

Fixed point theorems for (F, ψ, ϕ) – contractions on ordered S-Complete Hausdorff uniform spaces

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Abstract: The aim of this paper is to proved some new fixed point theorems for (F, ψ, ϕ) –weak contractions on ordered S-complete Hausdorff uniform spaces. Our results extend existing results in the literature.

Keywords: Fixed point, (F, ψ, ϕ) -contraction, C-class function, E-distance ,S-complete space.

1 Introduction

Aamri and El Moutawakil [2] proved some common fixed point theorems for some new contractive or expansive maps in uniform spaces by introducing the notions of an E-distance. Some other authors proved fixed point theorems using this concept ([1],[3-5],[8],[10],[11],[18],[19],[22],[23]).

Existence of fixed points in partially ordered metric spaces was first investigated in 2004 by Ran and Reurings [21] and then by Nieto and Lopez [17]. Recently, some results were proved in this direction ([9],[10],[13],[16],[20]).

Definition 1. ([2]) Let (X, ϑ) be a uniform space. A function $p : X \times X \longrightarrow \mathbb{R}^+$ is said to be an A-distance if for any $V \in \vartheta$, there exists $\delta > 0$, such that $p(z, x) \le \delta$ and $p(z, y) \le \delta$ for some $z \in X$ imply $(x, y) \in V$.

Definition 2. ([2]) Let (X, ϑ) be a uniform space. A function $p: X \times X \longrightarrow \mathbb{R}^+$ is said to be an E-distance if

(*p*1) *p* is an *A*-distance, (*p*2) $p(x,y) \le p(x,z) + p(z,y)$ for all $x, y, z \in X$.

Example 1. ([2]) Let $X = [0, +\infty)$ and $p(x, y) = \max\{x, y\}$. The function p is an A-distance. Also, p is an E-distance.

The following lemma embodies some useful properties of E – distance.

Lemma 1. ([1], [2]) Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be arbitrary sequences in X and $\{\alpha_n\}, \{\beta_n\}$ be sequences in \mathbb{R}^+ converging to 0. Then, for $x, y, z \in X$, the following holds

(a) If $p(x_n, y) \le \alpha_n$ and $p(x_n, z) \le \beta_n$ for all $n \in \mathbb{N}$, then y = z. In particular, if p(x, y) = 0 and p(x, z) = 0, then y = z.

(b) If $p(x_n, y_n) \le \alpha_n$ and $p(x_n, z) \le \beta_n$ for all $n \in \mathbb{N}$, then $\{y_n\}$ converges to z.

(c) If $p(x_n, x_m) \le \alpha_n$ for all m > n, then $\{x_n\}$ is a Cauchy sequence in (X, ϑ) .

Let (X, ϑ) be a uniform space equipped with E-distance p. A sequence in X is p-Cauchy if it satisfies the usual metric condition. There are several concepts of completeness in this setting.

Definition 3. ([1], [2]) Let (X, ϑ) be a uniform space and p be an E-distance on X.Then

- (i) X is said to be S-complete if for every p-Cauchy sequence $\{x_n\}$ there exists $x \in X$ with $\lim_{n \to \infty} p(x_n, x) = 0$,
- (ii) *X* is said to be *p*-Cauchy complete if for every *p*-Cauchy sequence $\{x_n\}$ there exists $x \in X$ with $\lim x_n = x$ with respect to $\tau(\vartheta)$.
- (iii) $f: X \longrightarrow X$ is *p*-continuous if $\lim_{n \to \infty} p(x_n, x) = 0$ implies $\lim_{n \to \infty} p(fx_n, fx) = 0$, (iv) $f: X \longrightarrow X$ is $\tau(\vartheta)$ -continuous if $\lim_{n \to \infty} x_n = x$ with respect to $\tau(\vartheta)$ implies $\lim_{n \to \infty} fx_n = fx$ with respect to $\tau(\vartheta)$.

Remark. ([2]) Let (X, ϑ) be a Hausdorff uniform space and let $\{x_n\}$ be a p-Cauchy sequence. Suppose that X is S-complete, then there exists $x \in X$ such that $\lim_{n \to \infty} p(x_n, x) = 0$. Lemma 4 (b) then gives $\lim_{n \to \infty} x_n = x$ with respect to the topology $\tau(\vartheta)$. Therefore S-completeness implies p-Cauchy completeness.

In 2014, the concept of C-class functions were introduced by H. Ansari in [6]. After some fixed point theorems were gived using this concept ([7],[12],[14]).

Definition 4. ([6]) A mapping $F : [0, \infty)^2 \to \mathbb{R}$ is called C-class function if it is continuous and satisfies following axioms.

- (1) $F(s,t) \le s;$
- (2) F(s,t) = s implies that either s = 0 or t = 0; for all $s, t \in [0,\infty)$.

Note for some F we have that F(0,0) = 0. We denote C-class functions as \mathscr{C} .

Example 2. The following functions $F : [0,\infty)^2 \to \mathbb{R}$ are elements of \mathscr{C} , for all $s,t \in [0,\infty)$.

- (1) $F(s,t) = s-t, F(s,t) = s \Rightarrow t = 0;$ (2) $F(s,t) = ms, 0 < m < 1, F(s,t) = s \Rightarrow s = 0;$ (3) $F(s,t) = \frac{s}{(1+t)^r}$; $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$ or t = 0; (4) $F(s,t) = \log(t+a^s)/(1+t), a > 1, F(s,t) = s \Rightarrow s = 0 \text{ or } t = 0;$ (5) $F(s,t) = \ln(1+a^s)/2, a > e, F(s,1) = s \Rightarrow s = 0;$ (6) $F(s,t) = (s+l)^{(1/(1+t)^r)} - l, l > 1, r \in (0,\infty), F(s,t) = s \Rightarrow t = 0;$ (7) $F(s,t) = s \log_{t+a} a, a > 1, F(s,t) = s \Rightarrow s = 0 \text{ or } t = 0;$ (8) $F(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t}), F(s,t) = s \Rightarrow t = 0;$ (9) $F(s,t) = s\beta(s), \beta: [0,\infty) \to (0,1)$, and is continuous, $F(s,t) = s \Rightarrow s = 0$; (10) $F(s,t) = s - \frac{t}{k+t}, F(s,t) = s \Rightarrow t = 0;$
- (11) $F(s,t) = s \varphi(s), F(s,t) = s \Rightarrow s = 0$, here $\varphi: [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;
- (12) $F(s,t) = sh(s,t), F(s,t) = s \Rightarrow s = 0$, here $h: [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function such that h(t,s) < 1 for all t, s > 0;
- (13) $F(s,t) = s (\frac{2+t}{1+t})t, F(s,t) = s \Rightarrow t = 0;$
- (14) $F(s,t) = \sqrt[n]{\ln(1+s^n)}, F(s,t) = s \Rightarrow s = 0;$
- (15) $F(s,t) = \phi(s), F(s,t) = s \Rightarrow s = 0$, here $\phi: [0,\infty) \to [0,\infty)$ is a upper semicontinuous function such that $\phi(0) = 0$, and $\phi(t) < t$ for t > 0;
- (16) $F(s,t) = \frac{s}{(1+s)^r}; r \in (0,\infty), F(s,t) = s \Rightarrow s = 0.$

We shall also state the following definition of altering distance function which is required in the sequel to establish a fixed point theorem in uniform space.

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Definition 5. ([15]) A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:

- (i) $\psi(0) = 0$,
- (ii) ψ is continuous and monotonically nondecreasing.

Remark. We denote set of altering distance functions by Ψ .

In this paper, we assume that

Definition 6. ([6])An ultra altering distance function is a continuous, nondecreasing mapping $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(t) > 0$, t > 0 and $\varphi(0) \ge 0$.

Remark. We denote set ultra altering distance functions by Φ_u .

2 Fixed point results

In this section, we prove some fixed point results using C-class function in ordered uniform spaces.

Theorem 1. Let (X, ϑ, \preceq) be an ordered Hausdorff uniform space and p be an E-distance on S-complete and p-bounded space X. Let $f, g: X \to X$ be two commuting p-continuous or $\tau(\vartheta)$ -continuous selfmappings such that

- (i) $f(X) \subseteq g(X)$,
- (ii) f is g-nondecreasing,
- (iii) $\psi(p(fx, fy)) \leq F(\psi(p(gx, gy)), \varphi(p(gx, gy)))$ for all $x, y \in X$ with $gx \leq gy$ where $\psi \in \Psi, \varphi \in \Phi_u$ and $F \in \mathscr{C}$

If there exists $x_0 \in X$ with $gx_0 \preceq fx_0$ then f and g have an unique common fixed point.

Proof. If $x_0 \in X$ such that $gx_0 \preceq fx_0$. Since $f(X) \subseteq g(X)$, we can choose $x_1 \in X$ such that $fx_0 = gx_1$. Then $gx_0 \preceq fx_0 = gx_1$. As f is g-nondecreasing, we get $fx_0 \preceq fx_1$. Continuing this process, we can construct a sequence $\{x_n\}$ in X such that

$$gx_n = fx_{n-1}, n = 1, 2, \dots$$

for which

$$gx_0 \leq fx_0 = gx_1 \leq fx_1 = gx_2 \leq \cdots \leq fx_{n-1} = gx_n \leq \cdots$$

From (iii),

$$\psi(p(fx_{n}, fx_{n+1})) \leq F(\psi(p(gx_{n}, gx_{n+1})), \varphi(p(gx_{n}, gx_{n+1})))$$

$$\leq \psi(p(gx_{n}, gx_{n+1})) = \psi(p(fx_{n-1}, fx_{n}))$$
(1)

so $p(fx_n, fx_{n+1}) \le p(fx_{n-1}, fx_n)$, therefore $\{p(fx_n, fx_{n+1})\}$, is decreasing so tend to $r \ge 0$,

In (1), on taking limit as $n \to \infty$, by definiton of *F*,

$$\boldsymbol{\psi}(r) \leq F(\boldsymbol{\psi}(r), \boldsymbol{\varphi}(r)) \leq \boldsymbol{\psi}(r).$$

So, $\psi(r) = 0$, or $\varphi(r) = 0$, therefore r = 0. Hence

$$\lim_{n \to \infty} p(x_{n+1}, x_n) = 0.$$

245



Similarly, we can show

$$\lim_{n\to\infty}p(x_n,x_{n+1})=0.$$

Next we show that $\{x_n\}$ is a *p*-Cauchy sequence. Assume $\{y_n = fx_n\}$ is not *p*-Cauchy. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{y_{m(k)}\}$ and $\{y_{n(k)}\}$ of $\{y_n\}$ with m(k) > n(k) > k such that

$$p\left(y_{n(k)}, y_{m(k)}\right) \ge \varepsilon. \tag{2}$$

Further, corresponding to n(k), we can choose m(k) in such a way that it is the smallest integer with m(k) > n(k) and satisfying (2). Hence,

$$p(y_{n(k)}, y_{m(k)-1}) < \varepsilon.$$

Then we have

$$\varepsilon \leq p\left(y_{n(k)}, y_{m(k)}\right) \leq p\left(y_{n(k)}, y_{m(k)-1}\right) + p\left(y_{m(k)-1}, y_{m(k)}\right)$$

that is

$$\boldsymbol{\varepsilon} \leq p\left(y_{n(k)}, y_{m(k)}\right) < \boldsymbol{\varepsilon} + p\left(y_{m(k)-1}, y_{n(k)}\right).$$

Taking the limit as $k \to \infty$, we have

$$\lim_{k \to \infty} p\left(x_{n(k)}, x_{m(k)}\right) = \varepsilon.$$
(3)

From (p2),

$$p(y_{n(k)}, y_{m(k)}) \le p(y_{n(k)}, y_{n(k)+1}) + p(y_{n(k)+1}, y_{m(k)+1}) + p(y_{m(k)+1}, y_{m(k)})$$

and

$$p\left(y_{n(k)+1}, y_{m(k)+1}\right) \le p\left(y_{n(k)+1}, y_{n(k)}\right) + p\left(y_{n(k)}, y_{m(k)}\right) + p\left(y_{m(k)}, y_{m(k)+1}\right)$$

Taking the limit as $k \rightarrow \infty$ we have

$$\lim_{k \to \infty} p\left(y_{n(k)+1}, y_{m(k)+1}\right) = \varepsilon.$$
(4)

From (iii),

$$\begin{split} \psi \left(p \left(y_{n(k)+1}, y_{m(k)+1} \right) \right) &= \psi \left(p \left(f x_{n(k)+1}, f x_{m(k)+1} \right) \right) \\ &\leq F(\psi \left(p \left(g x_{n(k)+1}, g x_{m(k)+1} \right) \right), \varphi \left(p \left(g x_{n(k)+1}, g x_{m(k)+1} \right) \right)) \\ &= F(\psi \left(p \left(f x_{n(k)}, f x_{m(k)} \right) \right), \varphi \left(p \left(f x_{n(k)}, f x_{m(k)} \right) \right)) \\ &= F(\psi \left(p \left(y_{n(k)}, y_{m(k)} \right) \right), \varphi \left(p \left(y_{n(k)}, y_{m(k)} \right) \right)). \end{split}$$

Letting $k \to \infty$ in the above inequality, using (3), (4), the continuities of ψ and φ and definition of F, we have

$$\psi(\varepsilon) \leq F(\psi(\varepsilon), \varphi(\varepsilon)) \leq \psi(\varepsilon)$$

So, $\psi(\varepsilon) = 0$ or $\varphi(\varepsilon) = 0$, therefore $\varepsilon = 0$ which is a contradiction. Hence $\{fx_n\}$ is a *p*-Cauchy sequence. Since *S*-completeness of *X*, there exists a $z \in X$ such that

$$\lim_{n\to\infty} p(fx_n, z) = 0 \text{ and } \lim_{n\to\infty} p(gx_n, z) = 0$$

Moreover, the p-continuity of f and g implies that

$$\lim_{n \to \infty} p\left(gfx_n, gz\right) = \lim_{n \to \infty} p\left(fgx_n, fz\right) = 0$$

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Since f and g are commuting, then fg = gf. So we have

$$\lim_{n \to \infty} p(fgx_n, gz) = \lim_{n \to \infty} p(fgx_n, fz) = 0$$

By Lemma 4(a), we have fz = gz. Since fg = gf, we have ffz = fgz = gfz = ggz. From (iii) and definition of F, ψ and φ ,

$$\psi(p(fz, ffz)) \le F(\psi(p(gz, gfz)), \varphi(p(gz, gfz)))$$

= $F(\psi(p(fz, ffz)), \varphi(p(fz, ffz)))$ (5)

so, $\psi(p(fz, ffz)) = 0$, or $\varphi(p(fz, ffz)) = 0$. Thus p(fz, ffz) = 0. Again From (iii), we have

$$\psi(p(fz,fz)) \le F(\psi(p(gz,gz)), \varphi(p(gz,gz)))$$

= $F(\psi(p(fz,fz)), \varphi(p(fz,fz)))$ (6)

so, $\psi(p(fz, fz)) = 0$, or $\varphi(p(fz, fz)) = 0$. Thus p(fz, fz) = 0. Thus from (5), (6) and Lemma 4(a), we get ffz = fz. Hence fz is common fixed point of f and g. The proof is similar when T is $\tau(\vartheta)$ -continuous.

Now, we show uniqueness. Suppose that there exists $u, t \in X$ such that fu = gu = u and ft = gt = t. Then by (iii),

$$\begin{aligned} \psi(p(u,t)) &= \psi(p(fu,ft)) \le F(\psi(p(gu,gt)), \varphi(p(gu,gt))) \\ &= F(\psi(p(u,t)), \varphi(p(u,t))) \end{aligned}$$

so, $\psi(p(u,t)) = 0$, or $\varphi(p(u,t)) = 0$. Thus p(u,t) = 0. Similarly, we show that p(t,u) = 0. By (p2)

$$p(u,u) \le p(u,t) + p(t,u)$$

and therefore p(u, u) = 0. By Lemma 4 (a), we have u = t.

Corollary 1. Let (X, ϑ, \preceq) be an ordered Hausdorff uniform space. Suppose p be an E-distance on S-complete and p-bounded space X. Let $f: X \to X$ be a p-continuous or $\tau(\vartheta)$ -continuous nondecreasing mapping such that for all comparable $x, y \in X$ with

 $\psi(p(fx, fy)) \le F(\psi(p(x, y)), \varphi(p(x, y)))$

where $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathscr{C}$.

If there exists $x_0 \in X$ with $x_0 \preceq f(x_0)$ then f has a fixed point.

Corollary 2. ([23]) Let (X, ϑ, \preceq) be a Hausdorff uniform space, " \preceq " be a partial order on X and p be an E-distance on S-complete space X. Let $f: X \to X$ be a p-continuous or $\tau(\vartheta)$ -continuous nondecreasing mapping such that for all comparable $x, y \in X$ with

$$\psi(p(fx, fy)) \le \psi(p(x, y)) - \varphi(p(x, y))$$

where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are altering distance functions.

If there exists $x_0 \in X$ *with* $x_0 \preceq f(x_0)$ *then f has a fixed point.*

Corollary 3. Let (X, ϑ, \preceq) be an ordered Hausdorff uniform space. Suppose p be an E-distance on S-complete and p-bounded space X. Let $f: X \to X$ be a p-continuous or $\tau(\vartheta)$ -continuous nondecreasing mapping such that for all

247

248 BISK

comparable $x, y \in X$ *with*

$$\psi(p(fx, fy)) \le k\psi(p(x, y))$$

where $\psi : [0, \infty) \to [0, \infty)$ is an altering distance function and 0 < k < 1.

If there exists $x_0 \in X$ *with* $x_0 \preceq f(x_0)$ *then* f *has a fixed point.*

Example 3. Let $F(s,t) = \ln(1+s)$, X = [0,1] equipped with usual metric d(x,y) = |x-y| and a partial order be defined as $x \leq y$ whenever $y \leq x$ and suppose

$$\vartheta = \{V \subset X \times X : \Delta \subset V\}.$$

Define the function p as p(x,y) = y for all x, y in X and $f,g: X \to X$ defined by $f(t) = \frac{3t}{4}$ and $g(t) = \frac{t}{16}$. Consider the function ψ and φ defined as follows

$$\psi(t) = \frac{t}{5}$$
 and $\varphi(t) = \frac{t}{3}$.

Definition of ϑ , $\cap_{V \in \vartheta} V = \Delta$ and this show that the uniform space (X, ϑ) is Hausdorff uniform space. And also X is S-complete.On the other hand, p is an E-distance. f, g are commuting, p-continuous and f is g-nondecreasing. We have that for all $x, y \in X$

$$p(fx, fy) \le \ln(1 + \psi(p(x, y))) = F(\psi(p(gx, gy)), \varphi(p(gx, gy))).$$

And 0 is the unique common fixed point of f and g.

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