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# Collocation Method with Quintic $B$-Spline Method for Solving the Hirota equation

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**Abstract:** In the present article, a numerical method is proposed for the numerical solution of the Hirota equation by using collocation method with the quintic $B$-spline. The method is shown to be unconditionally stable using von-Neumann technique. To test accuracy the error norms $L_2$, $L_\infty$ are computed. Two invariants of motion are predestined to determine the conservation properties of the problem, and the numerical scheme leads to careful and active results. Furthermore, interaction of two and three solitary waves is shown. These results show that the technique introduced here is plain to apply.

**Keywords:** Collocation method, quintic B-splines method, Hirota equation.

## 1 Introduction

The purpose of this article is to apply quintic $B$-Splines method to the Hirota equation. The Hirota equation in the form [1]

$$u_t + 3\alpha|u|^2 u_x + \gamma u_{xxx} = 0, \tag{1}$$

where $u$ is a complex valued function of the spatial assortment $x$ and the time $t$ and $\alpha, \gamma$ are positive real constants. Boundary conditions

$$u(x,t) = u_x(x,t) = 0, \qquad |x| \to \infty, \quad 0 \le t \le T. \tag{2}$$

And initial conditions

$$u(x,0) = f(x), \qquad -\infty < x < \infty. \tag{3}$$

The exact solution of Hitora equation (1) is

$$\begin{aligned} &u(x,t) = \beta \operatorname{sec}h[\kappa(x-s-vt)]\exp(i\varphi), \\ &\beta = \sqrt{\frac{2\gamma}{\alpha}}\kappa, \qquad \varphi = a(x-bt-s), \\ &v = \gamma(\kappa^2 - 3a^2), \quad b = \gamma(3\kappa^3 - a^2), \end{aligned} \tag{4}$$

where $\beta$ is the amplitude of the wave, $\kappa$ is related to the width of the wave envelope and $v$ is the velocity. The parameter $a$ is the wave number of the phase and $b$ is related to the frequency of the phase. We assume that [2, 3].

$$u(x,t) = u_1(x,t) + iu_2(x,t), \quad i^2 = -1, \tag{5}$$

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where $u_1(x,t)$ and $u_2(x,t)$ are real functions. By substituting in Eq. (1) we will reduce Hirota equation to the coupled system in this form

$$(u_1)_t + 3\alpha(u_1^2 + u_2^2)(u_1)_x + \gamma(u_1)_{xxx} = 0,$$
$$(u_2)_t + 3\alpha(u_1^2 + u_2^2)(u_2)_x + \gamma(u_2)_{xxx} = 0. \tag{6}$$

We can write this system in this form

$$(u)_t + 3\alpha z(u)(u)_x + \gamma(u)_{xxx} = 0, \tag{7}$$

where

$$z(u) = (u_1^2 + u_2^2), \quad u = [u_1, u_2]^T.$$

The Eq. (1) is an integrable equation and it is important because it has many physical implementation, such as the spread of optical pluses in nematic liquid crystal waveguides. The Hirota equation is regarding to both the nonlinear Schrodinger equation and modified Korteweg-de Vries (mKdV) equations, as it is complex popularization of the mKdV equation. In addition; the soliton solution of this equation has a very comparable form to the nonlinear Schrodinger equation soliton. The Eq. (1) has two-parameter soliton family, with amplitude and velocity.

The Hirota equation has been studied numerically by Hoseini S. M. and Marchant T. R. [1] and the Eq. (1) has been solved by W. G. Al. Harbi [4]. The numerical solution of nonlinear wave equations has been the subject of many studies in recent years. Such as the Korteweg-de Vries (KdV) equation has been studied by [5, 6, 7, 8] and the nonlinear Schrodinger equation has been solved by [9, 10]. Numerical solution of coupled partial differential equations, as an example, the coupled nonlinear Schrodinger equation admits soliton solution and it has many applications in communication, this system has been studied numerically by Ismail [11,12,13,14] and the coupled Korteweg-de Vries equation has been discussed numerically [15, 16, 17, 18]. The complex nonlinear partial differential equations have been studied in [2, 3, 19, 20].

The paper is marshaled as follows. In section 2, we have introduced dissection of our method. In section 3, we have been studied the stability for our scheme. In section 4, numerical results for problem and some related figures are given in order to show the efficiency as well as the accuracy of the proposed method and we introduced the interaction of two and three solitary waves. Finally, conclusions are followed in section 5.

## 2 Quintic $B$-spline functions

To construct numerical solution, consider nodal points $(x_j, t_n)$ defined in the region $[a,b] \times [0,T]$ where

$$a = x_0 < x_1 < \cdots < x_N = b, \quad h = x_{j+1} - x_j = \frac{b-a}{N}, \quad j = 0, 1, \cdots, N.$$

$$0 = t_0 < t_1 < \cdots < t_n < \cdots < T, \quad t_{j+1} - t_j = \Delta t,\ t_n = n\Delta t,\ n = 0, 1, \cdots.$$

The quintic $B$-spline basis functions at knots are given by:

$$B_j(x) = \frac{1}{h^5}\begin{cases} (x-x_{j-3})^5, & x_{j-3} \le x \le x_{j-2} \\ (x-x_{j-3})^5 - 6(x-x_{j-2})^5, & x_{j-2} \le x \le x_{j-1} \\ (x-x_{j-3})^5 - 6(x-x_{j-2})^5 + 15(x-x_{j-1})^5, & x_{j-1} \le x \le x_j \\ (-x+x_{j+3})^5 + 6(x-x_{j+2})^5 - 15(x-x_{j+1})^5, & x_j \le x \le x_{j+1} \\ (-x+x_{j+3})^5 + 6(x-x_{j+2})^5, & x_{j+1} \le x \le x_{j+2} \\ (-x+x_{j+3})^5, & x_{j+2} \le x \le x_{j+3} \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Using quintic $B$-spline basis function (8) the values of $B_j(x)$ and its derivatives at the knots points can be calculated, which are tabulated in Table 1.

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## 3 Solution of the Hirota equation

To apply the proposed method, we rewrite (6) as

$$(u_1)_t + 3\alpha(u_1^2 + u_2^2)(u_1)_x + \gamma(u_1)_{xxx} = 0,$$
$$(u_2)_t + 3\alpha(u_1^2 + u_2^2)(u_2)_x + \gamma(u_2)_{xxx} = 0.$$

we take the approximations $u_1(x,t) = (U_1)_j^n$ and $u_2(x,t) = (U_2)_j^n$, then from famous Cranckâ-Nicolson scheme and forward finite difference approximation for the derivative $t$ [21], we get

$$\left[\frac{(U_1)_{xxxj}^{n+1} + (U_1)_{xxxj}^{n}}{2}\right] + \frac{(U_1)_j^{n+1} - (U_1)_j^n}{k} + 3\alpha\left[\frac{(z(U))_j^{n+1} + (z(U))_j^n}{2}\right]\left[\frac{(U_1)_{xj}^{n+1} + (U_1)_{xj}^{n}}{2}\right] = 0, \tag{9}$$

$$\left[\frac{(U_2)_{xxxj}^{n+1} + (U_2)_{xxxj}^{n}}{2}\right] + \frac{(U_2)_j^{n+1} - (U_2)_j^n}{k} + 3\alpha\left[\frac{(z(U))_j^{n+1} + (z(U))_j^n}{2}\right]\left[\frac{(U_2)_{xj}^{n+1} + (U_2)_{xj}^{n}}{2}\right] = 0, \tag{10}$$

where $k = \delta t$ is the time step.

In the Crankâ-Nicolson scheme, the time stepping process is half explicit and half implicit. So the method is better than simple finite difference method.

Expressing $U_1(x,t)$ and $U_2(x,t)$ by using quintic $B$-spline functions $B_j(x)$ and the time dependent parameters $c_j(t)$ and $\delta_j(t)$ for $U_1(x,t)$ and $U_2(x,t)$ respectively, the approximate solution can be written as:

$$U_1^N(x,t) = \sum_{j=-2}^{N+2} c_j(t)B_j(x), \quad U_2^N(x,t) = \sum_{j=-2}^{N+2} \delta_j(t)B_j(x). \tag{11}$$

Using approximate function (11) and quintic $B$-spline functions (8), the approximate values $U_1(x), U_2(x)$ and their derivatives up to second order are determined in terms of the time parameters $c_j(t)$ and $\delta_j(t)$, respectively, as

$$\begin{aligned}
(U_1)_j &= (U_1)(x_j) = c_{j-2} + 26c_{j-1} + 66c_j + 26c_{j+1} + c_{j+2},\\
(U_1')_j &= (U_1')(x_j) = \frac{5}{h}(c_{j+2} + 10c_{j+1} - 10c_{j-1} - c_{j-2}),\\
(U_1'')_j &= (U_1'')(x_j) = \frac{20}{h^2}(c_{j-2} + 2c_{j-1} - 6c_j + 2c_{j+1} + c_{j+2}),\\
(U_1''')_j &= (U_1''')(x_j) = \frac{60}{h^3}(c_{j-2} + 2c_{j-1} - 2c_{j+1} + 2c_{j+12}),\\
(U_2)_j &= (U_2)(x_j) = \delta_{j-2} + 26\delta_{j-1} + 66\delta_j + 26\delta_{j+1} + \delta_{j+2},\\
(U_2')_j &= (U_2')(x_j) = \frac{5}{h}(\delta_{j+2} + 10\delta_{j+1} - 10\delta_{j-1} - \delta_{j-2}),\\
(U_2'')_j &= (U_2'')(x_j) = \frac{20}{h^2}(\delta_{j-2} + 2\delta_{j-1} - 6\delta_j + 2\delta_{j+1} + \delta_{j+2}),\\
(U_2''')_j &= (U_2''')(x_j) = \frac{60}{h^3}(\delta_{j-2} + 2c_{j-1} - 2\delta_{j+1} + 2\delta_{j+12}).
\end{aligned} \tag{12}$$

On substituting the approximate solution for $(U_1), (U_2)$ and its derivatives from Eq. (12) at the knots in Eqs. (9) and (10) yields the following difference equation with the variables $c_j(t)$ and $\delta_j(t)$.

$$\begin{aligned}
&A_1c_{j-2}^{n+1} + A_2c_{j-1}^{n+1} + A_3c_j^{n+1} + A_4c_{j+1}^{n+1} + A_5c_{j+2}^{n+1} =\\
&A_5c_{j-2}^{n} + A_4c_{j-1}^{n} + A_3c_j^{n} + A_2c_{j+1}^{n} + A_1c_{j+2}^{n},
\end{aligned} \tag{13}$$

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$$\begin{aligned}&A_1\delta_{j-2}^n+A_2\delta_{j-1}^n+A_3\delta_j^n+A_4\delta_{j+1}^n+A_5\delta_{j+2}^n=\\&A_5\delta_{j-2}^n+A_4\delta_{j-1}^n+A_3\delta_j^n+A_2\delta_{j+1}^n+A_1\delta_{j+2}^n,\end{aligned} \tag{14}$$

where

$$\begin{aligned}A_1&=1-\frac{30\Delta t}{h^3}-\frac{15\alpha\Delta t}{5h}(z_1)_j^n,\\A_2&=26+\frac{60\Delta t}{h^3}-\frac{150\alpha\Delta t}{5h}(z_1)_j^n,\\A_3&=66,\\A_4&=26-\frac{60\Delta t}{h^3}-\frac{150\alpha\Delta t}{5h}(z_1)_j^n,\\A_5&=1+\frac{30\Delta t}{h^3}-\frac{15\alpha\Delta t}{5h}(z_1)_j^n,\\(z_1)_j^n&=\left(\frac{(z(U))_j^{n+1}+(z(U))_j^n}{2}\right).\end{aligned}$$

The system thus obtained on simplifying Eqs. (13) and (14) consists of $(2N+2)$ linear equations in the $(2N+10)$ unknowns $(c_{-2},c_{-1},c_0,\cdots c_N,\ c_{N+1},c_{N+2}),\ (\delta_{-2},\delta_{-1},\delta_0,\cdots\delta_N,\delta_{N+1},\delta_{N+2})^T$. To obtain a unique solution to the resulting system four additional constraints are required. These are obtained by imposing boundary conditions. Eliminating $c_{-2},c_{-1},c_0,\ c_N,c_{N+1},c_{N+2}$ and $\delta_{-2},\delta_{-1},\delta_0,\delta_N,\delta_{N+1},\delta_{N+2}$ the system get reduced to a matrix system of dimension $(2N+2)\times(2N+2)$ which is the penta-diagonal system that can be solved by any algorithm.

## 4 Initial values

To find the initial parameters $c_j^0$ and $\delta_j^0$, the initial conditions and the derivatives at the boundaries are used in the following way

$$\begin{aligned}(U_1')(x_0,0)&=\frac{5}{h}(c_2+10c_1-10c_{-1}-c_{-2})=0,\\(U_1'')(x_0,0)&=\frac{20}{h^2}(c_{-2}+2c_{-1}-6c_0+2c_{j+1}+c_{j+2})=0,\\(U_1)(x_j,0)&=c_{j-2}+26c_{j-1}+66c_{j+1}+2c_{j+2}=0,\\(U_1')(x_N,0)&=\frac{5}{h}(c_{N+2}+10c_{N+1}-10c_{N-1}-c_{N-2})=0,\\(U_1'')(x_N,0)&=\frac{20}{h^2}(c_{N-2}+2c_{N-1}-6c_N+2c_{N+1}+c_{N+2})=0,\\(U_2')(x_0,0)&=\frac{5}{h}(\delta_2+10\delta_1-10\delta_{-1}-\delta_{-2})=0,\\(U_2'')(x_0,0)&=\frac{20}{h^2}(\delta_{-2}+2\delta_{-1}-6\delta_0+2\delta_{j+1}+\delta_{j+2})=0,\\(U_2)(x_j,0)&=\delta_{j-2}+26\delta_{j-1}+66\delta_{j+1}+2\delta_{j+2}=0,\\(U_2')(x_N,0)&=\frac{5}{h}(\delta_{N+2}+10\delta_{N+1}-10\delta_{N-1}-\delta_{N-2})=0,\\(U_2'')(x_N,0)&=\frac{20}{h^2}(\delta_{N-2}+2\delta_{N-1}-6\delta_N+2\delta_{N+1}+\delta_{N+2})=0.\end{aligned}$$

Which forms a linear block pintadiagonal system for unknown initial conditions $c_j^0$ and $\delta_j^0$ , of order $(2N+2)$ after eliminating the functions values of $c$ and $\delta$. This system can be solved by any algorithm. Once the initial vectors of parameters have been calculated, the numerical solution of the Hirota equation $U_1$ and $U_2$ can be determined from the

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time evaluation of the vectors $c_j^n$ and $\delta_j^n$, by using the recurrence relations

$$(U_1)(x_j,t_n) = c_{j-2}^n + 26c_{j-1}^n + 66c_j^n + 26c_{j+1}^n + c_{j+2}^n,$$

$$(U_2)(x_j,t_n) = \delta_{j-2}^n + 26\delta_{j-1}^n + 66\delta_j^n + 26\delta_{j+1}^n + \delta_{j+2}^n.$$

## 5 Stability analysis of the method

The stability analysis of nonlinear partial differential equations is not easy task to undertake. Most researchers copy with the problem by linearizing the partial differential equation. Our stability analysis will be based on the Von-Neumann concept in which the growth factor of a typical Fourier mode defined as

$$\begin{aligned} c_j^n &= A\xi^n \exp(ij\phi), \\ \delta_j^n &= B\xi^n \exp(ij\phi), \\ g &= \frac{\xi^{n+1}}{\xi^n}, \end{aligned} \tag{15}$$

where $A$ and $B$ are the harmonics amplitude, $\phi = kh$, $k$ is the mode number, $i = \sqrt{-1}$ and $g$ is the amplification factor of the schemes. We will be applied the stability of the quintic schemes by assuming the nonlinear term as a constants $(z_1)_j^n = \lambda_1$. At $x = x_j$ system (13) can be written as

$$\begin{aligned} & a_1 c_{j-2}^{n+1} + a_2 c_{j-1}^{n+1} + a_3 c_j^{n+1} + a_4 c_{j+1}^{n+1} + a_5 c_{j+2}^{n+1} = \\ & a_5 c_{j-2}^n + a_4 c_{j-1}^n + a_3 c_j^n + a_2 c_{j+1}^n + a_1 c_{j+2}^n, \end{aligned} \tag{16}$$

where

$$\begin{aligned} a_1 &= 1 - \frac{30\Delta t}{h^3} - \frac{15\alpha\Delta t}{2h}\lambda_1, \\ a_2 &= 26 + \frac{60\Delta t}{h^3} - \frac{150\alpha\Delta t}{2h}\lambda_1, \\ a_3 &= 66, \\ a_4 &= 26 - \frac{60\Delta t}{h^3} - \frac{150\alpha\Delta t}{2h}\lambda_1, \\ a_5 &= 1 + \frac{30\Delta t}{h^3} + \frac{15\aleph\Delta t}{2h}\lambda_1, \end{aligned}$$

Substituting (15) into the difference (16), we get

$$\xi^{n+1}\begin{bmatrix} [2\cos 2\phi + 62\cos\phi + 66] + \\ \\ i\left(\left(\frac{60\Delta t}{h^3} + \frac{30\alpha\Delta t}{2h}\lambda_1\right)\sin 2\phi + \left(-\frac{120\Delta t}{h^3} + \frac{300\alpha\Delta t}{2h}\lambda_1\right)\sin\phi\right) \end{bmatrix} =$$

$$\xi^{n}\begin{bmatrix} [2\cos 2\phi + 62\cos\phi + 66] - \\ \\ i\left(\left(\frac{60\Delta t}{h^3} + \frac{30\alpha\Delta t}{2h}\lambda_1\right)\sin 2\phi + \left(-\frac{120\Delta t}{h^3} + \frac{300\alpha\Delta t}{2h}\lambda_1\right)\sin\phi\right) \end{bmatrix},$$

we get

$$g = \frac{X + iY}{X - iY}, \tag{17}$$

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where

$$X = [2\cos 2\phi + 62\cos\phi + 66],$$

$$Y = \left( \left( \frac{60\Delta t}{h^3} + \frac{30\alpha\Delta t}{2h}\lambda_1 \right) \sin 2\phi + \left( -\frac{120\Delta t}{h^3} + \frac{300\alpha\Delta t}{2h}\lambda_1 \right) \sin\phi \right).$$

From (17) we get , $|g| = 1$ hence the scheme is unconditionally stable. It means that there is no restriction on the grid size, i.e. on $h$ and $\Delta t$, but we should choose them in such a way that the accuracy of the scheme is not degraded.

Similar results can be obtained from the difference (14), due to symmetric $u_1$ and $u_2$.

## 6 Numerical Tests and Results of Hirota equation

In this section, we present numerical example to test validity of our scheme for solving Hirota equation. The norms $L_2$-norm and $L_\infty$-norm are used to compare the numerical solution with the analytical solution [22].

$$\begin{aligned} L_2 &= \|u^E - u^N\| = \sqrt{h\sum_{i=0}^{N}(u_j^E - u_j^N)^2}, \\ L_\infty &= \max_j |u_j^E - u_j^N|, \qquad j = 0,1,2,\cdots,N, \end{aligned} \tag{18}$$

where $u^E$ is the exact solution $u$ and $u^N$ is the approximation solution $U_N$. And the quantities $I_1$ and $I_2$ are shown to measure conservation for the schemes.

$$\left.\begin{aligned} I_1 &= \int_{-\infty}^{\infty} |u(x,t)|^2\, dx|^2 \cong h\sum_{j=0}^{N} |U_j^n|^2, \\ I_2 &= \int_{-\infty}^{\infty} \left(|u(x,t)|^4 - |u_x(x,t)|^2\right)\, dx \cong h\sum_{j=0}^{N}\left(\frac{\alpha}{2}(|U|^4)_j^n - (|U_x|^2)_j^n\right) \end{aligned}\right\}, \tag{19}$$

where

$$u(x,t) = u_1(x,t) + iu_2(x,t), \qquad (U)_j^n = (U_1)_j^n + i(U_2)_j^n.$$

Now we consider this test problem.

### 6.1 Test problem:

We assume that the solution of the Hirota equation is negligible outside the interval $[x_L, x_R]$ together with all its $x$ derivatives tend to zero at the boundaries. Therefore, in our numerical study we replace Eq. (1) by

$$u_t + 3\alpha|u|^2 u_x + \gamma u_{xxx} = 0, \qquad x_L < x < x_R, \tag{20}$$

where $u$ is a complex valued function of the spatial coordinate $x$ and the time $t$ and $\alpha, \gamma$ are positive real constants. Boundary conditions

$$\begin{aligned} &u(x_L,t) = u(x_R,t) = 0 \\ &u_x(x_L,t) = u_x(x_R,t) = 0, \qquad 0 \le t \le T. \end{aligned} \tag{21}$$

And initial conditions.

$$u(x,0) = f(x), \qquad x_L < x < x_R. \tag{22}$$

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For our numerical work, we decompose the complex function $u$ into their real and imaginary parts by writing

$$u(x,t) = u_1(x,t) + iu_2(x,t), \qquad i^2 = -1, \tag{23}$$

where $u_1(x,t) and u_2(x,t)$ are real functions. This will reduce Hirota equation to the coupled system

$$\begin{aligned} &(u_1)_t + 3\alpha(u_1^2+u_2^2)(u_1)_x + \gamma(u_1)_{xxx} = 0, \\ &(u_2)_t + 3\alpha(u_1^2+u_2^2)(u_2)_x + \gamma(u_2)_{xxx} = 0. \end{aligned} \tag{24}$$

Then the exact solutions of system (24) is

$$\begin{aligned} &u_1(x,t) = \beta \sec h[\kappa(x-s-vt)]\cos(\phi), \\ &u_2(x,t) = \beta \sec h[\kappa(x-s-vt)]\sin(\phi), \\ &\beta = \sqrt{\frac{2\gamma}{\alpha}}\kappa, \quad \varphi = a(x-bt-s), \\ &v = \gamma(\kappa^2-3a^2) \quad b = \gamma(3\kappa^3-a^2), \end{aligned} \tag{25}$$

$\beta$ is the amplitude of the wave, $\kappa$ is related to the width of the wave envelope and $v$ is the velocity. The parameter $a$ is the wave number of the phase and bis related to the frequency of the phase. As well the solution is at $x = s$ at $t = 0$.In order to derive a numerical method for solving the system given in (24). The region $R = [x_L < x < x_R] \times [t > 0]$ with its boundary consisting of the ordinates $x_0 = x_L, x_N = x_R$ and the axis $t = 0$ is covered with a rectangular mesh of points with coordinates

$$\begin{aligned} &x = x_j = x_L + jh, \quad j = 0,1,2,\cdots,N, \\ &t = t_n = nk, \quad n = 0,1,2,\cdots, \end{aligned}$$

where $h$ and $k$ are the space and time increments, respectively.

To investigate the performance of the proposed schemes we consider solving the following problem.

### *6.2 Single soliton*

In previous section, we have provided four finite difference schemes for the Hirota equation, and we can take the following as an initial condition.

$$\begin{aligned} &u(x,0) = \beta \sec h[\kappa_j(x-s_j)]\exp(i\varphi), \\ &\beta = \sqrt{\frac{2\gamma}{\alpha}}\kappa_j, \qquad \varphi = a_j(x-bt-s_j), \\ &v = \gamma(\kappa_j^2-3a_j^2) \qquad b = \gamma(3\kappa_j^3-a_j^2). \end{aligned} \tag{26}$$

The norms $L_2$ and $L_\infty$ are used to compare the numerical results with the analytical values and the quantities $I_1$ and $I_2$ are shown to measure conservation for the schemes.

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Now, we consider two different cases to study the motion of single soliton.

**Case 1.** In this case we study the motion of single soliton by our scheme. In this case, we choose $h = 0.05, \alpha = 2, \gamma = 1, a = 0.5, \kappa = 0.5, x_L = -30, x_R = 30$, with different values of $k = 0.05, 0.1$ the simulations are done up to $t = 5$. The invariants $I_1$ and $I_2$ approach to zero percent, respectively at $k = 0.05, 0.1$ . Errors, also, are satisfactorily small $L_2$-error=$2.91598 \times 10^{-5}$ and $L_\infty$-error=$1.55616 \times 10^{-5}$, percent, respectively at $k = 0.05$. The invariants $I_1$ and $I_2$ approach to zero percent, respectively. Errors, also, are satisfactorily small $L_2$-error= $1.16219 \times 10^{-4}$ and $L_\infty$-error=$6.22412 \times 10^{-5}$, percent, respectively at $k = 0.1$. Our results are recorded in Table 2 and the motion of solitary wave is plotted at different time levels in Fig 1.

| $h,k$ | $T$ | $I_1$ | $I_2$ | $L_2$-norm | $L_\infty$-norm |
|---|---|---|---|---|---|
| $h = 0.05$ | 0.0 | 1.0000 | -0.166667 | 0.0 | 0.0 |
| $k = 0.05$ | 1.0 | 1.0000 | -0.166666 | 5.14542E-6 | 3.86206E-6 |
| | 2.0 | 1.0000 | -0.166664 | 1.12435E-5 | 6.96387E-6 |
| | 3.0 | 1.0000 | -0.166663 | 1.76432E-5 | 1.09366E-5 |
| | 4.0 | 1.0000 | -0.166661 | 2.37519E-5 | 1.35187E-5 |
| | 5.0 | 1.0000 | -0.166661 | 2.91598E-5 | .55616E-5 |
| $h = 0.05$ | 0.0 | 1.0000 | -0.166667 | 0.0 | 0.0 |
| $k = 0.1$ | 1.0 | 1.0000 | -0.166664 | 2.04474E-5 | 1.55067E-5 |
| | 2.0 | 1.0000 | -0.166657 | 4.48723E-5 | 3.08141E-5 |
| | 3.0 | 1.0000 | -0.166651 | 7.04433E-5 | 4.35989E-5 |
| | 4.0 | 1.0000 | -0.166644 | 9.48952E-5 | 5.43264E-5 |
| | 5.0 | 1.0000 | -0.166639 | 1.16219E-4 | 6.22412E-5 |

**Table 1:** Invariants and errors for single solitary wave , $h = 0.05, \alpha = 2, \gamma = 1 a = 0.5, \kappa = 0.5, x_L = -30, x_R = 30$.

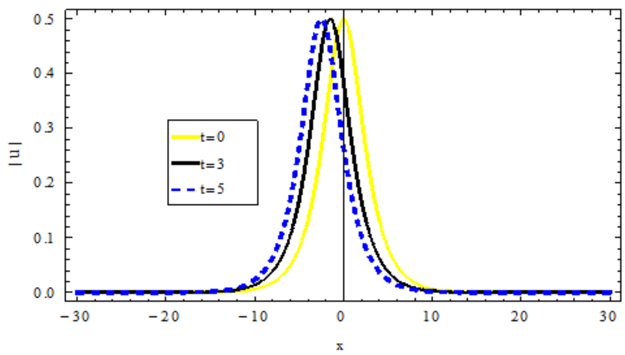


**Fig. 1:** Single solitary wave with $h = 0.05, k = 0.05, \alpha = 2, \gamma = 1$ and $a = 0.5, \kappa = 0.5, x_L = -30, x_R = 30, t = 0, 3, 5$ respectively.

**Case 2.** In this case we study the motion of single soliton by our scheme. In this case, we choose $h = 0.1, \alpha = 2, \gamma = 1, a = 0.5, \kappa = 0.5, x_L = -30, x_R = 30$, with different values of $k = 0.05, 0.1$ the simulations are done up to $t = 5$. The invariants $I_1$ and $I_2$ approach to zero percent, respectively at $k = 0.05, 0.1$ . Errors, also, are satisfactorily small $L_2$-error=$2.76053 \times 10^{-5}$ and $L_\infty$-error=$1.47542 \times 10^{-5}$, percent, respectively at $k = 0.05$, $t = 5$. The invariants $I_1$ and $I_2$ approach to zero percent, respectively. Errors, also, are satisfactorily small $L_2$-error= $1.14654 \times 10^{-4}$ and $L_\infty$-error=$6.14564 \times 10^{-5}$, percent, respectively at $k = 0.1$. Our results are recorded in Table 3 and the motion of solitary wave is plotted at different time levels in Fig 2.

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| $h,k$ | $T$ | $I_1$ | $I_2$ | $L_2$-norm | $L_\infty$-norm |
|---|---|---|---|---|---|
| $h=0.1$ | 0.0 | 1.0000 | -0.166667 | 0.0 | 0.0 |
| $k=0.05$ | 1.0 | 1.0000 | -0.166666 | 4.79837E-6 | 3.60953E-6 |
| | 2.0 | 1.0000 | -0.166664 | 1.05554 E-5 | 7.24351E-6 |
| | 3.0 | 1.0000 | -0.166663 | 1.66413E-5 | 1.03781E-5 |
| | 4.0 | 1.0000 | -0.166661 | 2.24578E-5 | 1.26539E-5 |
| | 5.0 | 1.0000 | 1.47542E-5 | 2.76053E-5 | .55616E-5 |
| $h=0.1$ | 0.0 | 1.0000 | -0.166667 | 0.0 | 0.0 |
| $k=0.1$ | 1.0 | 1.0000 | -0.166664 | 2.00874E-5 | 1.52293E-5 |
| | 2.0 | 1.0000 | -0.166657 | 4.41756 E-5 | 3.03681E-5 |
| | 3.0 | 1.0000 | -0.166651 | 5.94371E-5 | 4.32478E-5 |
| | 4.0 | 1.0000 | -0.166644 | 9.35896E-5 | 5.24549E-5 |
| | 5.0 | 1.0000 | -0.166641 | 1.14654E-4 | 6.14564E-5 |

**Table 2:** Invariants and errors for single solitary wave by the scheme, $\alpha = 2, \gamma = 1a = 0.5, \kappa = 0.5, x_L = -30, x_R = 30$.

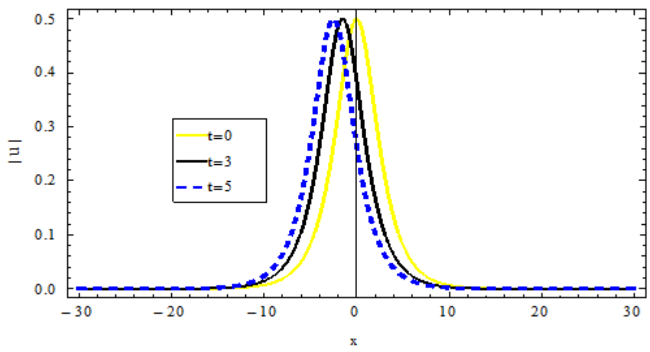


**Fig. 2:** Single solitary wave by our scheme with $h = 0.1, k = 0.05, \alpha = 2, \gamma = 1$ and $a = 0.5, \kappa = 0.5, x_L = -30, x_R = 30, t = 0, 3, 5$ respectively.

In the next table we make comparison between the results of our scheme and the results have been published in Search [4].

| Method | $I_1$ | $I_2$ | $L_2$-norm | $L_\infty$-norm |
|---|---|---|---|---|
| Analytical $h=0.1$ | 1.0000 | -0.166676 | 0.0 | 0.0 |
| Our scheme $h=0.1$ | 1.0000 | -0.166661 | 2.24578E-5 | 0.00001 |
| $[4]^a$ $h=0.05$ | 1.0000 | -0.166648 | - | 0.00001 |
| $[4]^b$ $h=0.05$ | 1.0000 | -0.166648 | - | 0.00001 |
| $[4]^c$ $h=0.05$ | 1.0000 | -0.166648 | | 0.00014 |

**Table 3:** Invariants and errors for single solitary wave , $k = 0.05, \alpha = 2, \gamma = 1a = 0.5, \kappa = 0.5, x_L = -30, x_R = 30$.

The results of our scheme are related with the results in [4].

## *6.3 Interaction of two solitary waves*

The interaction of two solitary waves having different amplitudes and traveling in the same direction is illustrated. We consider Hirota equation with initial conditions given by the linear sum of two well separated solitary waves of various

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amplitudes

$$u(x,0) = \beta \operatorname{sec}h[\kappa_j(x-s_j)]\exp(i\varphi),$$

$$\beta = \sqrt{\frac{2\gamma}{\alpha}}\kappa_j, \quad \varphi = a_j(x-bt-s_j), \tag{27}$$

$$v = \gamma(\kappa_j^2 - 3a_j^2) \quad b = \gamma(3\kappa_j^3 - a_j^2),$$

where, $j = 1,2, a_j$ and $s_j, \kappa_j$ are arbitrary constants. In our computational work.

Now, we choose $s_1 = -5, s_2 = 15, a_1 = 0.3, a_2 = 0.8, \kappa_1 = 0.2, \kappa_2 = 0.7, \gamma = 1, \alpha = 2, h = 0.05, k = 0.05$ with interval [-30, 30]. In Fig. 3, the interactions of these solitary waves are plotted at different time levels. We also, observe an appearance of a tail of small amplitude after interaction and the three invariants for this case are shown in Table 5. The invariants $I_1$ and $I_2$ are changed by less than $8.5\times10^{-4}$ and $2.23\times10^{-3}$, respectively for the scheme.

| $T$ | $I_1$ | $I_2$ |
|---|---|---|
| 0 | 1.82793 | 0.488457 |
| 3 | 1.82793 | 0.488535 |
| 5 | 1.82794 | 0.488411 |
| 8 | 1.82806 | 0.487765 |
| 22 | 1.82849 | 0.486227 |
| 25 | 1.82878 | 0.486362 |

**Table 4:** Invariants of interaction two solitary waves of Hirota equation $s_1 = -10, s_2 = 10, a_1 = 0.3, a_2 = 0.8, \kappa_1 = 0.2, \kappa_2 = 0.7, \gamma = 1, \alpha = 2, -30 < x < 30, h = 0.05, k = 0.05$.

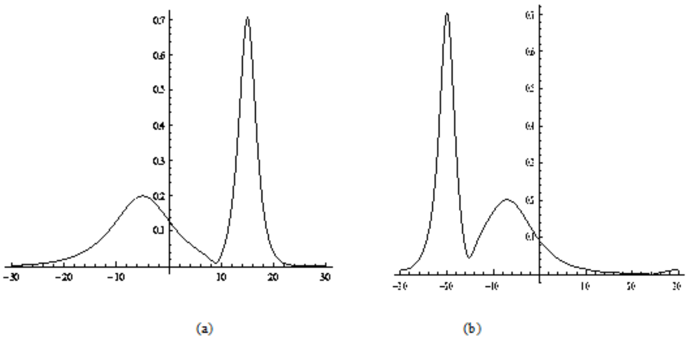

**Fig. 3:** interaction two solitary waves with $s_1 = -10, s_2 = 10, a_1 = 0.3, a_2 = 0.8, \kappa_1 = 0.2, \kappa_2 = 0.7, \gamma = 1, \alpha = 2, -30 < x < 30, h = 0.05, k = 0.05$ at time $t = 0, 15$ respectively.

## *6.4 Interaction of three solitary waves*

The interaction of three solitary waves having different amplitudes and traveling in the same direction is illustrated. We consider the Hirota equation with initial conditions given by the linear sum of three well separated solitary waves of

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various amplitudes:

$$u(x,0) = \beta \sec h[\kappa_j(x - s_j)] \exp(i\varphi),$$

$$\beta = \sqrt{\frac{2\gamma}{\alpha}}\kappa_j, \quad \varphi = a_j(x - bt - s_j), \tag{28}$$

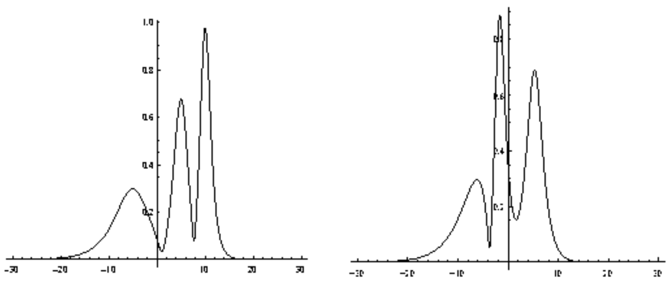
$$v = \gamma(\kappa_j^2 - 3a_j^2) \quad b = \gamma(3\kappa_j^3 - a_j^2),$$

where, $j = 1,2,3, a_j$ and $s_j, \kappa_j$ are arbitrary constants. In our computational work.

Now, we choose $s_1 = -5, s_2 = 10, a_1 = 0.2, a_2 = 0.4, a_3 = 0.7, \kappa_1 = 0.3, \kappa_2 = 0.7, \kappa_3 = 1, \gamma = 1, \alpha = 2, h = 0.05, k = 0.05$ with interval [-30, 30]. In Fig. 4, the interactions of these solitary waves are plotted at different time levels. We also, observe an appearance of a tail of small amplitude after interaction and the three invariants for this case are shown in Table 5. The invariants $I_1$ and $I_2$ are changed by less than $7.6 \times 10^{-4}$ and $8.5 \times 10^{-4}$, respectively for the scheme.

| $T$ | $I_1$ | $I_2$ |
|---|---|---|
| 0 | 3.51709 | 1.57696 |
| 2 | 3.51785 | 1.57611 |
| 4 | 3.51771 | 1.57667 |
| 6 | 3.51777 | 1.57633 |
| 12 | 3.51716 | 1.57613 |
| 10 | 3.51784 | 1.57623 |

**Table 5:** Invariants of interaction two solitary waves of Hirota equation $s_1 = -5, s_2 = 10, a_1 = 0.2, a_2 = 0.4, a_3 = 0.7 \kappa_1 = 0.3, \kappa_2 = 0.7, \kappa_3 = 1, \gamma = 1, \alpha = 2, h = 0.05, k = 0.05, -30 < x < 30$.



(a) (b)

**Fig. 4:** interaction three solitary waves with $s_1 = -5, s_2 = 5, s_3 = 10, a_1 = 0.2, a_2 = 0.4, a_3 = 0.7, \kappa_1 = 0.3, \kappa_2 = 0.7, \kappa_3 = 1, \gamma = 1, \alpha = 2, -30 < x < 30, h = 0.05, k = 0.05$ at time $t = 0, 14$ respectively.

## 7 Conclusions

In this paper a numerical treatment for the nonlinear Hirota equation is proposed using a collection method with the quintic B-splines. We show that the schemes are unconditionally stable. We tested our schemes through a single solitary wave in which the analytic solution is known, then extend it to study the interaction of solitons where no analytic solution is known during the interaction and its accuracy was shown by calculating error norms $L_2$ and $L_\infty$.

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