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Convergence of double singular integrals in weighted L_p spaces

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Abstract: The paper is devoted to the study of pointwise approximation of functions $f\in L_{p,\varphi}(D)$ by double singular integral operators with radial kernels at $p-$generalized Lebesgue points. Here, $\varphi>0$ is a weight function satisfying some sharp conditions and $L_{p,\varphi}(D)$ is the collection of all measurable and non-integrable functions for which $\left|\frac{f}{\varphi}\right|^p$ is integrable on D, where $D=\langle a,b;c,d\rangle$ is an arbitrary bounded open, semi open or closed region or $D=\mathbb{R}^2$.

Keywords: $p-$generalized Lebesgue point, double singular integral, radial kernel, weighted pointwise approximation.

1 Introduction

In Taberski's famous paper [18], the pointwise approximation of $2\pi-$ periodic Lebesgue integrable functions was investigated by the convolution type, linear singular integral operators of the form:

$$L_\lambda(f;x)=\int_{-\pi}^{\pi} f(t)K_\lambda(t-x)\,dt,\ x\in\langle-\pi,\pi\rangle,\ \lambda\in\Lambda\subset\mathbb{R}_0^+, \tag{1}$$

where $K_\lambda(t)$ is the kernel fulfilling appropriate conditions.

The papers [6] and [16], which are based on Taberski's study [18], are devoted to the study of pointwise convergence of the operators of type (1) on some planar sets consist of characteristic points (x_0,y_0) of various types. Besides, Bardaro [2] presented significant results about the rate of pointwise convergence of some classes of the linear singular integral operators. Distinctively, Esen [4,5] obtained some approximation results concerning the pointwise convergence and the rate of pointwise convergence of non-convolution type linear singular integral operators at $p-$Lebesgue points. Moreover, Karsli and Ibikli [8] extended the results in the articles [18], [6] and [16] by considering the more general integral operators of the form:

$$T_\lambda(f;x)=\int_a^b f(t)K_\lambda(t-x)\,dt,\ x\in\langle a,b\rangle,\ \lambda\in\Lambda\subset\mathbb{R}_0^+, \tag{2}$$

where $f\in L_1(a,b)$ and $\langle a,b\rangle$ denotes an arbitrary interval in $\mathbb{R}$ such as $[a,b]$, (a,b), $[a,b)$ or $(a,b]$. For some studies concerning approximation of functions by linear positive operators in several settings, the reader may see also e.g.

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[9]-[14].

In [19], Taberski studied the pointwise approximation of functions $f \in L_1(Q)$ by the three parameter family of convolution type double singular integral operators of the form:

$$V_\lambda(f;x,y) = \iint_Q f(t,s) K_\lambda(t-x,s-y)\, ds dt, \ \ (x,y) \in Q, \tag{3}$$

where $\lambda \in \Lambda \subset \mathbb{R}_0^+$ and Q denotes a given rectangle.

In [22], Yilmaz *et al.* investigated the pointwise convergence of double singular integral operators with radial kernels in the following setting:

$$L_\lambda(f;x,y) = \iint_D f(t,s) H_\lambda(t-x,s-y)\, ds dt, \ \ (x,y) \in D, \ \lambda \in \Lambda, \tag{4}$$

where $D = \langle a,b;c,d \rangle$ is an arbitrary bounded closed, semi-closed or open region or $D = \mathbb{R}^2$ and $f \in L_p(D)$ provided $1 \le p < \infty$. Besides, the region D was particularly chosen as $\langle -\pi,\pi;-\pi,\pi \rangle$ in this article.

It was natural to consider pointwise approximation in weighted Lebesgue spaces, as well as Lebesgue spaces. Therefore, Alexits [1], Mamedov [11] and Esen [4] presented necessary conditions satisfied by kernel functions in order to obtain a desired convergence, separately. Also, Taberski [20] studied the weighted pointwise convergence of double singular integral operators of type (3) using two dimensional counterparts of the conditions obtained by Alexits [1]. Later on, some weighted pointwise approximation results for the operator of type (4) were obtained in [21] using two dimensional counterpart of the approximation method presented by Esen [4].

This paper may be seen as a continuation and further generalization of [21]. In this paper, our main concern is to prove that the operators of type (4) converge to the function $f \in L_{p,\varphi}(D)$ at $p-$generalized Lebesgue point of it as (x,y,λ) tends to (x_0,y_0,λ_0). Here, $\varphi > 0$ is a weight function satisfying some sharp conditions and $L_{p,\varphi}(D)$ is the collection of all measurable and non-integrable functions for which $\left|\frac{f}{\varphi}\right|^p$ is integrable on D $(1 \le p < \infty)$, where $D = \langle a,b;c,d \rangle$ is open, semi open or closed bounded region or $D = \mathbb{R}^2$.

The paper is organized as follows: In Section 2, we give some preliminary concepts. In Section 3, the existence of the operators of type (4) is explored. In Section 4, main results are presented. In Section 5, the rate of pointwise convergence of the operators of type (4) is established.

2 Preliminaries

Definition 1. *A function $H \in L_1(\mathbb{R}^2)$ is said to be radial, if there exists a function $K : \mathbb{R}_0^+ \to \mathbb{R}$ such that $H(t,s) = K\left(\sqrt{t^2+s^2}\right)$ a.e. [3].*

Definition 2. *(Class A_φ) Let $H_\lambda : \mathbb{R}^2 \times \Lambda \to \mathbb{R}$ be a radial function i.e., there exists a function $K_\lambda : \mathbb{R}_0^+ \times \Lambda \to \mathbb{R}$ such that the following equality holds for $(t,s) \in \mathbb{R}^2$ a.e.: $H_\lambda(t,s) := K_\lambda\left(\sqrt{t^2+s^2}\right)$, where Λ is a given set of non-negative numbers with accumulation point λ_0. In addition, let $\phi(t,s) = \sup_{(x,y)\in D} \left|\frac{\varphi(t+x,s+y)}{\varphi(x,y)}\right|$ for every $(t,s) \in D$ and $\varphi : \mathbb{R}^2 \to \mathbb{R}^+$. $H_\lambda(t,s)$ belongs to class A_φ, if the following conditions are satisfied:*

(a) *$H_\lambda(t,s) = K_\lambda\left(\sqrt{t^2+s^2}\right)$ is non-negative and integrable as a function of (t,s) on $\mathbb{R}^2$ for each fixed $\lambda \in \Lambda$.*

(b) *For fixed $(x_0,y_0) \in D$, $K_\lambda\left(\sqrt{x_0^2+y_0^2}\right)$ tends to infinity as λ tends to λ_0.*

(c) $\lim_{\lambda\to\lambda_0}\left|\iint_{\mathbb{R}^2} K_\lambda\left(\sqrt{t^2+s^2}\right)dsdt-1\right|=0.$

(d) $\lim_{\lambda\to\lambda_0}\left[\sup_{\xi\le\sqrt{t^2+s^2}} K_\lambda\left(\sqrt{t^2+s^2}\right)\right]=0,\ \forall\xi>0.$

(e) $\lim_{\lambda\to\lambda_0}\left[\iint_{\xi\le\sqrt{t^2+s^2}} K_\lambda\left(\sqrt{t^2+s^2}\right)dsdt\right]=0,\ \forall\xi>0.$

(f) $\left\|\phi\left(.,.\right)K_\lambda\sqrt{(.)^2+(.)^2}\right\|_{L_1(\mathbb{R}^2)}\le M<\infty,\ \forall\lambda\in\Lambda.$

(g) $K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)$ *is non-increasing as a function of* t *on* $\langle x_0,b\rangle$ *and non-decreasing on* $\langle a,x_0\rangle$. *Similarly,* $K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)$ *is non-increasing as a function of* s *on* $\langle y_0,d\rangle$ *and non-decreasing on* $\langle c,y_0\rangle$, *for each fixed* $\lambda\in\Lambda$ *and for fixed* $(x_0,y_0)\in D$. *As a function of* (t,s), $K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)$ *is bimonotonically increasing on* $\langle x_0,b;y_0,d\rangle$ *and* $\langle a,x_0;c,y_0\rangle$. *Similarly,* $K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)$ *is bimonotonically decreasing on* $\langle a,x_0;y_0,d\rangle$ *and* $\langle x_0,b;c,y_0\rangle$.

Throughout this paper we suppose that the kernel function $H_\lambda(t,s)$ belongs to class A_φ.

Remark. For more information about the concept of bimonotonicity, the reader may see also e.g. [7].

3 Existence of the operators

Main results in this work are based on the following theorem.

Theorem 1. *Let* $1\le p<\infty$. *If* $f\in L_{p,\varphi}(D)$, *then* $L_\lambda(f;x,y)$ *defines a continuous transformation acting on* $L_{p,\varphi}(D)$.

Proof. Let $D=\langle a,b;c,d\rangle$ is an arbitrary bounded closed, semi-closed or open region and $1<p<\infty$. By the linearity of the operator $L_\lambda(f;x,y)$, it is sufficient to show that the following expression:

$$\|L_\lambda\|_\varphi=\sup_{f\neq 0}\frac{\|L_\lambda(f;x,y)\|_{L_{p,\varphi}(D)}}{\|f\|_{L_{p,\varphi}(D)}}$$

remains uniformly bounded.

We define a function such that

$$g(t,s)=\begin{cases} f(t,s), & (t,s)\in D,\\ 0, & (t,s)\in\mathbb{R}^2\backslash D.\end{cases}\tag{5}$$

The expression $\left(\iint_D\left|\frac{f(x,y)}{\varphi(x,y)}\right|^p dydx\right)^{\frac{1}{p}}$ defines the norm in the space $L_{p,\varphi}(D)$; see, for example, [20]. The following equality is obtained for the norm of the operator of type (4) i.e.:

$$\|L_\lambda(f;x,y)\|_{L_{p,\varphi}(D)}=\|L_\lambda(g;x,y)\|_{L_{p,\varphi}(D)}$$
$$=\left(\iint_D\frac{1}{[\varphi(x,y)]^p}\left|\iint_{\mathbb{R}^2} g(t,s)K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)dsdt\right|^p dydx\right)^{\frac{1}{p}}.$$

By using the generalized Minkowski inequality (see, e.g., [17]) and condition (f) of class A_φ, we obtain

$$\|L_\lambda(f;x,y)\|_{L_{p,\varphi}(D)}\le M\|f\|_{L_{p,\varphi}(D)}.$$

Note that the proof of the case $p=1$ is quite similar to the above one. In addition, one may prove the assertion for the case $D=\mathbb{R}^2$ analogous to the above proof. Thus the proof is completed.

4 Pointwise convergence

Theorem 2. *Suppose that $D=\langle a,b;c,d\rangle$ is an arbitrary bounded closed, semi-closed or open region. If $(x_0,y_0)\in D$ is a common $p-$generalized Lebesgue point of the functions $f\in L_{p,\varphi}(D)$ and $\varphi\in L_p(D)$, then*

$$\lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)} L_\lambda(f;x,y)=f(x_0,y_0)$$

under the conditions

$$\frac{\partial K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)}{\partial t}\times\frac{\partial\varphi(t,s)}{\partial t}>0,\ \text{for each fixed } (x,y)\in D, \tag{6}$$

$$\frac{\partial K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)}{\partial s}\times\frac{\partial\varphi(t,s)}{\partial s}>0,\ \text{for each fixed } (x,y)\in D, \tag{7}$$

and

$$\frac{\partial^2 K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)}{\partial t\partial s}\times\frac{\partial^2\varphi(t,s)}{\partial t\partial s}>0,\ \text{for each fixed } (x,y)\in D \tag{8}$$

providing that first and second order (mixed) partial derivatives of $K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)$ and $\varphi(t,s)$ w.r.t. (t,s) exist a.e. on $\mathbb{R}^2$, on any set Z on which the function

$$\int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta}\varphi(t,s)K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)\left|d\left[(x_0-t)^{(\alpha+1)}(s-y_0)^{(\alpha+1)}\right]\right|,$$

where $d\left[(x_0-t)^{(\alpha+1)}(s-y_0)^{(\alpha+1)}\right]$ is the Lebesgue-Stieltjes measure with respect to $(x_0-t)^{(\alpha+1)}(s-y_0)^{(\alpha+1)}$, is bounded as (x,y,λ) tends to (x_0,y_0,λ_0).

Proof. Let $(x_0,y_0)\in D$ be a common $p-$generalized Lebesgue point of the functions $f\in L_{p,\varphi}(D)$ and $\varphi\in L_p(D)$. Let $|x-x_0|<\frac{\delta}{2}$ and $|y-y_0|<\frac{\delta}{2}$ for a given $\delta>0$. The proof will be given for the case $0<x_0-x<\frac{\delta}{2}$ and $0<y_0-y<\frac{\delta}{2}$ for all $\delta>0$ satisfying $x_0+\delta<b$, $x_0-\delta>a$, $y_0+\delta<d$ and $y_0-\delta>c$. For the remaining cases, the proof follows a similar line. Also, for the simplicity, the proof will be stated for the case $1<p<\infty$. The proof of the case $p=1$ may be given using similar strategy.

According to the definition of $p-$generalized Lebesgue point given in [22], for all given $\varepsilon>0$, there exists a $\delta>0$ such that for all h,k satisfying $0<h,k\leq\delta$, the inequality

$$\int_{x_0-\delta}^{x_0}\int_{y_0-\delta}^{y_0}\left|\frac{f(t,s)}{\varphi(t,s)}-\frac{f(x_0,y_0)}{\varphi(x_0,y_0)}\right|^p dsdt<\varepsilon^p(hk)^{(\alpha+1)} \tag{9}$$

holds.

Set $I(x,y,\lambda) := |L_\lambda(f;x,y) - f(x_0,y_0)|$. We may easily write

$$\begin{aligned} I(x,y,\lambda) &= \left| \iint_D f(t,s) H_\lambda(t-x,s-y)\,dsdt - f(x_0,y_0) \right| = \left| \iint_D f(t,s) K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right) dsdt - f(x_0,y_0) \right| \\ &\le \iint_D \left| \frac{f(t,s)}{\varphi(t,s)} - \frac{f(x_0,y_0)}{\varphi(x_0,y_0)} \right| \varphi(t,s) K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right) dsdt \\ &+ \left| \frac{f(x_0,y_0)}{\varphi(x_0,y_0)} \right| \left| \iint_D \varphi(t,s) K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right) dsdt - \varphi(x_0,y_0) \right|. \end{aligned}$$

Since whenever m, n being positive numbers the inequality $(m+n)^p \le 2^p(m^p+n^p)$ holds (see, e.g., [15]), we have

$$\begin{aligned} [I(x,y,\lambda)]^p &\le 2^p \left(\left\{ \iint_{D\setminus B_\delta} + \iint_{B_\delta} \right\} \left| \frac{f(t,s)}{\varphi(t,s)} - \frac{f(x_0,y_0)}{\varphi(x_0,y_0)} \right| \varphi(t,s) K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right) dsdt \right)^p \\ &+ 2^p \left| \frac{f(x_0,y_0)}{\varphi(x_0,y_0)} \right|^p \left| \iint_D \varphi(t,s) K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right) dsdt - \varphi(x_0,y_0) \right|^p \\ &= I_1 + I_2, \end{aligned}$$

where $B_\delta := \left\{ (t,s) : (t-x_0)^2 + (s-y_0)^2 \le \delta^2,\ (x_0,y_0) \in D \right\}$.

From Theorem 4.1 in [22], we see that $I_2 \to 0$ as (x,y,λ) tends to (x_0,y_0,λ_0). Now, applying Hölder's inequality (for Hölder's inequality, see [15]) to the integral I_1, and then using the inequality given as $(m+n)^p \le 2^p(m^p+n^p)$, we obtain

$$\begin{aligned} I_1 &\le c(x,y,\lambda)\, 2^{2p} \left\{ K_\lambda\left(\frac{\delta}{\sqrt{2}}\right) \sup_{D\setminus B_\delta} \varphi(t,s) \left\{ \|f\|^p_{L_{p,\varphi}(D)} + \left| \frac{f(x_0,y_0)}{\varphi(x_0,y_0)} \right|^p |b-a|\,|c-d| \right\} \right\} \\ &+ c(x,y,\lambda)\, 2^p \iint_{B_\delta} \left| \frac{f(t,s)}{\varphi(t,s)} - \frac{f(x_0,y_0)}{\varphi(x_0,y_0)} \right|^p \varphi(t,s) K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right) dsdt \\ &= c(x,y,\lambda) \left\{ 2^{2p} I_{11} + 2^p I_{21} \right\}, \end{aligned}$$

where $c(x,y,\lambda) := \left(\iint_D \varphi(t,s) K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right) dsdt \right)^{\frac{p}{q}}$.

From Theorem 4.1 in [22], $c(x,y,\lambda) \to [\varphi(x_0,y_0)]^{\frac{p}{q}}$ as (x,y,λ) tends to (x_0,y_0,λ_0), and by condition (e) of class A_φ, $I_{11} \to 0$ as (x,y,λ) tends to (x_0,y_0,λ_0).

The following inequality holds for the integral I_{21}:

$$\begin{aligned} I_{21} &\le \left\{ \int_{x_0-\delta}^{x_0} \int_{y_0-\delta}^{y_0} + \int_{x_0}^{x_0+\delta} \int_{y_0-\delta}^{y_0} \right\} \left| \frac{f(t,s)}{\varphi(t,s)} - \frac{f(x_0,y_0)}{\varphi(x_0,y_0)} \right|^p \varphi(t,s) K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right) dsdt \\ &+ \left\{ \int_{x_0-\delta}^{x_0} \int_{y_0}^{y_0+\delta} + \int_{x_0}^{x_0+\delta} \int_{y_0}^{y_0+\delta} \right\} \left| \frac{f(t,s)}{\varphi(t,s)} - \frac{f(x_0,y_0)}{\varphi(x_0,y_0)} \right|^p \varphi(t,s) K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right) dsdt \\ &= I_{211} + I_{212} + I_{213} + I_{214}. \end{aligned}$$

Since

$$I_{21} \leq I_{211} + I_{212} + I_{213} + I_{214},$$

it is sufficient to show that the terms on the right hand side of the last inequality tends to zero as $(x,y,\lambda) \to (x_0,y_0,\lambda_0)$ on Z. Let us consider the integral I_{211}.

For this aim, we define the new function as follows:

$$F(t,s) := \int_t^{x_0} \int_s^{y_0} \left| \frac{f(u,v)}{\varphi(u,v)} - \frac{f(x_0,y_0)}{\varphi(x_0,y_0)} \right|^p dv du.$$

From (9), for all t and s satisfying $0 < x_0 - t \leq \delta$ and $0 < y_0 - s \leq \delta$ we have

$$|F(t,s)| \leq \varepsilon^p (x_0 - t)^{(\alpha+1)} (y_0 - s)^{(\alpha+1)}. \tag{10}$$

Now, we can concentrate the integral I_{211}. From Theorem 2.5 in [19], we have the following equality:

$$\begin{aligned} I_{211} &= \int_{x_0-\delta}^{x_0} \int_{y_0-\delta}^{y_0} \left| \frac{f(t,s)}{\varphi(t,s)} - \frac{f(x_0,y_0)}{\varphi(x_0,y_0)} \right|^p \varphi(t,s) K_\lambda \left(\sqrt{(t-x)^2 + (s-y)^2} \right) dsdt \\ &= (LS) \int_{x_0-\delta}^{x_0} \int_{y_0-\delta}^{y_0} \varphi(t,s) K_\lambda \left(\sqrt{(t-x)^2 + (s-y)^2} \right) dF(t,s), \end{aligned}$$

where (LS) denotes Lebesgue-Stieltjes integral.

Two-dimensional integration by parts (see Theorem 2.2, p.100 in [19]) gives us

$$\begin{aligned} &\int_{x_0-\delta}^{x_0} \int_{y_0-\delta}^{y_0} \varphi(t,s) K_\lambda \left(\sqrt{(t-x)^2 + (s-y)^2} \right) dF(t,s) \\ &= \int_{x_0-\delta}^{x_0} \int_{y_0-\delta}^{y_0} F(t,s)\, d \left[\varphi(t,s) K_\lambda \left(\sqrt{(t-x)^2 + (s-y)^2} \right) \right] \\ &+ \int_{x_0-\delta}^{x_0} F(t,y_0-\delta)\, d_t \left[\varphi(t,y_0-\delta) K_\lambda \left(\sqrt{(t-x)^2 + (y_0-y-\delta)^2} \right) \right] \\ &+ \int_{y_0-\delta}^{y_0} F(x_0-\delta,s)\, d_s \left[\varphi(x_0-\delta,s) K_\lambda \left(\sqrt{(x_0-x-\delta)^2 + (s-y)^2} \right) \right] \\ &+ F(x_0-\delta,y_0-\delta)\, \varphi(x_0-\delta,y_0-\delta) K_\lambda \left(\sqrt{(x_0-x-\delta)^2 + (y_0-y-\delta)^2} \right). \end{aligned}$$

From (10), we can write

$$|I_{211}| \le \varepsilon^p \int_{x_0-\delta}^{x_0} \int_{y_0-\delta}^{y_0} (x_0-t)^{(\alpha+1)} (y_0-s)^{(\alpha+1)} \left| d\left[\varphi(t,s) K_\lambda\left(\sqrt{(s-x)^2+(t-y)^2}\right)\right]\right|$$
$$+\varepsilon^p \delta^{(\alpha+1)} \int_{x_0-\delta}^{x_0} (x_0-t)^{(\alpha+1)} \left| d_t\left[\varphi(t,y_0-\delta) K_\lambda\left(\sqrt{(t-x)^2+(y_0-y-\delta)^2}\right)\right]\right|$$
$$+\varepsilon^p \delta^{(\alpha+1)} \int_{y_0-\delta}^{y_0} (y_0-s)^{(\alpha+1)} \left| d_s\left[\varphi(x_0-\delta,s) K_\lambda\left(\sqrt{(x_0-x-\delta)^2+(s-y)^2}\right)\right]\right|$$
$$+\varepsilon^p \delta^{2(\alpha+1)} \varphi(x_0-\delta,y_0-\delta) K_\lambda\left(\sqrt{(x_0-\delta-x)^2+(y_0-\delta-y)^2}\right).$$

From $(6)-(8)$, we see that the monotonicity properties of

$$K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right) \text{ and } \varphi(t,s) K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)$$

coincide for each fixed $(x,y) \in D$. Applying two-dimensional integration by parts to the last inequality, we get the following inequality:

$$|I_{211}| \le \varepsilon^p \int_{x_0-\delta}^{x_0} \int_{y_0-\delta}^{y_0} \varphi(t,s) K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right) d\left[(x_0-t)^{(\alpha+1)} (y_0-s)^{(\alpha+1)}\right].$$

For the integrals I_{212}, $I_{213,}$ and I_{214} the proof is similar to the above one. Thus we obtain following inequalities:

$$|I_{212}| \le -\varepsilon^p \int_{x_0}^{x_0+\delta} \int_{y_0-\delta}^{y_0} \varphi(t,s) K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right) d\left[(t-x_0)^{(\alpha+1)} (y_0-s)^{(\alpha+1)}\right],$$

$$|I_{213}| \le -\varepsilon^p \int_{x_0-\delta}^{x_0} \int_{y_0}^{y_0+\delta} \varphi(t,s) K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right) d\left[(x_0-t)^{(\alpha+1)} (s-y_0)^{(\alpha+1)}\right],$$

$$|I_{214}| \le \varepsilon^p \int_{x_0}^{x_0+\delta} \int_{y_0}^{y_0+\delta} \varphi(t,s) K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right) d\left[(t-x_0)^{(\alpha+1)} (s-y_0)^{(\alpha+1)}\right].$$

Collecting the estimates I_{212}, $I_{213,}$ and I_{214}, we have

$$|I_{21}| \le \varepsilon^p \int_{x_0-\delta}^{x_0+\delta} \int_{y_0-\delta}^{y_0+\delta} \varphi(t,s) K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right) \left| d\left[(x_0-t)^{(\alpha+1)} (s-y_0)^{(\alpha+1)}\right]\right|.$$

Therefore, if the points (x,y,λ) are sufficiently close to (x_0,y_0,λ_0), we have

$$I_{21} \le \varepsilon^p C,$$

where

$$C=\sup\left\{\int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta}\varphi(t,s)K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)\left|d\left[(x_0-t)^{(\alpha+1)}(s-y_0)^{(\alpha+1)}\right]\right|:(x,y,\lambda)\in Z\right\}.$$

From Theorem 4.1 in [22], $c(x,y,\lambda)\to[\varphi(x_0,y_0)]^{\frac{p}{q}}$ as (x,y,λ) tends to (x_0,y_0,λ_0). This completes the proof.

Theorem 3. *Suppose that $D=\mathbb{R}^2$, and the hypotheses (6)-(8) of Theorem 2 are satisfied. If $(x_0,y_0)\in\mathbb{R}^2$ is a common $p-$generalized Lebesgue point of the functions $f\in L_{p,\varphi}(\mathbb{R}^2)$ and $\varphi\in L_p(\mathbb{R}^2)$,then*

$$\lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)}L_\lambda(f;x,y)=f(x_0,y_0)$$

on any set Z on which the function

$$\int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta}\varphi(t,s)K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)\left|d\left[(x_0-t)^{(\alpha+1)}(s-y_0)^{(\alpha+1)}\right]\right|,$$

where $d\left[(x_0-t)^{(\alpha+1)}(s-y_0)^{(\alpha+1)}\right]$ is the Lebesgue-Stieltjes measure with respect to $(x_0-t)^{(\alpha+1)}(s-y_0)^{(\alpha+1)}$, is bounded as (x,y,λ) tends to (x_0,y_0,λ_0).

Proof. The proof of this theorem is quite similar to the proof of Theorem 2, and it is omitted.

5 Rate of convergence

In this section, two theorems concerning the rate of pointwise convergence of the operators of type (4) will be given.

Theorem 4. *Suppose that the hypotheses of Theorem 2 are satisfied. Let*

$$\Delta(x,y,\lambda,\delta)=\int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta}|x_0-t|\,|s-y_0|\,\varphi(t,s)K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)dsdt,$$

where $0<\delta\le\delta_0$ for a fixed (and finite!) positive number δ_0, and the following conditions are satisfied:

(i) $\Delta(x,y,\lambda,\delta)\to 0$ *as* (x,y,λ) *tends to* (x_0,y_0,λ_0) *for some* $\delta>0$.
(ii) $\forall\xi>0$ *and* $\forall\alpha\in(0,1)$, *we have* $K_\lambda(\xi)=o\left((\Delta(x,y,\lambda,\delta))^\alpha\right)$ *as* (x,y,λ) *tends to* (x_0,y_0,λ_0).
(iii) $\forall\alpha\in(0,1)$, $\left|\iint_D\varphi(t,s)K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)dsdt-\varphi(x_0,y_0)\right|=o\left((\Delta(x,y,\lambda,\delta))^\alpha\right)$ *as* (x,y,λ) *tends to* (x_0,y_0,λ_0).

Then, at each common $p-$generalized Lebesgue point of $f\in L_{p,\varphi}(D)$ and $\varphi\in L_p(D)$ we have

$$|L_\lambda(f;x,y)-f(x_0,y_0)|=o\left((\Delta(x,y,\lambda,\delta))^{\frac{\alpha}{p}}\right)$$

as (x,y,λ) tends to (x_0,y_0,λ_0).

Proof. By the hypotheses of Theorem 2, the following inequality holds:

$$\begin{aligned}
|L_\lambda(f;x,y)-f(x_0,y_0)|^p &\le 2^{2p}K_\lambda\left(\frac{\delta}{\sqrt{2}}\right)\sup_{D\setminus B_\delta}\varphi(t,s)\,\|f\|^p_{L_{p,\varphi}(D)}\left(\iint_D\varphi(t,s)K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)dsdt\right)^{\frac{p}{q}}\\
&+2^{2p}K_\lambda\left(\frac{\delta}{\sqrt{2}}\right)\left|\frac{f(x_0,y_0)}{\varphi(x_0,y_0)}\right|^p|b-a||c-d|\left(\iint_D\varphi(t,s)K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)dsdt\right)^{\frac{p}{q}}\\
&+2^p(\alpha+1)^2\varepsilon^p\int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta}|x_0-t|^\alpha|s-y_0|^\alpha\varphi(t,s)K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)dsdt\\
&\times\left(\iint_D\varphi(t,s)K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)dsdt\right)^{\frac{p}{q}}\\
&+2^p\left|\frac{f(x_0,y_0)}{\varphi(x_0,y_0)}\right|^p\left|\iint_D\varphi(t,s)K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)dsdt-\varphi(x_0,y_0)\right|^p.
\end{aligned}$$

Set

$$A(x,y,\lambda,\delta):=\int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta}|x_0-t|^\alpha|s-y_0|^\alpha\varphi(t,s)K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)dsdt.$$

Now, we can apply the proof method used by Mamedov [12] to the remaining part. Since $0<\alpha<1$, we have $\frac{1}{\alpha}>1$, and the conjugate of $\frac{1}{\alpha}$ is $\frac{1}{(1-\alpha)}$. Applying Hölder's inequality to the term $A(x,y,\lambda,\delta)$, we have

$$\begin{aligned}
A(x,y,\lambda,\delta)&=\int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta}|x_0-t|^\alpha|s-y_0|^\alpha\left(\varphi(t,s)K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)\right)^\alpha\\
&\times\left(\varphi(t,s)K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)\right)^{(1-\alpha)}dsdt\\
&\le\left\{\int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta}\left[|x_0-t|^\alpha|s-y_0|^\alpha\left(\varphi(t,s)K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)\right)^\alpha\right]^{\frac{1}{\alpha}}dsdt\right\}^\alpha\\
&\times\left\{\int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta}\left[\left(\varphi(t,s)K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)\right)^{(1-\alpha)}\right]^{\frac{1}{(1-\alpha)}}dsdt\right\}^{(1-\alpha)}\\
&\le(\Delta(x,y,\lambda,\delta))^\alpha\times\left\{\iint_D\varphi(t,s)K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)dsdt\right\}^{(1-\alpha)}.
\end{aligned}$$

Since $\iint_D\varphi(t,s)K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)dsdt\to\varphi(x_0,y_0)$ as (x,y,λ) tends to (x_0,y_0,λ_0), the rest of the proof is clear by the conditions (i)-(iii). Thus the proof is completed.

Theorem 5. *Suppose that the hypotheses of Theorem 3 are satisfied. Let*

$$\Delta(x,y,\lambda,\delta)=\int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta}|x_0-t||s-y_0|\varphi(t,s)K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)dsdt,$$

where $0<\delta\leq\delta_0$ for a fixed (and finite!) positive number δ_0, and the following conditions are satisfied:

(i) $\Delta(x,y,\lambda,\delta)\to 0$ *as* (x,y,λ) *tends to* (x_0,y_0,λ_0) *for some* $\delta>0$.

(ii) $\forall\xi>0$ *and* $\forall\alpha\in(0,1)$, *we have* $K_\lambda(\xi)=o\left((\Delta(x,y,\lambda,\delta))^\alpha\right)$ *as* (x,y,λ) *tends to* (x_0,y_0,λ_0).

(iii) $\forall\xi>0$ *and* $\forall\alpha\in(0,1)$, *we have* $\iint_{\xi\leq\sqrt{t^2+s^2}}K_\lambda\left(\sqrt{t^2+s^2}\right)dsdt=o\left((\Delta(x,y,\lambda,\delta))^\alpha\right)$ *as* (x,y,λ) *tends to* (x_0,y_0,λ_0).

(iv) $\forall\alpha\in(0,1)$, $\left|\iint_{\mathbb{R}^2}\varphi(t,s)K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)dsdt-\varphi(x_0,y_0)\right|=o\left((\Delta(x,y,\lambda,\delta))^\alpha\right)$ *as* (x,y,λ) *tends to* (x_0,y_0,λ_0).

Then, at each common $p-$generalized Lebesgue point of $f\in L_{p,\varphi}(\mathbb{R}^2)$ and $\varphi\in L_p(\mathbb{R}^2)$ we have

$$|L_\lambda(f;x,y)-f(x_0,y_0)|=o\left((\Delta(x,y,\lambda,\delta))^{\frac{\alpha}{p}}\right)$$

as (x,y,λ) tends to (x_0,y_0,λ_0).

Proof. The proof of this theorem is analogous to the proof of Theorem 4, and it is omitted.

6 Conclusion

In this paper, the weighted pointwise convergence of the convolution type double singular integral operators depending on three parameters is investigated. Since the approximation results and the character of the kernel functions are related, a special class of kernel functions has been defined. Therefore, the main results are presented as Theorem 1 and Theorem 2. By using main results, the rate of pointwise convergence of the indicated type operators is computed.

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References

[1] G. Alexits, Konvergenzprobleme der Orthogonalreihen, Verlag der Ungarischen Akademie der Wissenschaften, Budapest, 1960.

[2] C. Bardaro, On approximation properties for some classes of linear operators of convolution type, Atti Sem. Mat. Fis. Univ. Modena 33 (1984), 329-356.

[3] S. Bochner and K. Chandrasekharan, Fourier Transforms, Annals of Mathematics Studies, no. 19, Princeton Univ. Press, Princeton, N. J. Oxford Univ. Press, London, 1949.

[4] S. Esen, Convergence and the order of convergence of family of nonconvolution type integral operators at characteristic points, Ph. D. Thesis, Ankara University, Graduate School of Applied Science, Ankara, 2002.

[5] S. Esen, Approximation of functions by the family of integral operators with positive kernels, Trans. Acad. Sci. Azerb. Ser. Phys. Tech. Math. Sci. 22 (2002), no. 1, Math. Mech., 56–61, 253.

[6] A. D. Gadjiev, The order of convergence of singular integrals which depend on two parameters, In: Special Problems of Functional Analysis and their Appl. to the Theory of Diff. Eq. and the Theory of Func., Izdat. Akad. Nauk Azerbaidzan. SSR, (1968), 40–44.

[7] S. R. Ghorpade and B. V. Limaye, A Course in Multivariable Calculus and Analysis, Springer, New York, 2010.
[8] H. Karsli and E. Ibikli, On convergence of convolution type singular integral operators depending on two parameters, Fasc. Math. 38 (2007), 25-39.
[9] R. G. Mamedov, A study of the orders of convergence of higher-dimensional singular integrals, (Russian) Akad. Nauk Azerbaĭdžan. SSR Dokl. 18 (1962), no. 11, 9–13.
[10] R. G. Mamedov, A generalization of some results on the order of convergence of singular integrals, (Russian) Akad. Nauk Azerbaĭdžan. SSR Dokl. 18 (1962), no. 3, 3–7.
[11] R. G. Mamedov, On the order of convergence of m-singular integrals at generalized Lebesgue points and in the space $Lp(-\infty,\infty)$, Izv. Akad. Nauk SSSR Ser. Mat. 27 (1963), no. 2, 287-304.
[12] R. G. Mamedov, A study of orders of convergence of one-dimensional and multidimensional singular integrals, (Russian), In: Studies in Theory of Differential Equations and Theory of Functions (Russian), Izdat. Akad. Nauk Azerbaĭdžan. SSR, (1965), 92-108.
[13] V. N. Mishra and L. N. Mishra, Trigonometric approximation of signals (functions) in Lp-norm, Int. J. Contemp. Math. Sci. 7 (2012), no. 17-20, 909–918.
[14] P. Patel and V. N. Mishra, Approximation properties of certain summation integral type operators, Demonstr. Math. 48 (2015), no. 1, 77–90.
[15] W. Rudin, Real and Complex Analysis. Mc-Graw Hill Book Co., London, 1987.
[16] B. Rydzewska, Approximation des fonctions par des intégrales singulières ordinaires, Fasc. Math. 7 (1973), 71-81.
[17] E. M. Stein, Singular Integrals and Differentiability of Functions, Princeton Univ. Press, New Jersey, 1970.
[18] R. Taberski, Singular integrals depending on two parameters, Prace Mat. 7 (1962), 173-179.
[19] R. Taberski, On double integrals and Fourier Series, Ann. Polon. Math. 15 (1964), 97–115.
[20] R. Taberski, On double singular integrals, Comment. Math. Prace Mat. 19 (1976), no.1, 155-160.
[21] G. Uysal and M. M. Yilmaz, Some theorems on the approximation of non-integrable functions via singular integral operators, Proc. Jangjeon Math. Soc. 18 (2015), no. 2, 241-251.
[22] M. M. Yilmaz, G. Uysal and E. Ibikli, A note on rate of convergence of double singular integral operators, Adv. Difference Equ. (2014), no. 287, 1-13.