New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2016320382

Some characterizations of a timelike curve in R_1^3

Muslum Aykut Akgun and Ali Ihsan Sivridag

Department of Mathematics Inonu University, Malatya, Turkey

Received: 1 February 2016, Accepted: 10 March 2016
Published online: 13 August 2016.

Abstract: In this paper, we study the position vectors of a timelike curve in the Minkowski 3-space R_1^3. We give some characterizations for timelike curves which lie on some subspaces of R_1^3.

Keywords: Timelike curve, Minkowski space, Frenet frame.

1 Introduction

Investigating of special curves is one of the most attractive topic in differential geometry. Some of these special curves are spacelike curves, timelike curves and null curves. Spacelike curves and timelike curves were initially investigated and developed by several authors[1,2,3,6]. Later, this topic drew attention of several authors and they studied different kinds of curves in the Lorentzian manifolds R_1^3 and R_1^4. Some of articles related to this topic are as follows:

A. Fernandez, A. Gimenez and P. Lucas introduced a Frenet frame with curvature functions for a null curve in a Lorentzian manifold and studied null helices in Lorentzian space forms [2]. C. Coken and U. Ciftci studied null curves in the 4-dimensional Minkowski space R_1^4 and give some results for pseudo spherical null curves and Bertrand null curves[1].

K. Ilarslan and O. Boyacioglu studied position vectors of a timelike and a null helice in R_1^3 [6]. K. Ilarslan and E. Nesovic gave some characterizations for null curves in E^4 and they obtained some relations between null normal curves and null osculating curves as well as between null rectifying curves and null osculating curves [10].

K. Ilarslan studied spacelike normal curves in Minkowski space E_1^3 and gave some characterizations of spacelike normal curves with spacelike, timelike and null principal normal [7]. K. Ilarsalan, E. Nesovic and M. Petrovic-Torgasev characterized non-null and null rectifying curves, lying fully in the Minkowski 3-space [8].

A. T. Ali and M. Onder characterize rectifying spacelike curves in terms of their curvature functions in Minkowski spacetime [3]. M. Onder, H. Kocayigit and M. Kazaz gave some characterizations for spacelike helices in Minkowski spacetime and found the differential equations characterizing the spacelike helices in Minkowski 4-space [13]. H.H. Ugurlu studied timelike surfaces and give some characterizations for timelike curves on the timelike surfaces[5].

M. A. Akgun and A. I. Sivridag studied null Cartan curves in Minkowski 4-space and give some theorems for null Cartan curves to lie on some subspaces of R_1^4 [14]. M. A. Akgun and A. I. Sivridag studied spacelike and timelike curves to lie

© 2016 BISKA Bilisim Technology
* Corresponding author e-mail: maakgun@hotmail.com

on some subspaces of R_1^4 and give some theorems in [15] and [16].

This paper organized following: In section 2 we give some basic knowledge related with curves in Minkowski space-time. Section 3 is the original part of this paper. In this section we investigate the conditions for timelike curves to lie on some subspaces of R_1^3 and we give some characterizations and theorems for these curves.

2 Preliminaries

Let R_1^3 denote Minkowski space together with a flat Lorentz metric $\langle,\rangle$ of signature $(-,+,+)$. A vector X is said to be timelike if $\langle X,X\rangle < 0$, spacelike if $\langle X,X\rangle > 0$ and null(lightlike) if $\langle X,X\rangle = 0$ and $X \neq 0$. The norm of a vector $X \in R_1^3$ is denoted by $\|X\|$ and defined by $\|X\| = \sqrt{|\langle X,X\rangle|}$. Let M be a timelike surface given by $y = y(u,v)$ in R_1^3 and α is a timelike curve on M. Then there exists a Frenet frame $\{t,n,b\}$ given by (1).

$$\begin{aligned} t' &= k_1 n \\ n' &= k_1 t - k_2 b \\ b' &= k_2 n. \end{aligned} \tag{1}$$

Thus, we can choose another basis $\{t,g,N\}$ on M along α, where N is the unit normal vector and $g = t \wedge N$. We note that g and N are spacelike vectors along α. Comparing this frame with the Frenet frame $\{t,n,b\}$. Let φ denote the angle between n and N. In this case we have

$$\begin{aligned} N &= n cos\varphi + b sin\varphi \\ g &= n sin\varphi - b cos\varphi. \end{aligned} \tag{2}$$

A curve α in R_1^3 is called a null curve if $\langle\alpha'(s),\alpha'(s)\rangle = 0$ and $\alpha'(s) \neq 0$, timelike curve if $\langle\alpha'(s),\alpha'(s)\rangle < 0$ and spacelike curve if $\langle\alpha'(s),\alpha'(s)\rangle > 0$ for all $s \in R$. Here $t = \alpha'(s)$, $\langle t,t\rangle = -1$, $\langle n,n\rangle = \langle b,b\rangle = 1$ and $\langle t,n\rangle = \langle t,b\rangle = \langle n,b\rangle = 0$.

3 Some characterizations of a timelike curve in R_1^3

In this section we will give some characterizations of timelike curves to lie on some subspaces of R_1^3. Let M be a timelike surface in R_1^3 and α be a timelike curve on M with the frame $\{t,g,N\}$. Then there are three subspaces of R_1^3 which are spanned by $\{t,g\}$, $\{t,N\}$ and $\{g,N\}$. Now, we investigate the position vector of a timelike curve α to lie in these subspaces.

Case 1. First we will investigate the conditions under which the timelike curve α lies on the subspace spanned by $\{t,g\}$. In this case we can write

$$\alpha(s) = \lambda(s)t + \mu(s)g \tag{3}$$

for some differentiable functions λ and μ of the parameter s. Differentiating (3) with respect to s and by using the Frenet equations (1) we have

$$\begin{aligned}\alpha'(s) &= \left(\lambda'(s) - \mu(s)k_1(s)sin\varphi\right)t \\ &+ \left(\lambda(s)k_1(s) + \mu'(s)sin\varphi + \mu(s)cos\varphi\frac{d\varphi}{ds} - \mu(s)k_2(s)cos\varphi\right)n \\ &+ \left(-\mu(s)cos\varphi - \mu(s)k_2(s)sin\varphi + \mu(s)sin\varphi\frac{d\varphi}{ds}\right)b\end{aligned} \tag{4}$$

where $\alpha' = t$. So we can write

$$\begin{cases} \lambda'(s) - \mu(s)k_1(s)sin\varphi = 1, \\ \lambda(s)k_1(s) + \mu'(s)sin\varphi + \mu(s)cos\varphi\frac{d\varphi}{ds} - \mu(s)k_2(s)cos\varphi = 0, \\ \mu(s)cos\varphi + \mu(s)k_2(s)sin\varphi - \mu(s)sin\varphi\frac{d\varphi}{ds} = 0. \end{cases} \tag{5}$$

If we choose λ as a constant we can write

$$\mu(s) = -\frac{1}{k_1(s)sin\varphi}. \tag{6}$$

Hence we have

$$\alpha(s) = ct - \left(\frac{1}{k_1(s)sin\varphi}\right)g. \tag{7}$$

If we choose $\mu(s) = const.$ and $\varphi(s) = const.$ by using (3.3) we find

$$\lambda(s) = \frac{ck_2(s)cos\varphi}{k_1(s)}. \tag{8}$$

So we have

$$\alpha(s) = \left(\frac{ck_2(s)cos\varphi}{k_1(s)}\right)t + cg. \tag{9}$$

Thus we have the following theorem.

Theorem 1. *A timelike curve α in R_1^3 lies on the subspace spanned by $\{t, g\}$ if and only if it is in the form*

$$\alpha(s) = ct - \left(\frac{1}{k_1(s)sin\varphi}\right)g$$

or

$$\alpha(s) = \left(\frac{ck_2(s)cos\varphi}{k_1(s)}\right)t + cg$$

where $\varphi(s) = const.$

Case 2. We will investigate the conditions under which the timelike curve α lies on the subspace spanned by $\{t, N\}$. In this case we can write

$$\alpha(s) = \lambda(s)t + \mu(s)N \tag{10}$$

© 2016 BISKA Bilisim Technology

for some differentiable functions λ and μ of the parameter s. Differentiating (10) with respect to s and by using the Frenet equations (1) we have

$$\begin{aligned}\alpha'(s) = &\left(\lambda'(s) + \mu(s)k_1(s)cos\varphi\right)t\\ &+\left(\mu'(s)cos\varphi - \mu(s)sin\varphi\frac{d\varphi}{ds} + \mu(s)k_2(s)sin\varphi + \lambda(s)k_1(s)\right)n\\ &+\left(\mu'(s)sin\varphi - \mu(s)k_2(s)cos\varphi + \mu(s)cos\varphi\frac{d\varphi}{ds}\right)b\end{aligned} \tag{11}$$

where $\alpha' = t$. So we can write

$$\begin{cases}\lambda'(s) + \mu(s)k_1(s)cos\varphi = -1,\\ \mu'(s)cos\varphi - \mu(s)sin\varphi\frac{d\varphi}{ds} + \mu(s)k_2(s)sin\varphi + \lambda(s)k_1(s) = 0,\\ \mu'(s)sin\varphi - \mu(s)k_2(s)cos\varphi + \mu(s)cos\varphi\frac{d\varphi}{ds} = 0.\end{cases} \tag{12}$$

If we choose $\varphi = const.$ in (12) we find

$$\mu'(s)sin\varphi - \mu(s)k_2(s)cos\varphi = 0. \tag{13}$$

From the solution of the differential equation (13) we find

$$\mu(s) = c.e^{\int k_1(s)cot\varphi ds}. \tag{14}$$

If we write (14) in (12) we find

$$\lambda(s) = -\frac{c}{k_1(s)}e^{\int k_1(s)cot\varphi ds}(k_1(s)cot\varphi cos\varphi + 1). \tag{15}$$

Hence we have

$$\alpha(s) = \left[-\frac{c}{k_1(s)}e^{\int k_1(s)cot\varphi ds}(k_1(s)cot\varphi cos\varphi + 1)\right]t + \left[c.e^{\int k_1(s)cot\varphi ds}\right]N. \tag{16}$$

Thus we have the following theorem.

Theorem 2. *A timelike curve α in R_1^3 lies on the subspace spanned by $\{t, N\}$ if and only if it is in the form*

$$\alpha(s) = \left[-\frac{c}{k_1(s)}e^{\int k_1(s)cot\varphi ds}(k_1(s)cot\varphi cos\varphi + 1)\right]t + \left[c.e^{\int k_1(s)cot\varphi ds}\right]N$$

where $\varphi = const.$

Case 3. We will investigate the conditions under which the timelike curve α lies on the subspace spanned by $\{g, N\}$. In this case we can write

$$\alpha(s) = \lambda(s)g + \mu(s)N \tag{17}$$

© 2016 BISKA Bilisim Technology

for some differentiable functions λ and μ of the parameter s. Differentiating (17) with respect to s and by using the Frenet equations (1) we have

$$\begin{aligned}\alpha'(s) =&[\lambda(s)k_1(s)sin\varphi+\mu(s)k_1(s)cos\varphi]t\\&+[\lambda'(s)sin\varphi+\lambda(s)cos\varphi\frac{d\varphi}{ds}-\lambda(s)k_2(s)cos\varphi\\&+\mu'(s)cos\varphi-\mu(s)sin\varphi\frac{d\varphi}{ds}+\mu(s)k_2(s)sin\varphi]n\\&+[-\lambda(s)cos\varphi-\lambda(s)k_2(s)sin\varphi+\lambda(s)sin\varphi\frac{d\varphi}{ds}\\&+\mu'(s)sin\varphi-\mu(s)k_2cos\varphi+\mu(s)cos\varphi\frac{d\varphi}{ds}]b\end{aligned} \tag{18}$$

where $\alpha'=t$. So we can write

$$\begin{cases}\lambda(s)k_1(s)sin\varphi+\mu(s)k_1(s)cos\varphi=-1,\\ \lambda'(s)sin\varphi+\lambda(s)cos\varphi\frac{d\varphi}{ds}-\lambda(s)k_2(s)cos\varphi+\mu'(s)cos\varphi\\ -\mu(s)sin\varphi\frac{d\varphi}{ds}+\mu(s)k_2(s)sin\varphi=0,\\ -\lambda(s)cos\varphi-\lambda(s)k_2(s)sin\varphi+\lambda(s)sin\varphi\frac{d\varphi}{ds}\\ +\mu'(s)sin\varphi-\mu(s)k_2(s)cos\varphi+\mu(s)cos\varphi\frac{d\varphi}{ds}=0.\end{cases} \tag{19}$$

If we choose $\varphi=const.$ in (19) we obtain

$$\begin{cases}\lambda(s)sin\varphi+\mu(s)cos\varphi=-\frac{1}{k_1(s)},\\ \lambda'(s)sin\varphi-\lambda(s)k_2(s)cos\varphi+\mu'(s)cos\varphi+\mu(s)k_2(s)sin\varphi=0,\\ -\lambda(s)cos\varphi-\lambda(s)k_2(s)sin\varphi+\mu'(s)sin\varphi-\mu(s)k_2(s)cos\varphi=0.\end{cases} \tag{20}$$

From (20) we have

$$-k_2(s)(\lambda(s)sin\varphi+\mu(s)cos\varphi)-\lambda(s)cos\varphi+\mu'(s)sin\varphi=0. \tag{21}$$

By using (21) we obtain

$$\frac{k_2(s)}{k_1(s)}-\lambda(s)cos\varphi+\mu'(s)sin\varphi=0. \tag{22}$$

Furthermore from (20) we can easily obtain

$$\lambda'(s)sin\varphi+\mu'(s)cos\varphi=(-\frac{1}{k_1(s)})'. \tag{23}$$

By using (20) and (23) we find

$$\lambda(s)cos\varphi-\mu'(s)sin\varphi=\frac{1}{k_2(s)}(-\frac{1}{k_1(s)})'. \tag{24}$$

If we use (22) and (24) we find the differential equation

$$\mu'(s)-\mu(s)=\frac{k_1'(s)-k_2^2(s)k_1(s)}{k_2(s)k_1^2(s)sin\varphi}. \tag{25}$$

© 2016 BISKA Bilisim Technology

From the solution of (25) we obtain

$$\mu(s)=\frac{1}{e^{-s+c}sin\varphi}\int e^{-s+c}\frac{k_1'(s)-k_2^2(s)k_1(s)}{k_2(s)k_1^2(s)sin\varphi}ds. \tag{26}$$

Considering (20) and (26) we find

$$\lambda(s)=-\frac{1}{k_1(s)sin\varphi}-\frac{cot\varphi}{e^{-s+c}sin\varphi}\int e^{-s+c}\frac{k_1'(s)-k_2^2(s)k_1(s)}{k_2(s)k_1^2(s)sin\varphi}ds. \tag{27}$$

So we have

$$\begin{aligned}\alpha(s)=&\left[-\frac{1}{k_1(s)sin\varphi}-\frac{cot\varphi}{e^{-s+c}sin\varphi}\int e^{-s+c}\frac{k_1'(s)-k_2^2(s)k_1(s)}{k_2(s)k_1^2(s)sin\varphi}ds\right]g\\&+\left[\frac{1}{e^{-s+c}sin\varphi}\int e^{-s+c}\frac{k_1'(s)-k_2^2(s)k_1(s)}{k_2(s)k_1^2(s)sin\varphi}ds\right]N.\end{aligned} \tag{28}$$

Thus we have the following theorem.

Theorem 3. *A timelike curve α in R_1^3 lies on the subspace spanned by $\{g,N\}$ if and only if it is in the form*

$$\begin{aligned}\alpha(s)=&\left[-\frac{1}{k_1(s)sin\varphi}-\frac{cot\varphi}{e^{-s+c}sin\varphi}\int e^{-s+c}\frac{k_1'(s)-k_2^2(s)k_1(s)}{k_2(s)k_1^2(s)sin\varphi}ds\right]g\\&+\left[\frac{1}{e^{-s+c}sin\varphi}\int e^{-s+c}\frac{k_1'(s)-k_2^2(s)k_1(s)}{k_2(s)k_1^2(s)sin\varphi}ds\right]N\end{aligned}$$

where $\varphi=const$.

4 Conclusion

In this paper we give some characterizations for timelike curves which lie on some subspaces of R_1^3. We give some theorems about these curves and characterize the timelike curves in terms of their curvature functions in R_1^3.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] A.C. Coken, and U. Ciftci, *On the Cartan Curvatures of a Null Curve in Minkowski Spacetime*, Geometriae Dedicate 114 (2005), 71-78.

[2] A. Fernandez, A. Gimenez, and P. Lucas, *Null helices in Lorentzian space forms*, Int. J. Mod. Phys. A. 16 (2001), 4845-4863.

© 2016 BISKA Bilisim Technology

[3] A. T. Ali, M. Onder, *Some Characterizations of Rectifying Spacelike Curves in the Minkowski Space-Time*, Global J of Sciences Frontier Research Math, Vol 12, Is 1, 2012 2249-4626

[4] C. Camci, K. Ilarslan, L. Kula, H.H. Hacisalihoglu, *Harmonic Curvatures and Generalized Helices in* E^n, Chaos,Solitions and Fractals 40 (2009), 2590-2596.

[5] H.H. Ugurlu, *On The Geometry of Timelike Surfaces* , Communication, Ankara University, Faculty of Sciences, Dept.of Math., Series Al, Vol.46, pp. 211-223 (1997)

[6] K. Ilarslan, and O. Boyacioglu, *Position vectors of a timelike and a null helix in Minkowski 3-space*, Chaos,Solitions and Fractals (2008), 1383-1389.

[7] K. Ilarslan, *Spacelike Normal Curves in Minkowski* E_1^3, Turk. J. Math 29(2005), 53-63

[8] K. Ilarslan, E. Nesovic, M. Petrovic-Torgasev, *Some Characterizations of Rectifying Curves in Minkowski 3-space*, Novi Sad J Math 2003, 33(2), 23-32

[9] K. Ilarslan, E. Nesovic, *Spacelike and timelike normal curves*, Publications de l'Institut MathÃ©matique (Beograd) Nouvelle serie, tome 85(99) (2009) 111-118.

[10] K. Ilarslan, E. Nesovic, *Some characterizations of rectifying curves in the Euclidean space* E^4., Turkish J. Math. 32 (2008), no. 1, 21-30

[11] K. Ilarslan, and E. Nesovic, *Some characterizations of null osculating curves in the Minkowski space-time*, Proceedings of the Estonian Academy of Sciences, 61, I (2012), 1-8.

[12] K.L. Duggal, and D.H. Jin, *Null Curves and Hypersurfaces of Semi-Riemannian Manifolds*, Copyriht C 2007 by World Scientific Publishing Co. Pte. Ltd., (2007), Page 83.

[13] M. Onder, H. Kocayigit and M. Kazaz, *Spacelike Helices in Minkowski 4-space*, Ann. Univ. Ferrera 2010, 56, 335-343

[14] M. A. Akgun, A. I. Sivridag, On The Null Cartan Curves of R_1^4, Global Journal of Mathematics, Vol.1 No.1,2015,41-50

[15] M. A. Akgun, A. I. Sivridag, On The Characterizations of Timelike Curves in R_1^4, Global Journal of Mathematics, Vol.2 No.2,2015,116-127

[16] M. A. Akgun, A. I. Sivridag, Some Characterizations of a Spacelike Curve in R_1^4, Pure Mathematical Sciences, Hikari Ltd., Vol.4 No.1-4,2015,43-55

[17] S. Keles, S. Y. Perktas and E. Kilic, *Biharmonic Curves in LP-Sasakian Manifolds*, Bulletin of the Malasyian Mathematical Society, (2) 33(2), 2010, 325-344

© 2016 BISKA Bilisim Technology