Some Hermite-Hadamard-Fejer type inequalities for harmonically convex functions via fractional integral

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Abstract: In this paper, we gave the new general identity for differentiable functions. As a result of this identity some new and general inequalities for differentiable harmonically-convex functions are obtained.

Keywords: Harmonically-convex, Hermite-Hadamard-Fejer type inequality, fractional integral.

1 Introduction

The classical or the usual convexity is defined as follows,

Definition 1. A function $f : I \rightarrow \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on $I$ if inequality

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

A number of papers have been written on inequalities using the classical convexity and one of the most captivating inequalities in mathematical analysis is stated as follows,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2},$$

(1)

where $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a \leq b$. Both the inequalities hold in reversed direction if $f$ is concave. The inequalities stated in (1) are known as Hermite-Hadamard inequalities.

For more results on (1) which provide new proof, significantly extensions, generalizations, refinements, counterparts, new Hermite-Hadamard-type inequalities and numerous applications, we refer the interested reader to [2,3,5,6,8,9,12,13,15,16] and the references there in.

The usual notion of convex function have been generalized in diverse manners. One of them is the so called harmonically s-convex functions and is stated in the definition below.

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Definition 2. [5, 7] Let \( I \subseteq (0, \infty) \) be a real interval. A function \( f : I \rightarrow \mathbb{R} \) is said to be harmonically \( s \)-convex (concave), if
\[
f \left( \frac{xy}{tx + (1-t)y} \right) \leq (\geq) t f(y) + (1-t)f(x)
\]
holds for all \( x, y \in I \) and \( t \in [0,1] \), and for some fixed \( s \in (0,1] \).

It can be easily seen that for \( s = 1 \) in Definition 2 reduces to following Definition 3,

Definition 3. [6] A function \( f : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) is said to be harmonically-convex function, if
\[
f \left( \frac{xy}{tx + (1-t)y} \right) \leq tf(y) + (1-t)f(x)
\]
holds for all \( x, y \in I \) and \( t \in [0,1] \). If the inequality is reversed, then \( f \) is said to be harmonically concave.

Proposition 1. [6] Let \( I \subseteq \mathbb{R} \setminus \{0\} \) be a real interval and \( f : I \rightarrow \mathbb{R} \) is function, then:

(i) if \( I \subset (0,\infty) \) and \( f \) is convex and nondecreasing function then \( f \) is harmonically convex.
(ii) if \( I \subset (0,\infty) \) and \( f \) is harmonically convex and nonincreasing function then \( f \) is convex.
(iii) if \( I \subset (-\infty,0) \) and \( f \) is harmonically convex and nondecreasing function then \( f \) is convex.
(iv) if \( I \subset (-\infty,0) \) and \( f \) is convex and nonincreasing function then \( f \) is harmonically convex.

For the properties of harmonically-convex functions and harmonically-\( s \)-convex function, we refer the reader to [1, 5, 6, 7, 8, 10, 11] and the reference there in.

Most recently, a number of findings have been seen on Hermite-Hadamard type integral inequalities for harmonically-convex and for harmonically-\( s \)-convex functions.

In [14], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1).

Theorem 1. Let \( f : [a,b] \rightarrow \mathbb{R} \) be convex function. Then the inequality
\[
f \left( \frac{a+b}{2} \right) \int_a^b g(x)dx \leq \frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{f(a) + f(b)}{2} \int_a^b g(x)dx,
\]
holds, where \( g : [a,b] \rightarrow \mathbb{R} \) is nonnegative, integrable and symmetric to \( (a+b)/2 \).

For some results which generalize, improve, and extend the inequalities (1) and (2) see [15].

In [6], İşcan gave definition of harmonically convex functions and established following Hermite- Hadamard type inequality for harmonically-convex functions as follows.

Theorem 2. [15] Let \( f : (0,\infty) \rightarrow \mathbb{R} \) be a harmonically convex function and \( a, b \in I \) with \( a < b \). If \( f \in L[a,b] \) then the following inequalities hold:
\[
f \left( \frac{2ab}{a+b} \right) \leq \frac{ab}{b-a} \int_a^b f(x) \frac{dx}{x^2} \leq \frac{f(a) + f(b)}{2}.
\]

In [11], Iscan and Wu represented Hermite-Hadamard’s inequalities for harmonically convex functions in fractional integral form as follows.
Theorem 3. [11] Let \( f : I \subseteq \mathbb{R}^+ \to \mathbb{R} \) be a function such that \( f \in L[a,b] \), where \( a, b \in I \) with \( a < b \). If \( f \) is harmonically-convex on \([a,b]\), then the following inequalities for fractional integrals hold:

\[
\frac{2ab}{a+b} \leq \frac{\Gamma(\alpha+1)}{2^\alpha} \left( \frac{ab}{b-a} \right)^\alpha \left\{ f_{1/a}^\alpha (f \circ h)(1/b) + J_{1/b^+}^\alpha (f \circ h)(1/a) \right\} \leq \frac{f(a) + f(b)}{2},
\]

with \( \alpha > 0 \) and \( h(x) = 1/x \).

Definition 4. A function \( g : [a,b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R} \) is said to be harmonically symmetric with respect to \( 2ab/a+b \) if

\[
g(x) = g \left( \frac{1}{\frac{a}{x} + \frac{b}{x} - \frac{1}{x}} \right)
\]

holds for all \( x \in [a,b] \).

Theorem 4. In [1] Chan and Wu represented Hermite-Hadamard-Fejer inequality for harmonically convex functions as follows:

Theorem 5. Suppose that \( f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R} \) be harmonically-convex function and \( a, b \in I \), with \( a < b \). If \( f \in L[a,b] \) and \( g : [a,b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R} \) is nonnegative, integrable and harmonically symmetric with respect to \( 2ab/a+b \) then

\[
f \left( \frac{2ab}{a+b} \right) \int_a^b \frac{g(x)}{x^2} dx \leq \int_a^b \frac{f(x)g(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2} \int_a^b \frac{g(x)}{x^2} dx
\]

In [10] Iscan and Kunt represented Hermite-Hadamard-Fejer type inequality for harmonically convex functions in fractional integral forms and established following identity as follows:

Theorem 6. Let \( f : [a,b] \to \mathbb{R} \) be harmonically convex function with \( a < b \) and \( f \in L[a,b] \). If \( g : [a,b] \to \mathbb{R} \) is nonnegative, integrable and harmonically symmetric with respect to \( 2ab/a+b \), then the following inequalities for fractional integrals hold:

\[
f \left( \frac{2ab}{a+b} \right) \left[ f_{1/a}^\alpha (g \circ h)(1/b) + J_{1/b^+}^\alpha (g \circ h)(1/a) \right] \leq \left[ f_{1/a}^\alpha (fg \circ h)(1/b) + J_{1/b^+}^\alpha (fg \circ h)(1/a) \right] \leq \frac{f(a) + f(b)}{2} \left[ f_{1/a}^\alpha (g \circ h)(1/b) + J_{1/b^+}^\alpha (g \circ h)(1/a) \right]
\]

with \( \alpha > 0 \) and \( h(x) = 1/x, x \in \left[ \frac{1}{b}, \frac{1}{a} \right] \).

Definition 5. Let \( f \in L[a,b] \). The right-hand side and left-hand side Hadamard fractional integrals \( f_{a^+}^\alpha \) and \( f_{b^-}^\alpha \) of order \( \alpha > 0 \) with \( a \geq 0 \) are defined by

\[
f_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a
\]

\[
f_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b
\]

respectively where \( \Gamma(\alpha) \) is the Gamma function defined by \( \Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt \) and \( f_{a^+}^\alpha f(x) = f_{b^-}^\alpha f(x) = f(x) \).

Lemma 1. For \( 0 < \theta \leq 1 \) and \( 0 < a \leq b \) we have

\[
|a^\theta - b^\theta| \leq (b-a)^\theta.
\]
In [4] D. Y. Hwang found out a new identity and by using this identity, established a new inequalities. Then in [12] İ. İşcan and S. Turhan used this identity for GA-convex functions and obtain generalized new inequalities. In this paper, we established a new inequality similar to inequality in [12] and then we obtained some new and general integral inequalities for differentiable harmonically-convex functions using this lemma. The following sections, let the notion, $L(t) = \frac{bt}{t + \gamma t}$, $U(t) = \frac{bt}{\gamma t + t}$ and $H = H(a,b) = \frac{2ab}{a+b}$.

2 Main result

Throughout this section, let $\|g\|_{\infty} = \sup_{x \in [a,b]} |g(x)|$, for the continuous function $g : [a,b] \rightarrow [0,\infty)$ be differentiable mapping $P$, where $a, b \in I$ with $a \leq b$, and $h : [a,b] \rightarrow [0,\infty)$ be differentiable mapping.

**Lemma 2.** If $f' \in L[a,b]$ then the following inequality holds:

$$
|h(b) - 2h(a)| \frac{f'(a)}{2} + h(b) \frac{f(b)}{2} - \int_a^b f(x)h'(x)dx
$$

$$
= \frac{b-a}{4ab} \left\{ \frac{1}{2} [2h(L(t)) - h(b)] f'(L(t)) (L(t))^2 dt + \frac{1}{2} [2h(U(t)) - h(b)] f'(U(t)) (U(t))^2 dt \right\}.
$$

**Proof.** By the integration by parts, we have

$$
I_1 = \int_0^1 [2h(L(t)) - h(b)] d(f(L(t))) = [2h(L(t)) - h(b)] f(L(t))^2 \left\{ \frac{1}{a} - \frac{1}{b} \right\} \int_0^1 f(L(t)) h'(L(t)) (L(t))^2 dt
$$

and

$$
I_2 = \int_0^1 [2h(U(t)) - h(b)] d(f(U(t))) = [2h(U(t)) - h(b)] f(U(t))^2 \left\{ \frac{1}{a} - \frac{1}{b} \right\} \int_0^1 f(U(t)) h'(U(t)) (U(t))^2 dt.
$$

Therefore

$$
\frac{I_1 + I_2}{2} = [h(b) - 2h(a)] \frac{f'(a)}{2} + h(b) \frac{f(b)}{2} - \frac{b-a}{2ab} \left\{ \int_0^1 f(L(t)) h'(L(t)) (L(t))^2 dt + \int_0^1 f(U(t)) h'(U(t)) (U(t))^2 dt \right\}.
$$

This complete the proof.

**Lemma 3.** For $a, H, b > 0$, we have

$$
\zeta_1(a,b) = \int_0^1 [2h(L(t)) - h(b)] (1-t) (L(t))^2 dt
$$

$$
\zeta_2(a,b) = \int_0^1 t(L(t))^2 [2h(L(t)) - h(b)] dt + \int_0^1 t((U(t))^2 [2h(U(t)) - h(b)] dt
$$

$$
\zeta_3(a,b) = \int_0^1 [2h(U(t)) - h(b)] (1-t) (U(t))^2 dt.
$$

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Theorem 7. Let \( f : I \subseteq \mathbb{R} = (0, \infty) \rightarrow \mathbb{R} \) be differentiable mapping \( I' \), where \( a, b \in I \) with \( a < b \). If the mapping \( |f'| \) is harmonically-convex on \([a, b]\), then the following inequality holds:

\[
\left| h(b) - 2h(a) \right| \frac{f(a)}{2} + \frac{b}{2} f(h') - \frac{b}{a} \int_a^b f(x)dx \leq \frac{b-a}{4ab} \left[ \xi_1(a, b) |f'(a)| + \xi_2(a, b) |f'(H)| + \xi_3(a, b) |f'(b)| \right]
\]

where \( \xi_1(a, b), \xi_2(a, b), \xi_3(a, b) \) are defined in Lemma 2.

Proof. Continuing equality (8) in Lemma 2

\[
\left| h(b) - 2h(a) \right| \frac{f(a)}{2} + \frac{b}{2} f(h') - \frac{b}{a} \int_a^b f(x)dx \leq \frac{b-a}{4ab} \left\{ \int_0^1 |2h(L(t)) - h(b)| |f'(L(t))| (L(t))^2 dt + \int_0^1 2h(U(t)) - h(b) |f'(U(t)) (U(t))^2 dt \right\}.
\]

Using \( |f'| \) is harmonically-convex in (14).

\[
\left| h(b) - 2h(a) \right| \frac{f(a)}{2} + \frac{b}{2} f(h') - \frac{b}{a} \int_a^b f(x)dx \leq \frac{b-a}{4ab} \left\{ \int_0^1 |2h(L(t)) - h(b)| \{ t |f'(H)| + (1 - t) |f'(a)| \} (L(t))^2 dt \\
+ \int_0^1 2h(U(t)) - h(b) \{ t |f'(H)| + (1 - t) |f'(b)| \} (U(t))^2 dt \right\},
\]

by (15) and Lemma 2, this proof is complete.

Corollary 1. Let \( h(t) = \int_{1/t}^{1} \left( -\frac{1}{2} x^{\alpha-1} + \left( \frac{1}{2} - x \right)^{\alpha-1} \right) g \circ \varphi(x) dx \) for all \( 1/t \in \left[ \frac{1}{2}, \frac{1}{a} \right] \), \( \alpha > 0 \) and \( g : [a, b] \rightarrow [0, \infty) \) be continuous positive mapping and symmetric to \( \frac{2ab}{\alpha+2} \) in Theorem 7, we obtain:

\[
\left| \left( \frac{f(a)}{2} + f(b) \right) \left[ J_{1/b+}^a g \circ \varphi(1/a) + J_{1/a-}^a g \circ \varphi(1/b) \right] - \left[ J_{1/b+}^a (fg \circ \varphi)(1/a) + J_{1/a-}^a (fg \circ \varphi)(1/b) \right] \right|
\]

\[
\leq \frac{(b-a)^{\alpha+1} \|g\|_{\alpha}}{2^{\alpha+1}(ab)^{\alpha+1} \Gamma (\alpha + 1)} \left[ C_1(\alpha) |f'(a)| + C_2(\alpha) |f'(H)| + C_3(\alpha) |f'(b)| \right]
\]

where

\[
C_1(\alpha) = \int_0^1 (1-t) [(1+t)^a - (1-t)^a] (L(t))^2 dt
\]

\[
C_2(\alpha) = \int_0^1 t [(1+t)^a - (1-t)^a] [(L(t))^2 + (U(t))^2] dt
\]

\[
C_3(\alpha) = \int_0^1 (1-t) [(1+t)^a - (1-t)^a] (L(t))^2 dt.
\]
Specially in (16) and using Lemma 1, for $0 < \alpha \leq 1$ we have:

\[
\left| \frac{f(a) + f(b)}{2} \right| \left[ J_{1/b+}^a g \circ \varphi (1/a) + J_{1/a-}^a g \circ \varphi (1/b) \right] - \left[ J_{1/b+}^a (fg \circ \varphi) (1/a) + J_{1/a-}^a (fg \circ \varphi) (1/b) \right] \\
\leq \frac{(b-a)^{\alpha+1} \|g\|_{\infty}}{2(ab)\Gamma(\alpha+1)} \left[ C_1(\alpha) |f'(a)| + C_2(\alpha) |f'(H)| + C_3(\alpha) |f'(b)| \right]
\]

where

\[
C_1(\alpha) = \int_0^1 (1-t) t^{\alpha} (L(t))^2 dt, 
C_2(\alpha) = \int_0^1 t^{\alpha+1} [(L(t))^2 + (U(t))^2] dt, 
C_3(\alpha) = \int_0^1 (1-t) t^{\alpha} (U(t))^2 dt.
\]

Proof. By left side of inequality (15) in Theorem 7, when we write $h(t) = \int_0^1 (x-t)^{\alpha-1} + (\frac{a}{b} - x)^{\alpha-1} g \circ \varphi (x) dx$ for all $x \in [1/b, 1/a]$ and $\varphi(x) = 1/x$, we have

\[
\Gamma(\alpha) \left( \frac{f(a) + f(b)}{2} \right) \left[ J_{1/b+}^a g \circ \varphi (1/a) + J_{1/a-}^a g \circ \varphi (1/b) \right] - \Gamma(\alpha) \left[ J_{1/b+}^a (fg \circ \varphi) (1/a) + J_{1/a-}^a (fg \circ \varphi) (1/b) \right].
\]

On the other hand, right side of inequality (15), with

\[
\Psi(x,a,b) = \left( x - \frac{1}{b} \right)^{\alpha-1} + \left( \frac{1}{a} - x \right)^{\alpha-1}
\]

\[
\leq \frac{b-a}{4ab} \left\{ \frac{1}{2} \int_0^{1/a} \|\Psi(x,a,b)\| g \circ \varphi(x) dx - \int_{1/b}^{1/a} \|\Psi(x,a,b)\| g \circ \varphi(x) dx \right\} \left\{ t |f'(H)| + (1-t) |f'(a)| \right\} (L(t))^2 dt
\]

\[
+ \int_0^1 \frac{1}{2} \int_{1/U(t)}^{1/a} \|\Psi(x,a,b)\| g \circ \varphi(x) dx - \int_{1/b}^{1/a} \|\Psi(x,a,b)\| g \circ \varphi(x) dx \right\} \left\{ t |f'(H)| + (1-t) |f'(b)| \right\} (U(t))^2 dt
\]

Since $g(x)$ is symmetric to $x = \frac{2ab}{a+b}$, we have

\[
\int_0^{1/a} \|\Psi(x,a,b)\| g \circ \varphi(x) dx = \int_{1/b}^{1/a} \|\Psi(x,a,b)\| g \circ \varphi(x) dx
\]

and

\[
\int_{1/U(t)}^{1/a} \|\Psi(x,a,b)\| g \circ \varphi(x) dx = \int_{1/b}^{1/a} \|\Psi(x,a,b)\| g \circ \varphi(x) dx
\]

for all $t \in [0,1]$. By (18)-(20), we have

\[
\left| \frac{f(a) + f(b)}{2} \right| \left[ J_{1/b+}^a g \circ \varphi (1/a) + J_{1/a-}^a g \circ \varphi (1/b) \right] - \left[ J_{1/b+}^a (fg \circ \varphi) (1/a) + J_{1/a-}^a (fg \circ \varphi) (1/b) \right]
\]

\[
\leq \frac{b-a}{4ab\Gamma(\alpha)} \left\{ \frac{1}{2} \int_{1/U(t)}^{1/a} \|\Psi(x,a,b)\| g \circ \varphi(x) dx \right\} \left\{ t |f'(H)| + (1-t) |f'(a)| \right\} (L(t))^2 dt
\]
By Lemma 1, we have
\begin{align*}
+ \int_{0}^{1/L(t)} \left\{ \frac{1}{1/U(t)} \right\} \int_{0}^{1/U(t)} |\nabla x(a,b)| g \circ \varphi(x) dx \left\{ t \left| f'(H) \right| + (1-t) \left| f'(b) \right| \right\} \left( U(t) \right)^{2} dt \\
\leq \frac{(b-a)\|g\|_{\infty}}{4ab^{\frac{1}{2}}} \left\{ \int_{0}^{1/L(t)} \left\{ \frac{1}{1/U(t)} \right\} \int_{0}^{1/U(t)} |\nabla x(a,b)| dx \left\{ t \left| f'(H) \right| + (1-t) \left| f'(a) \right| \right\} \left( L(t) \right)^{2} dt \\
+ \int_{0}^{1/L(t)} \left\{ \frac{1}{1/U(t)} \right\} \int_{0}^{1/U(t)} |\nabla x(a,b)| dx \left\{ t \left| f'(H) \right| + (1-t) \left| f'(b) \right| \right\} \left( U(t) \right)^{2} dt \right\}.
\end{align*}

In the last inequality,
\begin{align*}
\int_{1/U(t)}^{1/L(t)} |\nabla x(a,b)| dx = \int_{1/U(t)}^{1/L(t)} \left( x - \frac{1}{b} \right)^{a-1} dx + \int_{1/U(t)}^{1/L(t)} \left( \frac{1}{a} - x \right)^{a-1} dx = \frac{2^{1-a}}{a} \left( \frac{b-a}{ab} \right)^{a} (1+t)^{a} - (1-t)^{a}.
\end{align*}

By Lemma 1, we have
\begin{align*}
\int_{1/U(t)}^{1/L(t)} |\nabla x(a,b)| dx = \int_{1/U(t)}^{1/L(t)} \left( x - \frac{1}{b} \right)^{a-1} dx + \int_{1/U(t)}^{1/L(t)} \left( \frac{1}{a} - x \right)^{a-1} dx \leq \frac{2}{a} \left( \frac{b-a}{ab} \right)^{a} t^{a}
\end{align*}

A combination of (21) and (22), we have (16). This complete is proof.

**Corollary 2.** In Corollary 1,

(i) If $\alpha = 1$ is in corollary, we obtain following Hermite-Hadamard-Fejer Type inequality for harmonically-convex function which is related the left-hand side of (17):

\begin{align*}
\left| \frac{f(a) + f(b)}{2} \right| \int_{a}^{b} \frac{g(x)}{x^{2}} dx - \int_{a}^{b} f(x) \frac{g(x)}{x^{2}} dx \leq \frac{(b-a)^{2}}{4(ab)^{2}} \|g\|_{\infty} \left[ C_{1}(1) \left| f'(a) \right| + C_{2}(1) \left| f'(H) \right| + C_{3}(1) \left| f'(b) \right| \right]
\end{align*}

where for $a,b,H > 0$, we have

\begin{align*}
C_{1}(1) &= \int_{0}^{1} (1-t) \left( L(t) \right)^{2} dt \\
C_{2}(1) &= \int_{0}^{1} (1-t) \left( L(t) \right)^{2} + (U(t) \right)^{2} dt \\
C_{3}(1) &= \int_{0}^{1} (1-t) \left( U(t) \right)^{2} dt
\end{align*}
(ii) If \( g(x) = 1 \) is in corollary, we obtain following Hermite-Hadamard-Fejer Type inequality for harmonically-convex function which is related the left-hand side of (16):

\[
\left| \frac{f(a) + f(b)}{2} \right| \leq \frac{(b-a)}{2(ab-a^\alpha \eta_4)} \left[ I_{[a, b]} \Gamma(\alpha + 1) \left( f \circ \varphi \right)(1/a) + J_{[a, b]}(f \circ \varphi)(1/b) \right] \]

(24)

(iii) If \( g(x) = 1 \) and \( \alpha = 1 \) is in corollary, we obtain following Hermite-Hadamard-Fejer Type inequality for harmonically-convex function which is related the left-hand side of (17):

\[
\left| \frac{f(a) + f(b)}{2} \right| \leq \frac{(b-a)}{2(ab-a^\alpha \eta_4)} \left[ C_1(1) |f'(a)| + C_2(1) |f'(H)| + C_3(1) |f'(b)| \right] .
\]

(25)

**Theorem 8.** Let \( f : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) be differentiable mapping \( F' \), where \( a, b \in I \) with \( a < b \). If the mapping \( |f'|^q \) is harmonically-convex on \([a, b]\), then the following inequality holds:

\[
\left| \frac{h(b) - 2h(a)}{2} + h(b) \frac{f(a)}{2} - \int_a^b f(x)h'(x)dx \right| \leq \frac{b-a}{4ab} \left\{ \eta_1^q + \eta_2^q + \eta_3^q + \eta_4^q \right\}
\]

(26)

where

\[
\eta_1 = \left\{ \int_0^1 (2h(L(t)) - h(b)) |dt| \right\},
\]

\[
\eta_2 = \left\{ \int_0^1 \left( 2h(L(t)) - h(b) \right) |dt| \times \left( t \left( L(t) \right)^{2q} \right) \right\},
\]

\[
\eta_3 = \left\{ \int_0^1 \left( 2h(U(t)) - h(b) \right) |dt| \right\},
\]

\[
\eta_4 = \left\{ \int_0^1 \left( 2h(U(t)) - h(b) \right) |dt| \times \left( t \left( U(t) \right)^{2q} \right) \right\}.
\]

**Proof.** Continuing from (14) in Theorem 7, we use Hölder Inequality and we use that \( |f'|^q \) is harmonically-convex. Thus this proof is complete.

**Corollary 3.** Let \( h(t) = \int_0^1 \left[ \left( x \frac{1}{x} - \frac{1}{x} \right)^{\alpha-1} + \left( \frac{1}{x} - x \right)^{\alpha-1} \right] \left( g \circ \varphi \right)(x)dx \) for all \( t \in [a, b] \) and \( g : [a, b] \rightarrow [0, \infty) \) be continuous positive mapping and symmetric to \( \frac{2ab}{\alpha+1} \) in Theorem 8, we obtain:

\[
\left| \frac{f(a) + f(b)}{2} \right| \leq \frac{(b-a)}{2(ab-a^\alpha \eta_4)} \left[ C_1(\alpha, q) |f'(a)|^q + C_2(\alpha, q) |f'(H)|^q + C_3(\alpha, q) |f'(b)|^q \right]
\]

(27)
where for \( q > 1 \)

\[
C_1 (\alpha, q) = \int_0^1 [(1 + t)\alpha - (1 - t)\alpha] t (L(t))^{2q} \, dt
\]

\[
C_2 (\alpha, q) = \int_0^1 [(1 + t)\alpha - (1 - t)\alpha] (1 - t) \left( (L(t))^{2q} + (U(t))^{2q} \right) \, dt
\]

\[
C_3 (\alpha, q) = \int_0^1 [(1 + t)\alpha - (1 - t)\alpha] t (U(t))^{2q} \, dt.
\]

**Proof.** Continuing from (22) of Corollary 1 and (26) in Theorem 8,

\[
\left| \frac{(f(a) + f(b))}{2} \left[ \zeta_1 - \zeta_2 \right] \right| \leq \frac{(b - a)^{\alpha + 1}}{2^{\alpha + 1} \Gamma (\alpha + 1)} \{ \ell_1 \times \ell_2 + \ell_1 \times \ell_3 \}
\]

\[
\leq \frac{(b - a)^{\alpha + 1} \|g\|_\infty}{2^{\alpha + 1} (ab)^{\alpha + 1} \Gamma (\alpha + 1)} (\zeta_0)^{1 - \frac{1}{q}} [\ell_2 + \ell_3]
\]

where

\[
\zeta_0 = \frac{2^{\alpha + 1} - 2}{\alpha + 1},
\]

\[
\zeta_1 = J_{\alpha+1}^b g(b) + J_{\alpha+1}^a g(a),
\]

\[
\zeta_2 = J_{\alpha+1}^b (fg)(b) + J_{\alpha+1}^a (fg)(a),
\]

\[
\ell_1 = \frac{1}{0} [(1 + t)^\alpha - (1 - t)^\alpha] \, dt^{1 - \frac{1}{q}},
\]

\[
\ell_2 = \frac{1}{0} [(1 + t)^\alpha - (1 - t)^\alpha] \left( t (L(t))^{2q} |f'(a)|^q + (1 - t) (L(t))^{2q} |f'(H)|^q \right) \, dt^{\frac{1}{q}},
\]

\[
\ell_3 = \frac{1}{0} [(1 + t)^\alpha - (1 - t)^\alpha] \left( t (U(t))^{2q} |f'(b)|^q + (1 - t) (U(t))^{2q} |f'(H)|^q \right) \, dt^{\frac{1}{q}},
\]

By the power-mean inequality \((a' + b' < 2^{1-r}(a + b))^{r} for \ a > 0, b > 0, \ r < 1) and \(\frac{1}{p} + \frac{1}{q} = 1\) we have

\[
\frac{(b - a)^{\alpha + 1} \|g\|_\infty}{2^{\alpha + 1} (ab)^{\alpha + 1} \Gamma (\alpha + 1)} (\zeta_0)^{1 - \frac{1}{q}} [\ell_4 + \ell_5] \leq \frac{(b - a)^{\alpha + 1} \|g\|_\infty}{2^{\alpha + 1} (ab)^{\alpha + 1} \Gamma (\alpha + 1)} \left( \frac{2^{r}(2^r - 1)}{\alpha + 1} \right)^{\frac{1}{q}} \left\{ \frac{1}{0} (\xi_1 + \xi_2 + \xi_3) \, dt \right\}^{\frac{1}{q}},
\]

where

\[
\xi_1 = [(1 + t)^\alpha - (1 - t)^\alpha] t (L(t))^{2q} |f'(a)|^q,
\]

\[
\xi_2 = [(1 + t)^\alpha - (1 - t)^\alpha] (1 - t) \left( (L(t))^{2q} + (U(t))^{2q} \right) |f'(H)|^q,
\]

\[
\xi_3 = [(1 + t)^\alpha - (1 - t)^\alpha] t (U(t))^{2q} |f'(b)|^q.
\]
Corollary 4. When $\alpha = 1$ and $g(x) = 1$ is taken in Corollary 3, we obtain:

$$
\left| \frac{f(a) + f(b)}{2} \right| - \frac{ab}{(b-a)} \int_a^b \frac{f(x)}{x^2} \, dx \leq \frac{(b-a)}{2^{\frac{1}{q}} (ab)} \left[ C_1(1,q) |f'(a)|^q + C_2(1,q) |f'(H)|^q + C_3(1,q) |f'(b)|^q \right]^\frac{1}{q}.
$$

(30)

This proof is complete.

References