpha)), \exp(xRe(\alpha))\right\} \max\left\{\exp(sRe(\alpha)), \exp(yRe(\alpha))\right\} ds dt. \tag{20}$$

Because of $Re(\alpha) > 0$ in condition of theorem, we have

$$\begin{aligned} &\int_a^b \max\left\{\exp(tRe(\alpha)), \exp(xRe(\alpha))\right\} dt \\ &= \int_a^x \max\left\{\exp(tRe(\alpha)), \exp(xRe(\alpha))\right\} dt + \int_x^b \max\left\{\exp(tRe(\alpha)), \exp(xRe(\alpha))\right\} dt \\ &= \int_a^x \exp(xRe(\alpha)) dt + \int_x^b \exp(tRe(\alpha)) dt \\ &= \exp(xRe(\alpha))(x-a) + \frac{\exp(bRe(\alpha)) - \exp(xRe(\alpha))}{Re(\alpha)} \\ &= B_\infty(a,b,x,\alpha) \end{aligned} \tag{21}$$

and similarly,

$$\int_c^d \max\left\{\exp(sRe(\alpha)), \exp(yRe(\alpha))\right\} ds = B_\infty(c,d,y,\alpha) \tag{22}$$

If we substitute the identities (22) and (21) in (20), then we get the third part of (13).Thus, the proof is completed.

Remark. If $Re(\alpha) < 0$, then a similar result may be stated. Furthermore, if taken the equality (2) instead of the equality (3) in Theorem 6, then a similar inequality may be obtained. However the details are left to the interested reader.

Corollary 2. *$f : \Delta \to \mathbb{C}$ be an absolutely continuous function such that the partial derivative of order 2 exists for all $(t,s) \in \Delta$. Then for any $(t,s), (x,y) \in \Delta$ with $x \neq y \neq t \neq s$, we have*

$$\begin{aligned} &\Bigg| (\exp(b) - \exp(a))(\exp(d) - \exp(c)) f(x,y) - (\exp(b) - \exp(a))\exp(y) \int_c^d f(x,s) ds \\ &- (\exp(d) - \exp(c))\exp(x) \int_a^b f(t,y) dt + \exp(x)\exp(y) \int_a^b \int_c^d f(t,s) ds dt \Bigg| \\ &\leq \begin{cases} \|F_1\|_\infty B_1(a,b,x) B_1(c,d,y), & \text{if } F_1 \in L_\infty(\Delta), \\ q^{\frac{2}{q}} (b-a)^{\frac{1}{p}} (d-c)^{\frac{1}{p}} \|F_1\|_p \times \left|B_q(a,b,x)\right|^{\frac{1}{q}} \left|B_q(c,d,y)\right|^{\frac{1}{q}}, & \text{if } F_1 \in L_p(\Delta), \ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ \|F_1\|_1 B_\infty(a,b,x) B_\infty(c,d,y), & \text{if } F_1 \in L_1(\Delta) \end{cases} \end{aligned} \tag{23}$$

where $F_1 = f_{uv} - f_u - f_v + f$,

$$B_q(a,b,x) := 2\left[\exp(xq)\left(x - \frac{a+b}{2}\right) + \frac{1}{q}\left(\frac{\exp(bq) + \exp(aq)}{2} - \exp(xq)\right)\right]$$

for $q \geq 1$ and

$$B_\infty(a,b,x) := \exp(x)(x-a) + \exp(b) - \exp(x).$$

Remark. If we take $(x,y) = \left(\frac{a+b}{2}, \frac{c+d}{2}\right)$ in (23), we get

$$\Bigg|(\exp(b)-\exp(a))(\exp(d)-\exp(c))\,f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - (\exp(b)-\exp(a))\exp\left(\frac{c+d}{2}\right)\int_c^d f\left(\frac{a+b}{2},s\right)ds$$
$$-(\exp(d)-\exp(c))\exp\left(\frac{a+b}{2}\right)\int_a^b f\left(t,\frac{c+d}{2}\right)dt + \exp\left(\frac{a+b}{2}\right)\exp\left(\frac{c+d}{2}\right)\int_a^b\int_c^d f(t,s)dsdt\Bigg|$$
$$\leq \begin{cases} \|F_1\|_\infty B_1(a,b)B_1(c,d), & \text{if } F_1 \in L_\infty(\Delta) \\ q^{\frac{2}{q}}\|F_1\|_p (b-a)^{\frac{1}{p}}(d-c)^{\frac{1}{p}} \times \left|B_q(a,b)\right|^{\frac{1}{q}}\left|B_q(c,d)\right|^{\frac{1}{q}}, & \text{if } F_1 \in L_p(\Delta), \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \|F_1\|_1 B_\infty(a,b)B_\infty(c,d), & \text{if } F_1 \in L_1(\Delta) \end{cases}$$

where $F_1 = f_{uv} - f_u - f_v + f$,

$$B_q(a,b) := \frac{2}{q}\left(\frac{\exp(bq)+\exp(aq)}{2} - \exp\left(\frac{a+b}{2}q\right)\right)$$

for $q \geq 1$ and

$$B_\infty(a,b) := \frac{b-a}{2}\exp\left(\frac{a+b}{2}\right) + \exp(b) - \exp\left(\frac{a+b}{2}\right).$$

Theorem 8. *$f : \Delta \to \mathbb{C}$ be an absolutely continuous function such that the partial derivative of order 2 exists for all $(t,s) \in \Delta$ and $\alpha \in \mathbb{C}$ with $Re(\alpha) = 0$ and $Im(\alpha) \neq 0$. Then for any $(t,s),(x,y) \in \Delta$ with $x \neq y \neq t \neq s$, we have*

$$\Bigg|-\frac{(\exp(iIm(\alpha)b)-\exp(iIm(\alpha)a))(\exp(iIm(\alpha)d)-\exp(iIm(\alpha)c))}{Im(\alpha)}f(x,y)$$
$$-\frac{(\exp(iIm(\alpha)b)-\exp(iIm(\alpha)a))}{iIm(\alpha)}\exp(iIm(\alpha)y)\int_c^d f(x,s)ds \tag{24}$$
$$-\frac{(\exp(iIm(\alpha)d)-\exp(iIm(\alpha)c))}{iIm(\alpha)}\exp(iIm(\alpha)x)\int_a^b f(t,y)dt + \exp(iIm(\alpha)x)\exp(iIm(\alpha)y)\int_a^b\int_c^d f(t,s)dsdt\Bigg|$$
$$\leq \begin{cases} (b-a)^2(d-c)^2\|F_2\|_\infty \times \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{4}+\left(\frac{y-\frac{c+d}{2}}{d-c}\right)^2\right], & \text{if } F_2 \in L_\infty(\Delta), \\ \frac{q^2(b-a)^{\frac{q+1}{q}}(d-c)^{\frac{q+1}{q}}}{(q+1)^2}\|F_2\|_p\left[\left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}}+\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}}\right]\left[\left(\frac{d-x}{d-c}\right)^{\frac{q+1}{q}}+\left(\frac{x-c}{d-c}\right)^{\frac{q+1}{q}}\right], & \text{if } F_2 \in L_p(\Delta), \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ (b-a)(d-c)\|F_2\|_1, & \text{if } F_2 \in L_1(\Delta) \end{cases}$$

where $F_2 = f_{uv} - iIm(\alpha)f_u - iIm(\alpha)f_v - Im^2(\alpha)f$.

Proof. Utilizing the inequality (12) we have

$$
\begin{aligned}
&\Bigg| -\frac{(\exp(iIm(\alpha)b)-\exp(iIm(\alpha)a))(\exp(iIm(\alpha)d)-\exp(iIm(\alpha)c))}{Im(\alpha)} f(x,y) \\
&-\frac{(\exp(iIm(\alpha)b)-\exp(iIm(\alpha)a))}{iIm(\alpha)}\exp(iIm(\alpha)y)\int_c^d f(x,s)ds \\
&-\frac{(\exp(iIm(\alpha)d)-\exp(iIm(\alpha)c))}{iIm(\alpha)}\exp(iIm(\alpha)x)\int_a^b f(t,y)dt+\exp(iIm(\alpha)x)\exp(iIm(\alpha)y)\int_a^b\int_c^d f(t,s)dsdt\Bigg| \\
&\leq \int_a^b\int_c^d |\exp(iIm(\alpha)t)\exp(iIm(\alpha)s)f(x,y)-\exp(iIm(\alpha)t)\exp(iIm(\alpha)y)f(x,s) \\
&\quad -\exp(iIm(\alpha)x)\exp(iIm(\alpha)s)f(t,y)+\exp(iIm(\alpha)x)\exp(iIm(\alpha)y)f(t,s)|\,dsdt \\
&\leq \begin{cases}
\left\|f_{uv}-iIm(\alpha)f_u-iIm(\alpha)f_v-Im^2(\alpha)f\right\|_\infty \int_a^b\int_c^d |x-t|\,|y-s|\,dsdt \\
\left\|f_{uv}-iIm(\alpha)f_u-iIm(\alpha)f_v-Im^2(\alpha)f\right\|_p \int_a^b\int_c^d |x-t|^{\frac{1}{q}}\,|y-s|^{\frac{1}{q}}\,dsdt \\
\left\|f_{uv}-iIm(\alpha)f_u-iIm(\alpha)f_v-Im^2(\alpha)f\right\|_1 \int_a^b\int_c^d dsdt.
\end{cases}
\end{aligned}
$$

Since

$$
\int_a^b |x-t|\,dt=\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2
$$

and

$$
\int_c^d |y-s|\,ds=\left[\frac{1}{4}+\left(\frac{y-\frac{c+d}{2}}{d-c}\right)^2\right](d-c)^2
$$

we obtain the first inequality in (24).

Since

$$
\int_a^b\int_c^d |x-t|^{\frac{1}{q}}\,|y-s|^{\frac{1}{q}}\,dsdt=\frac{q(b-a)^{\frac{q+1}{q}}}{q+1}\left[\left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}}+\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}}\right]
$$

and

$$
\int_a^b\int_c^d |x-t|^{\frac{1}{q}}\,|y-s|^{\frac{1}{q}}\,dsdt=\frac{q(d-c)^{\frac{q+1}{q}}}{q+1}\left[\left(\frac{d-x}{d-c}\right)^{\frac{q+1}{q}}+\left(\frac{x-c}{d-c}\right)^{\frac{q+1}{q}}\right]
$$

we obtain the second inequality in (24). Hence, the proof of theorem is completed.

Corollary 3. *$f:\Delta\to\mathbb{C}$ be an absolutely continuous function such that the partial derivative of order 2 exists for all $(t,s)\in\Delta$. Then for any $(t,s),(x,y)\in\Delta$ with $x\neq y\neq t\neq s$, we have*

$$\Bigg|-(\exp(ib)-\exp(ia))(\exp(id)-\exp(ic))f(x,y)-\frac{(\exp(ib)-\exp(ia))}{i}\exp(iy)\int_c^d f(x,s)ds \tag{25}$$

$$-\frac{(\exp(id)-\exp(ic))}{i}\exp(ix)\int_a^b f(t,y)dt+\exp(ix)\exp(iy)\int_a^b\int_c^d f(t,s)dsdt\Bigg|$$

$$\leq\begin{cases}\|F_3\|_\infty(b-a)^2(d-c)^2\times\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{4}+\left(\frac{y-\frac{c+d}{2}}{d-c}\right)^2\right], & \text{if } F_3\in L_\infty(\Delta),\\ \frac{q^2(b-a)^{\frac{q+1}{q}}(d-c)^{\frac{q+1}{q}}}{(q+1)^2}\|F_3\|_p\left[\left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}}+\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}}\right]\left[\left(\frac{d-x}{d-c}\right)^{\frac{q+1}{q}}+\left(\frac{x-c}{d-c}\right)^{\frac{q+1}{q}}\right], & \text{if } F_3\in L_p(\Delta),\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ (b-a)(d-c)\|F_3\|_1, & \text{if } F_3\in L_1(\Delta)\end{cases}$$

where $F_3=f_{uv}-if_u-if_v-f$.

Remark. If we take $(x,y)=\left(\frac{a+b}{2},\frac{c+d}{2}\right)$ in (25), we get

$$\Bigg|-(\exp(ib)-\exp(ia))(\exp(id)-\exp(ic))f(\frac{a+b}{2},\frac{c+d}{2})-\frac{(\exp(ib)-\exp(ia))}{i}\exp(i\frac{c+d}{2})\int_c^d f(\frac{a+b}{2},s)ds$$

$$-\frac{(\exp(id)-\exp(ic))}{i}\exp(i\frac{a+b}{2})\int_a^b f(t,\frac{c+d}{2})dt+\exp(i\frac{a+b}{2})\exp(i\frac{a+b}{2})\int_a^b\int_c^d f(t,s)dsdt\Bigg|$$

$$\leq\begin{cases}\frac{1}{16}\|F_3\|_\infty(b-a)^2(d-c)^2, \text{ if } F_3\in L_\infty(\Delta),\\ \\ \frac{q^2(b-a)^{\frac{q+1}{q}}(d-c)^{\frac{q+1}{q}}}{(q+1)^2 2^{\frac{2}{q}}}\|F_3\|_p, & \text{if } F_3\in L_p(\Delta),\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1.\end{cases}$$

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