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Integral transform involving Bessel's function

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Abstract: The main object of this paper is to obtain an integral transform involving Bessel's function into Appell's function, which generalize a well known class of hypergeometric function of some Kampe' de Fe'riet, Srivastava function $F^{(3)}$, Appell's function F_2, F_4 and Horn function H_3. A number of known and new transformations are also discussed as the special cases of our main result.

Keywords: Bessel function, Appell's function, Horn function and Laplace transform.

1 Introduction

Many integral formulas involving product of Bessel functions have been presented and play an important role in several physical problems. In fact, Bessel functions are associated with a wide range of problems in divers areas of mathematical physics. These connections of Bessel functions with various other research areas have led many researchers to the field of special functions.

Erde'lyi [3] defined the Bessel functions in terms of confluent hypergeometric functions by the relation.

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}\, e^{-iz}\, {}_1F_1\left(\nu+\frac{1}{2};2\nu+1;2iz\right). \tag{1}$$

and Horn function H_3 [4] defined as

$$H_3[\alpha,\beta;\gamma;x,y] = \sum_{n=0}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_n x^m y^n}{(\gamma)_{m+n} m!n!}$$
$$|\, x \,| < r, |\, y \,| < s, \quad r+\left(\frac{s-1}{2}\right)^2 = \frac{1}{4}. \tag{2}$$

Srivastava and Karlsson [5] gave a reduction formula of Appell's double series F_2 into ${}_4F_3$ in the following form

$$F_2\left(\alpha,\beta,\beta';2\beta,2\beta';x,-x\right) = {}_4F_3\left(\begin{matrix}\frac{\alpha}{2},\frac{\alpha}{2}+\frac{1}{2},\frac{1}{2}(\beta+\beta'),\frac{1}{2}(\beta+\beta'+1); & \\ & x^2 \\ \beta+\frac{1}{2},\beta'+\frac{1}{2},\beta+\beta' \qquad ; & \end{matrix}\right). \tag{3}$$

Another transformations of F_2 is given by Bailey [1] involving F_2 and F_4 (see also [6])

$$F_2\left[a,b,b';2b,2b';2x,2y\right] = (1-x-y)^{-a}\, F_4\left[\frac{a}{2},\frac{a+1}{2};b+\frac{1}{2},b'+\frac{1}{2};\frac{x^2}{(1-x-y)^2},\frac{y^2}{(1-x-y)^2}\right], \tag{4}$$

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where F_4 is Appell's function [6] defined as

$$F_4\left[\alpha,\beta;\gamma,\delta;x,y\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\gamma')_n}\frac{x^m}{m!}\frac{y^n}{n!}. \tag{5}$$

Also, a general triple hypergeometric series $F^{(3)}[x,y,z]$ of Srivastava's [6] defined as

$$F^{(3)}\begin{bmatrix}(a)::(b);(b');(b'');(e);(e');\ (e'') & ; \\ & x,y,z \\ (d)::(g);(g');(g'');(h);(h');(h'') & ;\end{bmatrix}$$
$$=\sum_{m,n,p=0}^{\infty}\frac{[(a)]_{m+n+p}[(b)]_{m+n}[(b')]_{n+p}[(b'')]_{p+m}[(e)]_m[(e')]_n[(e'')]_p}{[(d)]_{m+n+p}[(g)]_{m+n}[(g')]_{n+p}[(g'')]_{p+m}[(h)]_m[(h')]_n[(h'')]_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \tag{6}$$

where (a) is the sequence of A parameters $a_1,a_2,.....a_A$ and

$$[(a)]_n=\prod_{j=1}^{A}(a_j)_m=\frac{\Gamma(a_j+n)}{\Gamma(a_j)}. \tag{7}$$

2 Main integral transform

In this section, we established an integral transform involving two Bessel functions, which are expressed in terms of Appell function F_2 as

$$\int_0^\infty t^{\lambda-1/2}e^{-pt}\ {}_1F_2\left(\begin{matrix}b & ; \\ c,d; & \end{matrix}-u^2t^2\right)J_{\nu_1}(\beta_1 t)J_{\nu_2}(\beta_2 t)dt=\frac{2^{-\nu_1-\nu_2}\beta_1{}^{\nu_1}\beta_2{}^{\nu_2}\Gamma(A+2k)}{\Gamma(\nu_1+1)\Gamma(\nu_2+1)(p+i(\beta_1+\beta_2))^{A+2k}}\sum_{k=0}^{\infty}\frac{(-1)^k(b)_k\,u^{2k}}{(c)_k(d)_k\,k!}$$
$$F_2\left[A+2k,\nu_1+\frac{1}{2},\nu_2+\frac{1}{2};2\nu_1+1,2\nu_2+1;\frac{2\beta_1 i}{p+i(\beta_1+\beta_2)},\frac{2\beta_2 i}{p+i(\beta_1+\beta_2)}\right] \tag{8}$$

where $A=\lambda+\nu_1+\nu_2+\frac{1}{2}$, $\mathrm{Re}(\mathrm{p}\pm\mathrm{i}(\beta_1+\beta_2)>0$, $\mathrm{Re(A)}>0$, and F_2 is Appell's function [6] defined by

$$F_2\left[a,b,b';c,c';x,y\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n x^m y^n}{(c)_m(c')_n\ m!n!};$$
$$|\,x\,|+|\,y\,|<1.$$

Equation (8) can be obtained by expanding ${}_1F_2$ in series and integrating term by term with the help of the integral transform [1].

In view of a known transformation of Bailey [1].

$$F_2\left[a,b,b';2b,2b';2x,2y\right]=(1-x-y)^{-a}F_4\left[\frac{a}{2},\frac{a+1}{2};b+\frac{1}{2},b'+\frac{1}{2};\frac{x^2}{(1-x-y)^2},\frac{y^2}{(1-x-y)^2}\right], \tag{9}$$

expanding F_4 in series and making use of the Legendre's duplication formula [6],

$$(a)_{2n}=2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{a+1}{2}\right)_n, n=0,1,2...$$

then equation (8) reduce to

$$\int_0^\infty t^{\lambda-1/2}e^{-pt}\ {}_1F_2\left(\begin{matrix} b & ; \\ c,d; \end{matrix} -u^2t^2\right) J_{\nu_1}(\beta_1 t)J_{\nu_2}(\beta_2 t)dt = \frac{2^{-\nu_1-\nu_2}\beta_1^{\nu_1}\ \beta_2^{\nu_2}\Gamma(A)}{\Gamma(\nu_1+1)\Gamma(\nu_2+1)p^A}$$
$$F^{(3)}\left[\begin{matrix} \frac{A}{2},\frac{A}{2}+\frac{1}{2} :: -;-;-;b;-;-; \\ - :: -;-;-;c,d;\nu_1+1,\nu_2+1; \end{matrix} \frac{-4u^2}{p^2}, \frac{-\beta_1^2}{p^2}, \frac{-\beta_2^2}{p^2}\right]. \tag{10}$$

where $A=\lambda+\nu_1+\nu_2+\frac{1}{2}, Re(A)>0, Re(p+i(\alpha+\beta))>0$.

3 Special cases

In this section, we obtained some special cases of the main transformation (8) are given below,

(i) On setting $\nu_1=\gamma, \nu_2=\delta, \beta_1=w, \beta_2=x$ and $p=2-i(w+x)$ in (8) and comparing it with equation (10), we get

$$\sum_{k=0}^\infty \frac{(b)_k(-1)^k u^{2k}(A)_{2k}}{(c)_k(d)_k\ k!(2)^{A+2k}} F_2\left[A+2k,\gamma+\frac{1}{2},\delta+\frac{1}{2};2\gamma+1,2\delta+1;wi,xi\right]$$
$$=(2-i(w+x))^{-A}\left[\begin{matrix} \frac{A}{2},\frac{A}{2}+\frac{1}{2} :: -;-;-;b;-;-; \\ - :: -;-;-;c,d;\gamma+1,\delta+1; \end{matrix}, \frac{-u^2}{(2-i(w+x))^2}, \frac{-w^2}{(2-i(w+x))^2}, \frac{-x^2}{(2-i(w+x))^2}\right]. \tag{11}$$

(ii) On setting $i=1$ and $x=-w$ in (11), we get

$$\sum_{k=0}^\infty \frac{(b)_k(-1)^k u^{2k}(A)_{2k}}{2^{A+2k}(c)_k(d)_k\ k!} F_2\left[A+2k,\gamma+\frac{1}{2},\delta+\frac{1}{2};2\gamma+1,2\delta+1;w,-w\right]$$
$$=2^{-A}\ F^{(3)}\left[\begin{matrix} \frac{A}{2},\frac{A}{2}+\frac{1}{2} :: -;-;-;b;-;-; \\ - :: -;-;-;c,d;\gamma+1,\delta+1; \end{matrix} -u^2, \frac{-w^2}{4}, \frac{-w^2}{4}\right]. \tag{12}$$

In view of a known transformation of Srivastava and Karlsson [5]

$$F_2\left[\alpha,\beta,\beta';2\beta,2\beta';x,-x\right] = {}_4F_3\left(\begin{matrix} \frac{\alpha}{2},\frac{\alpha}{2}+\frac{1}{2},\frac{1}{2}(\beta+\beta'),\frac{1}{2}(\beta+\beta'+1); \\ \beta+\frac{1}{2},\beta'+\frac{1}{2},\beta+\beta' \qquad ; \end{matrix} x^2\right), \tag{13}$$

equation (12) can be put in the form

$$\sum_{k=0}^\infty \frac{(b)_k(-1)^k u^{2k}(A)_{2k}}{2^{A+2k}(c)_k(d)_k\ k!}\ {}_4F_3\left(\begin{matrix} \frac{A}{2}+k,\frac{A}{2}+\frac{1}{2}+k,\frac{1}{2}(\gamma+\delta+1),\frac{1}{2}(\gamma+\delta+2); \\ \gamma+1,\delta+1,\gamma+\delta+1\ ; \end{matrix} w^2\right)$$
$$=2^{-A}\ F^{(3)}\left[\begin{matrix} \frac{A}{2},\frac{A}{2}+\frac{1}{2} :: -;-;-;b;-;-; \\ - :: -;-;-;c,d;\gamma+1,\delta+1; \end{matrix} -u^2, \frac{-w^2}{4}, \frac{-w^2}{4}\right]. \tag{14}$$

(iii) On setting $x=0$ and $b=c$ in (3.1), we get

$$\sum_{k=0}^\infty \frac{(-1)^k u^{2k}(A)_{2k}}{2^{A+2k}(d)_k\ k!}\ {}_2F_1\left(\begin{matrix} A+2k,\gamma+\frac{1}{2}\ ; \\ 2\gamma+1 \qquad ; \end{matrix} wi\right) = (2-iw)^{-A}\ F^{2:0:0}_{0:1:1}\left[\begin{matrix} \frac{A}{2},\frac{A}{2}+\frac{1}{2} : -;-; \\ - : d,\gamma+1 \quad ; \end{matrix} \frac{-4u^2}{(2-iw)^2}, \frac{-w^2}{(2-iw)^2}\right]. \tag{15}$$

Again, by using a known transformation of Srivastava and Karlsson [5]

$$F^{2:0:0}_{0:1:1} = F_4,$$

equation (15) reduces to

$$\sum_{k=0}^{\infty} \frac{(-1)^k u^{2k} (A)_{2k}}{2^{A+2k} (d)_k \, k!} \, {}_2F_1 \begin{pmatrix} A+2k, \gamma+\frac{1}{2} \; ; & \\ & wi \\ 2\gamma+1 \quad ; & \end{pmatrix} = (2-iw)^{-A} \, F_4 \left[\frac{A}{2}, \frac{A}{2}+\frac{1}{2}; d, \gamma+1; \frac{-4u^2}{(2-iw)^2}, \frac{-w^2}{(2-iw)^2} \right]. \quad (16)$$

(iv) On setting $u = iu, w = iw$ in (16) and using a known result of Srivastava and Karlsson [5],

$$F_4 \left[\frac{\alpha}{2}, \frac{\alpha}{2}+\frac{1}{2}; \gamma, \gamma'; \, x^2, y^2 \right] = \frac{1}{\Gamma\alpha} \int_0^\infty e^{-t} t^{\alpha-1} \, {}_0F_1 \left[\begin{matrix} -; & \frac{1}{4}x^2t^2 \\ \gamma; & \end{matrix} \right] \, {}_0F_1 \left[\begin{matrix} -; & \frac{1}{4}y^2t^2 \\ \gamma'; & \end{matrix} \right] dt, \quad (17)$$

where $Re(x) + Re(y) < 1, \; Re(\alpha) > 0$; equation (16) reduces to

$$\sum_{k=0}^{\infty} \frac{(-1)^{2k} u^{2k} (A)_{2k}}{2^{A+2k} (d)_k \, k!} \, {}_2F_1 \begin{pmatrix} A+2k, \gamma+\frac{1}{2} \; ; & \\ & -w \\ 2\gamma+1 \quad ; & \end{pmatrix}$$

$$= (2+w)^{-A} \frac{1}{\Gamma A} \int_0^\infty e^{-t} t^{A-1} \, {}_0F_1 \left[\begin{matrix} -; & \frac{u^2t^2}{(2+w)^2} \\ d; & \end{matrix} \right] \, {}_0F_1 \left[\begin{matrix} -; & \frac{1}{4} \frac{w^2t^2}{(2+w)^2} \\ \gamma+1; & \end{matrix} \right] dt \quad (18)$$

where $A = \lambda + \nu_1 + \nu_2 + \frac{1}{2}, \; Re(u) + Re(w) < 1$ and $Re(A) > 0$.

(v) On setting $u^2 = -u, w^2 = -w$ and $\nu + 1 = \frac{A}{2} + \frac{1}{2}$ in (16) and using a known result [5],

$$F_4[\alpha, \beta; \gamma, \beta; x, y] = (1-x-y)^{-\alpha} H_3 \left[\alpha, \gamma-\beta; \gamma; \frac{xy}{(x+y-1)^2}, \frac{x}{(x+y-1)} \right], \quad (19)$$

equation (16) reduces to

$$\sum_{k=0}^{\infty} \frac{(-1)^{2k} u^{k} (A)_{2k}}{2^{A+2k} (d)_k \, k!} \, {}_2F_1 \begin{pmatrix} A+2k, \frac{A}{2} \; ; & \\ & -\sqrt{w} \\ A \quad ; & \end{pmatrix}$$

$$= (4(1-u+\sqrt{w})^{\frac{-A}{2}} \; H_3 \left[\frac{A}{2}, d - \frac{A}{2} - \frac{1}{2}; d; \frac{4uw}{4(u-1-\sqrt{w})^2}, \frac{4u}{(u-1-\sqrt{w})} \right]. \quad (20)$$

(vi) On setting $u = \nu_2 = 0$ in equation (8), we get

$$\int_0^\infty t^{\lambda-1/2} e^{-pt} J_{\nu_1}(\beta_1 t) J_0(\beta_2 t) dt = \frac{2^{-\nu_1} \beta_1^{\nu_1} \Gamma(\lambda+\nu_1+\frac{1}{2})}{\Gamma(\nu_1+1)(p+i(\beta_1+\beta_2))^{\lambda+\nu_1+\frac{1}{2}}} \, {}_2F_1 \begin{pmatrix} \lambda+\nu_1+\frac{1}{2}, \nu_1+\frac{1}{2} \; ; & \\ & \frac{2\beta_1 i}{(p+i(\beta_1+\beta_2))} \\ 2\nu_1+1 \quad ; & \end{pmatrix}. \quad (21)$$

In the above equation (21), expressing $J_0(\beta_2)$ in terms of series and using the integral transform [2], we obtain

$$\int_0^\infty t^{\lambda-1/2} e^{-pt} J_{\nu_1}(\beta_1 t) J_0(\beta_2 t) dt = \frac{2^{-\nu_1} \beta_1^{\nu_1} \Gamma(\lambda+\nu_1+2r+\frac{1}{2})}{\Gamma(\nu_1+1) p^{\lambda+\nu_1+2r+\frac{1}{2}}} \sum_{r=0}^{\infty} \frac{(-1)^r (\beta_2)^{2r}}{2^{2r} (r!)^2} \, {}_2F_1 \begin{pmatrix} \frac{a}{2}+r, \; \frac{a+1}{2}+r \; ; & \\ & \frac{-\beta_1^2}{p^2} \\ \nu_1+1 \quad ; & \end{pmatrix}, \quad (22)$$

where $a = \lambda + \nu_1 + \frac{1}{2}$. Re(a) $> -$Re(β_1).

4 Series expansion

In this section, we present the series expansion of the main integral transform (8), which reduces to a general triple hypergeometric series $F^{(3)}$ introduced by Srivastava [6].

The generalized hypergeometric series of power t defined by [6].

$$\sum_{k=0}^{\infty} \frac{t^k}{k!}\ {}_{p+2}F_q \left(\begin{matrix} \frac{(-k)}{2}, \frac{(1-k)}{2}, (a_p) \ ; \\ (b_q) \qquad ; \end{matrix} \ u \right) = e^t \ {}_pF_q \left(\begin{matrix} (a_p); \\ (b_q); \end{matrix} \ \frac{t^2 u}{4} \right), \tag{23}$$

For $p = 1, q = 2$, equation (23) reduces

$$\sum_{k=0}^{\infty} \frac{t^k}{k!}\ {}_3F_2 \left(\begin{matrix} \frac{(-k)}{2}, \frac{(1-k)}{2}, (a_1) \ ; \\ (b_1), (b_2) \qquad ; \end{matrix} \ u \right) = e^t \ {}_1F_2 \left(\begin{matrix} (a_1) \qquad ; \\ (b_1), (b_2); \end{matrix} \ \frac{t^2 u}{4} \right). \tag{24}$$

Multiplying both the sides of (24) by $t^{\lambda-1/2}e^{-pt}J_{\nu_1}(\beta_1 t)J_{\nu_2}(\beta_2 t)$ and integrating term by term with the help of the result [3], we obtain

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(A)_k}{p^k k!} F^{(3)} \left[\begin{matrix} - :: \frac{1}{2}(A+k), \frac{1}{2}(A+k)+\frac{1}{2} : -; -; -; \frac{(-k)}{2}, \frac{(1-k)}{2}, a_1; \\ - :: -; -; \nu_1+1, \nu_2+1; b_1, b_2; \end{matrix} \ \frac{-\beta_1^2}{p^2}, \frac{-\beta_2^2}{p^2}, u \right] \\ &= \left(\frac{p}{p-1} \right)^A F^{(3)} \left[\begin{matrix} \frac{A}{2}, \frac{1}{2}(A+1) :: -; -; -; -; -; a_1 \ ; \\ - :: -; -; -; \nu_1+1, \nu_2+1; b_1, b_2; \end{matrix} \ \frac{-\beta_1^2}{(p-1)^2}, \frac{-\beta_2^2}{(p-1)^2}, \frac{u}{4(p-1)^2} \right]. \end{aligned} \tag{25}$$

If we set $u = 0$ in (25), we get

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(A)_k}{p^k k!} F^{(2)} \left[\begin{matrix} - :: \frac{1}{2}(A+k), \frac{1}{2}(A+k)+\frac{1}{2} : -; -; -; -; -; \\ - :: -; -; -; \nu_1+1, \nu_2+1; \end{matrix} \ \frac{-\beta_1^2}{p^2}, \frac{-\beta_2^2}{p^2} \right] \\ &= \left(\frac{p}{p-1} \right)^A F^{(2)} \left[\begin{matrix} \frac{A}{2}, \frac{1}{2}(A+1) : -; -; \\ - : \nu_1+1, \nu_2+1; \end{matrix} \ \frac{-\beta_1^2}{(p-1)^2}, \frac{-\beta_2^2}{(p-1)^2} \right]. \end{aligned} \tag{26}$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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