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Quarter-symmetric metric connection on a Lorentzian α-Sasakian manifold

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Abstract: In the present paper we study locally ϕ-symmetric, locally projective ϕ-symmetric, ϕ-recurrent and ϕ-projectively flat Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection. Further, the existence of a Lorentzian α-Sasakian manifold admitting quarter-symmetric metric connection is shown by constructing an example.

Keywords: Lorentzian α-Sasakian manifold, quarter-symmetric metric connection, locally ϕ-symmetric, locally projective ϕ-symmetric, ϕ-recurrent, ϕ-projectively flat.

1 Introduction

In 1924, the idea of semi-symmetric linear connection on a differentiable manifold was introduced by Friedmann and Schouten [5]. In 1930, Bartolotti [2] gave a geometrical meaning of such a connection. Further, the concept of metric connection with torsion on a Riemannian manifold was defined by Hayden [7]. Later Yano[18] studied some curvature and derivational conditions for semi-symmetric metric connection in Riemannian manifolds. As a generalization of semi-symmetric connection, Golab [6] introduced quarter symmetric connection on a differentiable manifold.

A linear connection $\tilde{\nabla}$ on an n-dimensional differentiable manifold is said to be a quarter-symmetric connection [6] if its torsion tensor T is of the form

$$T(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1}$$

where η is a 1-form and ϕ is a tensor field of type (1). In particular, if we replace ϕX by X and ϕY by Y, then the quarter-symmetric connection reduces to a semisymmetric connection [5]. Further if quarter-symmetric linear connection $\tilde{\nabla}$ satisfies the condition

$$(\tilde{\nabla}_X g)(Y,Z) = 0, \tag{2}$$

for all $X,Y,Z \in \chi(M^n)$, where $\chi(M^n)$ is the Lie algebra of vector fields on a manifold M, then $\tilde{\nabla}$ is said to be a quarter-symmetric metric connection. Quarter-symmetric metric connection on different manifold have been studied by many authors like Rastogi [13,14], Mishra and Pandey [8], Yano and Imai [19], De et al. [3,9], Pradeep Kumar et al.[11, 12,16] and others.

In 2005, Yildiz and Murathan [21] introduced Lorentzian α-Sasakian manifold. Later, many geometers have studied Lorentzian α-Sasakian manifolds with different curvature restrictions in the papers [1,4,21,22] and others.

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Motivated by the above studies, we have made an attempt to study quarter-symmetric metric connection on a Lorentzian α-Sasakian manifold. The present paper is structured as follows: The section 2 is equipped with some preliminaries of Lorentzian α-Sasakian manifold. Section 3 and 4 are devoted respectively to the study of locally ϕ-symmetric and locally projective ϕ-symmetric Lorentzian α-Sasakian manifold admitting quarter-symmetric metric connection. And in sections 5 and 6, we have proved that ϕ-recurrent and ϕ-projectively flat Lorentzian α-Sasakian manifolds with respect to quarter-symmetric metric connection is a generalized η-Einstein manifold. In the last section, we have constructed an example of a Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection.

2 Preliminaries

A n-dimensional differentiable manifold M is called a Lorentzian α-Sasakian manifold [21], if it admits a (1) tensor field φ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g which satisfy,

$$\varphi^2 X = X + \eta(X)\xi, g(X,\xi) = \eta(X), \eta(\xi) = -1, \tag{3}$$

$$g(\varphi X, \varphi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{4}$$

$$\varphi\xi = 0, \eta(\varphi X) = 0 \tag{5}$$

Also a Lorentzian α -Sasakian manifold M satisfies [21,17],

$$\nabla_X \xi = \alpha\varphi X, \tag{6}$$

$$(\nabla_X \eta)(Y) = \alpha g(\varphi X, Y), \tag{7}$$

$$\eta(R(X,Y)Z) = \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{8}$$

$$R(\xi,X)Y = \alpha^2[g(X,Y)\xi - \eta(Y)X], \tag{9}$$

$$R(\xi,X)\xi = \alpha^2[\eta(X)\xi + X], \tag{10}$$

$$R(X,Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y], \tag{11}$$

$$(X\varphi)(Y) = \alpha[g(X,Y)\xi + \eta(Y)X], \tag{12}$$

$$S(X,\xi) = (n-1)\alpha^2\eta(X), \tag{13}$$

$$S(\varphi X, \varphi Y) = S(X,Y) + (n-1)\alpha^2\eta(X)\eta(Y), \tag{14}$$

for any vector fields $X,Y,Z \in M$ and where ∇ denotes the operator of covariant differentiation with respect to Lorentzian metric g and $\alpha \in R$. The relation between quarter-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection∇ on M

[10] is given by,

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X. \tag{15}$$

By virtue of (15), the relation between Riemannian curvature tensor $\tilde{R}$, Ricci tensor $\tilde{S}$ and scalar curvature $\tilde{r}$ with respect to quarter-symmetric metric connection and the Riemannian curvature tensor R, Ricci tensor R and scalar curvature r with respect to Levi-Civita connection in a Lorentzian α-Sasakian manifold [10] are given as follows.

$$\tilde{R}(X,Y)Z = R(X,Y)Z + \alpha[g(\varphi X,Z)\varphi Y - g(\varphi Y,Z)\varphi X] + \alpha\eta(Z)[\eta(Y)X - \eta(X)Y], \tag{16}$$

$$\tilde{S}(Y,Z) = S(Y,Z) + \alpha[g(Y,Z) + n\eta(Y)\eta(Z)], \tag{17}$$

$$\tilde{r} = r. \tag{18}$$

A Lorentzian α -Sasakian manifold M^n is said to be a generalized η -Einstein manifold [23] if the following condition

$$S(X,Y) = \lambda g(X,Y) + \mu\eta(X)\eta(Y) + \nu\Omega(X,Y), \tag{19}$$

holds on M. Where λ , μ and ν are smooth functions and Ω (X ,Y) = g(φ X ,Y). If $\nu = 0$ then *the* manifold is an η -Ein*stein* man*ifold*.

3 Locally ϕ-symmetric Lorentzian α-Sasakian manifold admitting quarter-symmetric metric connection

Definition 1. *A Lorentzian α -Sasakian manifold M is said to be locally ϕ-symmetric if*

$$\phi^2((\nabla_w R)(X,Y)Z) = 0 \tag{20}$$

for all vector fields X,Y,Z and W orthogonal to ξ on M. This notion was introduced by Takahashi for Sasakian manifold

Definition 2. *A Lorentzian α-Sasakian manifold M is said to be locally φ-symmetric with respect to quarter symmetric metric connection if*

$$\phi^2((\tilde{\nabla}_w \tilde{R})(X,Y)Z) = 0 \tag{21}$$

for all vector fields X,Y,Z and W orthogonal to ξ on M.

Theorem 1. *A Lorentzian $\alpha-$Sasakian manifold is locally ϕ-symmetric with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ if and only if it is locally ϕ-symmetric with respect to the Levi-Civita connection.*

*Proof.*From equation (15) and (16), we have

$$(\tilde{\nabla}_w \tilde{R})(X,Y)Z = (\nabla_w \tilde{R})(X,Y)Z + \eta(\tilde{R}(X,Y)Z)\phi W. \tag{22}$$

Differentiating (16) covariantly with respect to W, we obtain

$$\begin{aligned}(\nabla_w \tilde{R})(X,Y)Z = (\nabla_w R)(X,Y)Z &+ \alpha\{g((\nabla_w\phi)(X),Z)\phi Y + g(\phi X,Z)(\nabla_w\phi)(Y)\} \\ &-\alpha\{g((\nabla_w\phi)(Y),Z)\phi X\} + g(\phi Y,Z)(\nabla_w\phi)(X)\} + \alpha(\nabla_w\eta)(Z)[\eta(Y)X \\ &-\eta(X)Y] + \alpha\{\eta(Z)[(\nabla_w\eta)(Y)X - (\nabla_w\eta)(X)Y]\}.\end{aligned} \tag{23}$$

Using (7) and (12), (23) reduces to

$$\begin{aligned}(\nabla_w \tilde{R})(X,Y)Z = (\nabla_w R)(X,Y)Z - \alpha^2\{g(W,Y)\eta(Z)+g(W,Z)\eta(Y)\}\phi X\\ +\alpha\{g(W,X)\eta(Z)+g(W,Z)\eta(X)\}\phi Y+\alpha^2\{g(\phi X,Z)g(W,Y)\xi+g(\phi X,Z)\eta(Y)W\}\\ -\alpha^2\{g(\phi Y,Z)g(W,X)\xi+g(\phi Y,Z)\eta(X)W\}+\alpha^2 g(\phi W,Z)[\eta(Y)X\\ -\eta(X)Y]+\alpha^2[g(\phi W,Y)X-g(\phi W,X)Y]\eta(Z).\end{aligned} \tag{24}$$

Taking inner product of (16) with x and then using (3), (5) and (8), we obtain

$$\eta\left(\tilde{R}(X,Y)Z\right) = \alpha^2\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}. \tag{25}$$

Making use of equations (24) and (25), equation (22) becomes

$$\begin{aligned}(\tilde{\nabla}_W \tilde{R})(X,Y)Z = (\tilde{\nabla}_W R)(X,Y)Z - \alpha^2\{g(W,Y)\eta(Z)+g(W,Z)\eta(Y)\}\phi X\\ +\alpha^2\{g(W,X)\eta(Z)+g(W,Z)\eta(X)\}\phi Y+\alpha^2\{g(\phi X,Z)g(W,Y)\xi+g(\phi X,Z)\eta(Y)W\}\\ -\alpha^2\{g(\phi Y,Z)g(W,X)\xi+g(\phi Y,Z)\eta(X)W\}+\alpha^2 g(\phi W,Z)[\eta(Y)X-\eta(X)Y]+\alpha^2[g(\phi W,Y)X\\ -g(\phi W,X)Y]\eta(Z)+\alpha^2\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\phi W.\end{aligned} \tag{26}$$

By applying ϕ^2 on both sides of (26) and using (3), we obtain

$$\begin{aligned}\phi^2(\tilde{\nabla}_W \tilde{R})(X,Y)Z = \phi^2(\nabla_W R)(X,Y)Z - \alpha^2\{g(W,Y)\eta(Z)+g(W,Z)\eta(Y)\}\phi^2(\phi X)+\alpha^2\{g(W,X)\eta(Z)\\ +g(W,Z)\eta(X)\}\phi^2\phi Y+\alpha^2 g(\phi X,Z)\eta(Y)\phi^2 W-\alpha^2 g(\phi Y,Z)\eta(X)\phi^2 W\\ +\alpha^2 g(\phi W,Z)[\eta(Y)X-\eta(X)Y]+\alpha^2[g(\phi W,Y)X+g(\phi W,Y)\eta(X)\xi-g(\phi W,X)Y\\ -g(\phi W,X)\eta(Y)\xi]\eta(Z)+\alpha^2\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\phi^2\phi W.\end{aligned} \tag{27}$$

Assuming X,Y, Z and W are orthogonal to ξ , (27) reduces to,

$$\phi^2(\tilde{\nabla}_W \tilde{R})(X,Y)Z = \phi^2(\nabla_W R)(X,Y)Z \tag{28}$$

This completes the proof of the theorem.

4 Locally projectiveϕ-symmetric Lorentzian α-Sasakian manifold admitting quarter-symmetric metric connection

Definition 3. *An n-dimensional Lorentzian α-Sasakian manifold M is said to be locally projective ϕ -symmetric if*

$$\phi^2((\tilde{\nabla}_W P)(X,Y)Z) = 0, \tag{29}$$

for all vector fields X,Y,Z,W orthogonal to ξ and the projective curvature tensor P [20] is given by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1} - [S(Y,Z)X - S(X,Z)Y]. \tag{30}$$

Definition 4. *An n-dimensional Lorentzian α-Sasakian manifold M is said to be locally projective ϕ -symmetric with respect to quarter-symmetric metric connection if*

$$\phi^2((\tilde{\nabla}_W \tilde{P})(X,Y)Z) = 0 \tag{31}$$

for all vector fields X,Y,Z and W orthogonal to ξ. Here $\tilde{P}$ is the projective curvature tensor with respect to quarter-symmetric metric connection given by

$$\tilde{P}(X,Y)Z = \tilde{P}(X,Y)Z - \frac{1}{N-1}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y]. \tag{32}$$

With the help of (16), we can write

$$(\tilde{\nabla}_W\tilde{P})(X,Y)Z = (\nabla_W\tilde{P})(X,Y)Z + \eta(\tilde{P}(X,Y)Z)\phi W. \tag{33}$$

Theorem 2. *An n-dimensional Lorentzian α-Sasakian manifold M is locally projective ϕ-symmetric with respect to the quarter symmetric metric connection if and only if M is locally projective ϕ-symmetric with respect to the Levi-Civita connection.*

Proof. Differentiating (32) with respect to$W$, we get

$$(\tilde{\nabla}_W\tilde{P})(X,Y)Z = (\nabla_W\tilde{R})(X,Y)Z - \frac{1}{n-1}[(\nabla W\tilde{S})(Y,Z)X - (\nabla_W\tilde{S})(X,Z)Y]. \tag{34}$$

In view of equations (17) and (24), (34) reduces to

$$\begin{aligned}(\nabla_W\tilde{P})(X,Y)Z = &(\nabla_W R)(X,Y)Z - \alpha^2\{g(W,Y)\eta(Z) + g(W,Z)\eta(Y)\}\phi X\\ &+\alpha^2\{g(W,X)\eta(Z) + g(W,Z)\eta(X)\}\phi Y + \alpha^2\{g(\phi X,Z)g(W,Y)\xi\\ &+g(\phi X,Z)\eta(Y)W\} - \alpha^2\{g(\phi Y,Z)g(W,X)\xi + g(\phi Y,Z)\eta(X)W\}\\ &+\alpha^2 g(\phi W,Z)[\eta(Y)X - \eta(X)Y] + \alpha^2[g(\phi W,Y)X - g(\phi W,X)Y]\eta(Z)\\ &+\tfrac{\alpha^2 n}{n-1}[g(\varphi W,Z)\eta(Y)X - \eta(X)Y\} + g(\phi W,Y)Xg(\phi W,X)Y\eta(Z)]\\ &-\tfrac{1}{n-1}[(\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y].\end{aligned} \tag{35}$$

Substituting (32) in (35), we get

$$\begin{aligned}(\nabla_W\tilde{P})(X,Y)Z = &(\nabla_W P)(X,Y)Z - \alpha^2\{g(W,Y)\eta(Z) + g(W,Z)\eta(Y)\}\varphi X\\ &+\alpha^2\{g(W,X)\eta(Z) + g(W,Z)\eta(X)\}\phi Y + \alpha^2\{g(\phi X,Z)g(W,Y)\xi\\ &+g(\phi X,Z)\eta(Y)W\} - \alpha^2\{g(\phi Y,Z)g(W,X)\xi + g(\phi Y,Z)\eta(X)W\}\\ &+\alpha^2 g(\phi W,Z)[\eta(Y)X - \eta(X)Y] + \alpha^2[g(\varphi W,Y)X - g(\phi W,X)Y]\eta(Z)\\ &+\tfrac{\alpha^2 n}{n-1}[g(\phi W,Z)\{\eta(Y)X - \eta(X)Y\} + \{g(\phi W,Y)X - g(\phi W,X)Y\}\eta(Z)].\end{aligned} \tag{36}$$

By using the equations (16) and (17) in (32), we obtain

$$\begin{aligned}\tilde{P}(X,Y)Z = &R(X,Y)Z + \alpha[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X] - \tfrac{1}{n-1}[S(Y,Z)X - (X,Z)Y]\\ &-\tfrac{\alpha}{n-1}[g(Y,Z)X - g(X,Z)Y] - \tfrac{\alpha}{n-1}[\eta(Y)X - \eta(X)Y]\eta(Z).\end{aligned} \tag{37}$$

In virtue of (30), (37) becomes

$$\begin{aligned}\tilde{P}(X,Y)Z = &P(X,Y)Z + \alpha[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X] - \tfrac{\alpha}{n-1}[g(Y,Z)X - g(X,Z)Y]\\ &-\tfrac{\alpha}{n-1}[\eta(Y)X - \eta(X)Y]\eta(Z).\end{aligned} \tag{38}$$

Taking inner product of (37) with ξ and then using (3),(5) and (8), we get

$$\eta(\tilde{P}(X,Y)Z) = (\alpha^2 - \frac{\alpha}{n-1})[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] - \frac{1}{n-1}[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)]. \tag{39}$$

Now using equations (36) and (39), (33) reduces to

$$\begin{aligned}(\tilde{\nabla}_W \tilde{P})(X,Y)Z = {} & (\nabla_W P)(X,Y)Z - \alpha^2\{g(W,Y)\eta(Z) + g(W,Z)\eta(Y)\}\varphi X \\ & +\alpha^2\{g(W,X)\eta(Z) + g(W,Z)\eta(X)\}\phi Y + \alpha^2\{g(\phi X,Z)g(W,Y)\xi \\ & +g(\phi X,Z)\eta(Y)W\} - \alpha^2\{g(\phi Y,Z)g(W,X)\xi + g(\phi Y,Z)\eta(X)W\} \\ & +\alpha^2 g(\phi W,Z)[\eta(Y)X - \eta(X)Y] + \alpha^2[g(\phi W,Y)X - g(\phi W,X)Y]\eta(Z) \\ & +\tfrac{\alpha^2 n}{n-1}[g(\phi W,Z)\{\eta(Y)X - \eta(X)Y\} + \{g(\phi W,Y)X - g(\phi W,X)Y\}\eta(Z)] \\ & +(\alpha^2 - \tfrac{\alpha}{n-1})[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\phi W - \tfrac{1}{n-1}[S(Y,Z)\eta(X) \\ & -S(X,Z)\eta(Y)]\varphi W.\end{aligned} \tag{40}$$

By applying ϕ^2 on both sides of (40) and assuming that X,Y,Z and W are orthogonal to ξ , we get

$$\phi^2\left(\left(\tilde{\nabla}_W \tilde{P}\right)(X,Y)Z\right) = \phi^2\left((\nabla_W P)(X,Y)Z\right) \tag{41}$$

Hence the proof.

5 ϕ-Recurrent Lorentzian α-Sasakian manifold admitting quarter-symmetric metric connection

Definition 5. *An n-dimensional Lorentzian α-Sasakian manifold M is said to be ϕ-recurrent if there exists a non-zero 1-form A such that*

$$\phi^2\left((\nabla_W P)(X,Y)Z\right) = A(W)R(X,Y)Z, \tag{42}$$

for arbitrary vector fields X,Y,Z and M. If the 1-form A vanishes, then the manifold reduces to a ϕ-symmetric manifold [15].

Definition 6. *An n-dimensional Lorentzian α-Sasakian manifold M is said to be ϕ -recurrent with respect to quarter-symmetric metric connection if there exists a non-zero 1-form A such that*

$$\phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) = A(W)\tilde{R}(X,Y)Z, \tag{43}$$

for arbitrary vector fields X,Y,Z and W.

Theorem 3. *If a Lorentzian α-Sasakian manifold is ϕ-recurrent with respect to the quarter-symmetric metric connection then the manifold is a generalized η-Einstein manifold with respect to the Levi-Civita connection.*

Proof. Suppose M is ϕ-recurrent with respect to quarter-symmetric metric connection. Then in view of (3), we can write (43) as

$$(\tilde{\nabla}_W \tilde{R})(X,Y)Z + \eta((\tilde{\nabla}_W \tilde{R})(X,Y)Z)\xi = A(W)(\tilde{R}(X,Y)Z). \tag{44}$$

Taking inner product of (44) with U, we get

$$g((\tilde{\nabla}_W \tilde{R})(X,Y)Z,U) + \eta((\tilde{\nabla}_W \tilde{R})(X,Y)Z)\eta(U) = A(W)g(\tilde{R}(X,Y)Z,U). \tag{45}$$

Using (22) in (45), we get

$$g((\tilde{\nabla}_W \tilde{R})(X,Y)Z,U) + \eta((\tilde{\nabla}_W \tilde{R})(X,Y)Z)\eta(U) + \eta(\tilde{R}(X,Y)Z)g(\phi W,U) = A(W)g(\tilde{R}(X,Y)Z,U). \tag{46}$$

Making use of (16), (24) and (25) in (46), we obtain

$$\begin{aligned}&g((\nabla_W R)(X,Y)Z,U)-\alpha^2\{g(W,Y)\eta(Z)+g(W,Z)\eta(Y)\}g(\phi X,U)+\alpha^2\{g(W,X)\eta(Z)+g(W,Z)\eta(X)\}g(\phi Y,U)\\&+\alpha^2\{g(\phi X,Z)g(W,Y)\eta(U)+g(\phi X,Z)\eta(Y)g(W,U)\}-\alpha^2\{g(\phi Y,Z)g(W,X)\eta(U)+g(\phi Y,Z)\eta(X)g(W,U)\}\\&+\alpha^2 g(\phi W,Z)[\eta(Y)g(X,U)-\eta(X)g(Y,U)]+\alpha^2[g(\phi W,Y)g(X,U)-g(\phi W,X)g(Y,U)]\eta(Z)\\&+\eta((\nabla_W R)(X,Y)Z)\eta(U)+\alpha^2\{g(\phi X,Z)\eta(Y)\eta(W)-g(\phi X,Z)g(W,Y)\}\eta(U)-\alpha^2\{g(\phi Y,Z)\eta(X)\eta(W)\\&-g(\phi Y,Z)g(W,X)\}\eta(U)+\alpha^2\{g(\phi W,Y)\eta(Z)\eta(X)-g(\phi W,X)\eta(Y)\eta(Z)\}\eta(U)+\alpha^2\{g(Y,Z)\eta(X)\\&-g(X,Z)\eta(Y)\}g(\phi W,U)=A(W)g(R(X,Y)Z,U)+\alpha A(W)\{g(\phi X,Z)g(\phi Y,U)-g(\phi Y,Z)g(\phi X,U)\\&+g(X,U)\eta(Y)\eta(Z)-g(Y,U)\eta(X)\eta(Z)\}.\end{aligned}\tag{47}$$

Plugging $Z=\xi$ in (47) and then using equations (3) and (5), we get

$$\begin{aligned}&g((\nabla WR)(X,Y)\xi,U)+\alpha^2\{g(W,Y)-\eta(W)\eta(Y)\}g(\phi X,U)-\alpha^2\{g(W,X)\\&-\eta(W)\eta(X)\}g(\phi Y,U)-\alpha^2\{g(\phi W,Y)g(X,U)-g(\phi W,X)g(Y,U)\}\\&+\eta((\nabla WR)(X,Y)\xi)\eta(U)+\alpha^2\{g(\phi W,X)\eta(Y)-g(\phi W,Y)\eta(X)\}\eta(U)\\&=A(W)g(R(X,Y)\xi,U)-\alpha A(W)\{g(X,U)\eta(Y)-g(Y,U)\eta(X)\}.\end{aligned}\tag{48}$$

Taking $X=U=e_i$ in (48) and summing over $i, 1\le i\le n$, we get

$$\begin{aligned}&\nabla_W S(Y,\xi)+\sum_{i=1}^{n-1} g((\nabla_W R)(e_i,Y)\xi,\xi)g(e_i,\xi)+\alpha^2 trace\phi\{g(W,Y)-\eta(W)\eta(Y)\}\\&-\alpha^2(n-1)g(\phi W,Y)=A(W)S(Y,\xi)-\alpha(n-1)A(W)\eta(Y).\end{aligned}\tag{49}$$

We denote the second term of the left hand side of the above equation by G. In this case G vanishes. Therefore (49) reduces to

$$\begin{aligned}\nabla_W S(Y,\xi)&=-\alpha^2 trace\phi\{g(W,Y)-\eta(W)\eta(Y)\}+\alpha^2(n-1)g(\phi W,Y)\\&+A(W)S(Y,\xi)-\alpha(n-1)A(W)\eta(Y).\end{aligned}\tag{50}$$

We have

$$(\nabla_W S)(Y,\xi)=\nabla_W S(Y,\xi)-S(\nabla_W Y,\xi)-S(Y,\nabla_W\xi).$$

Using (6), (7) and (13) in the above equation, we find

$$(\nabla_W S)(Y,\xi)=(n-1)\alpha^3 g(\phi W,Y)-\alpha S(\alpha W,Y).\tag{51}$$

Equating (50) and (51), we obtain

$$\begin{aligned}(n-1)\alpha^3 g(\phi W,Y)-\alpha S(\phi W,Y)&=-\alpha^2 trace\phi\{g(W,Y)-\eta(W)\eta(Y)\}+\alpha^2(n-1)g(\phi W,Y)+A(W)S(Y,\xi)\\&-\alpha(n-1)A(W)\eta(Y).\end{aligned}\tag{52}$$

Replacing Y by ϕY in (52) and then using (4) and (14), we get

$$S(Y,W)=\lambda g(Y,W)+\mu\eta(Y)\eta(W)+vg(W,\phi Y),\tag{53}$$

where $\lambda=(n-1)(\alpha 2-\alpha), \mu=(1-n)\alpha$ and$v=\alpha trace\phi$. Thus, the proof of the theorem completes.

6 ϕ-projectively flat Lorentzian α-Sasakian manifold admitting quarter-symmetric metric connection

Definition 7. *An n-dimensional Lorentzian α -Sasakian manifold is said to be ϕ -projectively flat with respect to quarter-symmetric metric connection if it satisfies*

$$\phi^2(\tilde{P}(\phi X,\phi Y)\phi Z) = 0. \tag{54}$$

Let M be ϕ-projectively flat with respect to quarter-symmetric metric connection. Then it is easy to see that $\phi^2(\tilde{P}(\phi X,\phi Y)\phi Z) = 0$holds if and only if

$$g(\tilde{P}(\phi X,\phi Y)\phi W) = 0, \tag{55}$$

for every$X,Y,Z,W \in TM$.

Theorem 4. *An n-dimensional ϕ-projectively flat Lorentzian α-Sasakian manifold admitting the quarter-symmetric metric connection is a generalized η-Einstein manifold.*

Proof. Suppose M is ϕ-projectively flat with respect to quarter symmetric metric connection. Then we have from (32) that

$$g(\tilde{R}(\phi X,\phi Y)\phi Z,\phi W) = \frac{1}{n-1}[\tilde{S}(\phi Y,\phi Z)g(\phi X,\phi W) - \tilde{S}(\phi X,\phi Z)g(\phi Y,\phi W)]. \tag{56}$$

By using (3), (5), (16) and (17) in (56), we get

$$\begin{aligned} g(R(\phi X,\phi Y)\phi Z,\phi W) &+\alpha[g(X,\phi Z)g(Y,\phi W) - g(Y,\phi Z)g(X,\phi W)] = \tfrac{1}{n-1}[S(\phi Y,\phi Z)g(\phi X,\phi W) \\ &+\alpha g(\phi Y,\phi Z)g(\phi X,\phi W) - S(\phi X,\phi Z)g(\phi Y,\phi W) - \alpha g(\phi X,\phi Z)g(\phi Y,\phi W)]. \end{aligned} \tag{57}$$

Let $\{e_1,e_2,....,e_{n-1},\xi\}$ be a local orthonormal basis of vector fields in M. Using the fact that $\{\phi e_1,\phi e_2,....,\phi e_{n-1},\xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (57) and summing over i, we get

$$\begin{aligned} &\sum_{i=1}^{n-1} g(R(\phi e_i,\phi Y)\phi Z,\phi e_i) + \alpha[g(e_i,\phi Z)g(Y,\phi e_i) - g(Y,\phi Z)g(e_i,\phi e_i)] = \tfrac{1}{n-1}\sum_{i=1}^{n-1}[S(\phi Y,\phi Z)g(\phi e_i,\phi e_i) \\ &+\alpha g(\phi Y,\phi Z)g(\phi e_i,\phi e_i) - S(\phi e_i,\phi Z)g(\phi Y,\phi e_i) - \alpha g(\phi e_i,\phi Z)g(\phi Y,\phi e_i)]. \end{aligned} \tag{58}$$

It is verified that[22]

$$\sum_{i=1}^{n-1} g(R(\phi e_i,\phi Y)\phi Z,\phi e_i) = S(\phi Y,\phi Z) + g(\phi Y,\phi Z), \tag{59}$$

$$\sum_{i=1}^{n-1} S(\phi e_i,\phi e_i) = r - (n-1)\alpha^2, \tag{60}$$

$$\sum_{i=1}^{n-1} g(\phi e_i,\phi Z)S(\phi Y,\phi e_i) = S(\phi Y,\phi Z), \tag{61}$$

$$\sum_{i=1}^{n-1} g(\phi e_i,\phi e_i) = n-1, \tag{62}$$

$$\sum_{i=1}^{n-1} g(\phi e_i,\phi Z)g(\phi Y,\phi e_i) = g(\phi Y,\phi Z). \tag{63}$$

Making use of the equations (59)-(63), (58) becomes

$$S(\phi Y,\phi Z)=\{-\alpha-n+1\}g(\phi Y,\phi Z)+\alpha(n-1)trace\varphi g(Y,\phi Z). \tag{64}$$

Applying equations (4) and (14) in (64), we finally obtain

$$S(Y,Z)=\lambda g(Y,Z)+\mu\eta(Y)\eta(Z)+\nu g(Y,\phi Z), \tag{65}$$

where $\lambda=\{-\alpha-n+1\},\mu=\{(-(n-1)(\alpha 2+1)-\alpha\}$ and $\nu=\alpha(n-1)trace\varphi$. This completes the proof.

7 Example of a Lorentzian α-Sasakian manifold admitting quarter-symmetric metric connection

Example 1. In this section, we consider a 3-dimensional manifold $M=\{(x,y,z)\in R^3\}$, where (x,y,z) are the standard coordinates in R^3. We choose the vector fields

$$e_1=e^z\frac{\partial}{\partial y},e_2=e^z(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}),e_3=\alpha\frac{\partial}{\partial z},$$

which are linearly independent at each point of M and α is constant. Let g be a Lorentzian metric defined by

$$g(e_1,e_3)=g(e_2,e_3)=g(e_1,e_2)=0,$$
$$g(e_1,e_1)=g(e_2,e_2)=1,g(e_3,e_3)=-1.$$

Let η be a 1-form defined by

$$\eta(Z)=g(Z,e_3),$$

for any $Z\in\chi(M)$. Let ϕ be the $(1,1)$ tensor field defined by

$$\phi e_1=-e_1,\phi e_2=-e_2,\phi e_3=0.$$

Using the linearity of ϕ and g, we have

$$\eta(e_3)=-1,\phi^2 Z=Z+\eta(Z)e_3,g(\phi Z,\phi W)=g(Z,W)+\eta(Z)\eta(W),$$

For any $Z,W\in\chi(M)$. Then we have

$$[e_1,e_2]=0,[e_1,e_3]=-\alpha e_1,[e_2,e_3]=-\alpha e_2.$$

Using the Koszul's formula, we have

$$\nabla_{e_1}e_1=-\alpha e_1,\ \nabla_{e_1}e_2=0,\ \nabla_{e_1}e_3=-\alpha e_1,$$
$$\nabla_{e_2}e_1=0,\ \nabla_{e_2}e_2=-\alpha e_3,\ \nabla_{e_2}e_3=-\alpha e_2,$$
$$\nabla_{e_3}e_1=0,\ \nabla_{e_3}e_2=0,\ \nabla_{e_3}e_3=0.$$

By using (15) in the above equation, we obtain

$$\begin{aligned}&\tilde{\nabla}_{e_1} e_1 = -\alpha e_3,\ \tilde{\nabla}_{e_1} e_2 = 0,\ \tilde{\nabla}_{e_1} e_3 = (\alpha - 1)e_1,\\ &\tilde{\nabla}_{e_2} e_1 = 0,\ \tilde{\nabla}_{e_2} e_2 = -\alpha e_3,\ \tilde{\nabla}_{e_2} e_3 = (\alpha - 1)e_2,\\ &\tilde{\nabla}_{e_3} e_1 = 0,\ \tilde{\nabla}_{e_3} e_2 = 0,\ \tilde{\nabla}_{e_3} e_3 = 0.\end{aligned}$$

Using (1), the torsion tensor $\tilde{T}$ of the quarter symmetric metric connection $\tilde{\nabla}$ may be expressed as follows.

$$\tilde{T}(e_i, e_i) = 0, \text{ for } i = 1, 2, 3, \quad \tilde{T}(e_1, e_2) = 0, \tilde{T}(e_1, e_3) = e_1, \quad \tilde{T}(e_1, e_3) = e_2.$$

Also,

$$(\tilde{\nabla} e_1 g)(e_2, e_3) = (\tilde{\nabla} e_2 g)(e_3, e_1) = (\tilde{\nabla} e_3 g)(e_2, e_1) = 0.$$

Hence, we proved the existence of a Lorentzian α-Sasakian manifold admitting quarter symmetric metric connection through an example.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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