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A new approach for determination of boundary function for diffusion equation by fourier method

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Abstract: We consider a linear diffusion equation with a nonlocal boundary condition. We attempt to recover the boundary condition and the solution of diffusion equation for a problem by making use of an over-determination condition of integral type. Explicit solutions for these unknowns are derived by employing Fourier method. We obtain sufficient conditions for the existence and uniqueness of the solution and determination of boundary condition.

Keywords: Determination of boundary function, diffusion equation, inverse problem.

1 Introduction

Inverse problems have an important role in mathematics, physics, geophysics, medical imaging, astronomy and engineering. To find the unknown coefficient function, source function, initial and boundary conditions, some analytical methods and numerical methods are studied [15,10,14,6,5,9,11,22,23,16,17,4,3,20,21,1].

In this article, we are concerned with the linear diffusion equation

$$u_t(x,t) - u_{xx}(x,t) = 0, \quad (x,t) \in D_T \tag{1}$$

with initial and non-local boundary conditions

$$u(x,0) = \varphi(x), \quad x \in (0,1) \tag{2}$$

$$u(0,t) = u(1,t), \quad u_x(1,t) = g(t), \quad t \in (0,T) \tag{3}$$

where $D_T = (0,1) \times (0,T]$.

The inverse problem consists of the determination of boundary function $g(t)$ and the solution $u(x,t)$ from the initial condition under the over-determination condition

$$\int_0^1 u(x,t)dx = k(t). \tag{4}$$

It's well known that the equation (1) is used to explain various physical phenomenon. Moreover, non-local boundary conditions arise from many applications in science, see in [7,8,2,18].

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The solution of inverse problem is a pair of functions $\{u(x,t), g(t)\}$ where

$$u(.,t) \in C^2[(0,1),\mathbb{R}] \quad g(t) \in C^4[[0,T],\mathbb{R}].$$

We first define a new function $v(x,t)$ to make non-local boundary conditions equal to zero and we form corresponding boundary value problems.

Since we use the Fourier method, i.e. separation of variables, we have to consider the spectral problem. If the boundary-value problem consists of non-self adjoint operator, eigenvectors of the spectral problem are not complete in the space $L^2(0,1)$. By adding the associated eigenvectors we make the set of eigenvectors complete in $L^2(0,1)$. Moreover, we obtain another complete set of eigenvectors for the adjoint problem. These sets form a biorthogonal system for the space $L^2(0,1)$ and by using the biorthogonal system we expand the unknown functions into series to solve the inverse problem. By using the biorthogonal system, the inverse problem of determining unknown functions has been already considered in the literature, see for example [12,13,19].

In section 2, we transform the inverse problem (1-3) into another inverse problem by using convenient functions and we prove the existence and uniqueness of its solution. In section 3, we give an example to illustrate this study.

2 Existence and uniqueness of the solution for the inverse problem

Let us first consider the following inverse problem:

$$w_t(x,t) - w_{xx}(x,t) = F(x,t) \quad (x,t) \in D_T \tag{5}$$

$$w(x,0) = \psi(x) \quad x \in (0,1) \tag{6}$$

$$w(0,t) = w(1,t) \quad w_x(1,t) = 0 \quad t \in (0,T) \tag{7}$$

where $w \in C^2[(0,1),\mathbb{R}]$ and $F \in C^4[D_T,\mathbb{R}]$ are unknown which will be determined under the over-determination condition

$$\int_0^1 w(x,t)dx = k(t).$$

For this problem, we have the following non-selfadjoint spectral problem

$$X'' = -\lambda X \quad X \in (0,1)$$
$$X(0) = X(1) \quad X'(1) = 0$$

which has the adjoint problem:

$$Y'' = -\lambda Y \quad Y \in (0,1)$$
$$Y'(0) = Y'(1) \quad Y(0) = 0 \ .$$

The sets of functions

$$S = \{2, \{4\cos(2\pi nx)\}_{n=1}^{\infty}, \{4(1-x)\sin(2\pi nx)\}_{n=1}^{\infty}\} \tag{8}$$

and

$$\widetilde{S} = \{x, \{x\cos(2\pi nx)\}_{n=1}^{\infty}, \{\sin(2\pi nx)\}_{n=1}^{\infty}\} \tag{9}$$

are obtained as the solutions of spectral problem and its adjoint problem respectively. The sets of functions (8) and (9) are complete in $L_2(0,1)$ and each of them forms a Riesz basis in $L_2(0,1)$. Furthermore, the sets of functions (8) and (9) constitute a biorthogonal system [19].

In this method, the solvability of the inverse problem is based on the expansion of the solution $u(x,t)$ by employing the biorthogonal system of functions obtained from the spectral problem and its adjoint problem.

Theorem 1. *Suppose that the following conditions hold.*

(1) $\psi \in C^4([0,1])$, $\psi(1) = \psi(0)$, $\psi'(1) = 0$, $\psi''(1) = \psi''(0)$, $\psi'''(1) = 0$.
(2) $F \in C^4([D_T, \mathbb{R}])$, $F(0,t) = F(1,t)$, $F_x(1,t) = 0$, $F_{xx}(0,t) = F_{xx}(1,t)$, $F_{xxx}(1,t) = 0$, $\int_0^1 F(x,t)dx \neq 0$ *and*

$$0 < \frac{1}{M} < \left| \int_0^1 F(x,t)dx \right|$$

(3) $k \in AC([0,T])$, *and* $k(t)$ *satisfies the consistency condition* $\int_0^1 \psi(x)dx = k(0)$, *then the inverse problem (5-7) has a unique solution.*

Proof. See in [19].

We give a new theorem for the existence and uniqueness of the solution of the inverse problem (1-3). In this theorem, we define a new function to construct a relation between the system (1-3) and the system (5-7).

Theorem 2. *Let* $\varphi \in C^4([0,1])$ *and* $k \in AC([0,T])$ *be such that*

$$\varphi(1) = \varphi(0), \varphi'(1) = g(0), \varphi''(1) = \varphi''(0), \varphi'''(1) = g'(0), \varphi^{v}(1) = g''(0), \int_0^1 \varphi(x)dx = k(0). \tag{10}$$

Then $u(.,t) \in C^2([0,1])$ *and* $g(t) \in C^4([0,T])$ *form the unique classical solution and boundary of (1-3), respectively.*

Proof. Let us define the following function,

$$v(x,t) = u(x,t) - g(t)r(x) - g'(t)s(x) - g''(t)a(x) \tag{11}$$

where $r(x) = x(x-1)$, $s(x) = \frac{1}{12}x^2(x-1)^2$ and $a(x) = \frac{1}{720}x^2(x-1)^2(2x^2-2x-1)$.

The problem (1-3) is rewritten in terms of $v(x,t)$ as follows,

$$v_t(x,t) - v_{xx}(x,t) = 2g(t) + \frac{1}{6}g'(t) - \frac{1}{360}g''(t) - g'''(t)a(x) \tag{12}$$

$$v(x,0) = \varphi(x) - \varphi'(1)a(x) - \varphi'''(1)s(x) - \varphi^{v}(1)a(x) \tag{13}$$

$$v(0,t) = v(1,t) \tag{14}$$

$$v_x(1,t) = 0 \tag{15}$$

It is clearly shown that

$$\widetilde{F}(x,t) = 2g(t) + \frac{1}{6}g'(t) - \frac{1}{360}g''(t) - g'''(t)a(x)$$

and

$$\widetilde{\psi}(x) = \varphi(x) - g(0)a(x) - g'(0)s(x) - g''(0)a(x)$$

satisfy the conditions (1), (2) and (3). Thus the inverse problem has a unique solution $\{v(x,t), \widetilde{F}(x,t)\}$ based on Theorem 1.

As a second step, we show that if the inverse problem (12-15) has a unique solution $\{v(x,t),\widetilde{F}(x,t)\}$, then the inverse problem (1-3) under the over-determination condition (4) has a unique solution $\{u(x,t),g(t)\}$.

We first write the series expansion of solution of problem $v(x,t)$ and the right hand side function $\widetilde{F}(x,t)$ in the basis (8)

$$v(x,t) = 2V_0(t) + 4\sum_{n=1}^{\infty} V_n(t)\cos(2\pi nx) + 4\sum_{n=1}^{\infty} \widetilde{V}_n(t)(1-x)\sin(2\pi nx) \tag{16}$$

$$\widetilde{F}(x,t) = 2f_0(t) + 4\sum_{n=1}^{\infty} f_n(t)\cos(2\pi nx) + 4\sum_{n=1}^{\infty} \widetilde{f}_n(t)(1-x)\sin(2\pi nx)$$

where $V_0(t),\ V_n(t),\ \widetilde{V}_n(t),\ f_0(t),\ f_n(t)$ and $\widetilde{f}_n(t)$ are unknown functions. By using the biorthogonality of sets S and $\widetilde{S}$, we obtain the unknown functions $f_0(t),\ f_n(t)$ and $\widetilde{f}_n(t)$ as follows:

$$f_0(t) = \int_0^1 \widetilde{F}(x,t)x\,dx = g(t) - \frac{g'(t)}{12} - \frac{g''(t)}{720} - \frac{g'''(t)}{30240}$$

$$f_n(t) = \int_0^1 \widetilde{F}(x,t)x\cos(2\pi nx)dx = -\frac{g'''(t)}{(2\pi n)^6}$$

$$\widetilde{f}_n(t) = \int_0^1 \widetilde{F}(x,t)\sin(2\pi nx)dx = 0$$

where $(f,g) = \int_0^1 f(x)g(x)dx$ is the scalar product in $L^2(0,1)$. In addition, we have derivatives of $v(x,t)$ as follows:

$$v_t(x,t) = 2V_0'(t) + 4\sum_{n=1}^{\infty} V_n'(t)\cos(2\pi nx) + 4\sum_{n=1}^{\infty} \widetilde{V}_n'(t)(1-x)\sin(2\pi nx)$$

$$v_{xx}(x,t) = -4\pi\sum_{n=1}^{\infty}[\pi n^2 V_n(t) + n\widetilde{V}_n(t)]4\cos(2\pi nx) - 4\pi^2\sum_{n=1}^{\infty} n^2\widetilde{V}_n(t)4(1-x)\sin(2\pi nx).$$

Hence, using properties of the biorthogonal system, the following ordinary differential equations are obtained as follows.

$$V_0'(t) = g(t) - \frac{g'(t)}{12} - \frac{g''(t)}{720} - \frac{g'''(t)}{30240} \tag{17}$$

$$V_n'(t) + 4\pi^2 n^2 V_n(t) + 4\pi n\widetilde{V}_n(t) = -\frac{g'''(t)}{(2\pi n)^6} \tag{18}$$

$$\widetilde{V}_n'(t) + 4\pi^2 n^2\widetilde{V}_n(t) = 0. \tag{19}$$

From initial condition (13), we have the following initial conditions of the ordinary differential equations respectively.

$$V_0(0) = \varphi_0 + \frac{\varphi'(1)}{12} - \frac{\varphi'''(1)}{720} + \frac{\varphi^{v}(1)}{30240} \tag{20}$$

$$V_n(0) = \varphi_n - \frac{\varphi'(1)}{4\pi^2 n^2} + \frac{3\varphi'''(1)}{4\pi^4 n^4} - \frac{45\varphi^{v}(1)}{4\pi^6 n^6} \tag{21}$$

$$\widetilde{V}_n(0) = \widetilde{\varphi}_n. \tag{22}$$

where φ_0, φ_n and $\widetilde{\varphi}_n$ are the coefficients of the series expansion of initial condition $\varphi(x)$ in the basis (8).

The solution of ordinary differential equation (19) with initial condition (22) is obtained as

$$\widetilde{V}_n(t) = \widetilde{\varphi}_n e^{-4\pi^2 n^2 t}. \tag{23}$$

Moreover, the over-determination condition (4) and definition (11) leads to the following condition

$$\int_0^1 v(x,t)dx = k(t) + \frac{1}{6}g(t) - \frac{1}{360}g'(t) + \frac{1}{15120}g''(t).$$

We obtain unknown function $g(t)$ by using equation (12) as follows:

$$\int_0^1 (\widetilde{F}(x,t) + v_{xx}(x,t))dx = k'(t) + \frac{1}{6}g'(t) - \frac{1}{360}g''(t) + \frac{1}{15120}g'''(t) \tag{24}$$

$$2g(t) + \frac{1}{6}g'(t) - \frac{1}{360}g''(t) + \frac{1}{15120}g'''(t) + v_x(1,t) - v_x(0,t) = k'(t) + \frac{1}{6}g'(t) - \frac{1}{360}g''(t) + \frac{1}{15120}g'''(t) \tag{25}$$

$$g(t) = \frac{1}{2}(k'(t) + v_x(0,t))$$

$$g(t) = \frac{1}{2}k'(t) + 4\pi \sum_{n=1}^{\infty} \widetilde{\varphi}_n n e^{-4\pi^2 n^2 t} \tag{26}$$

where the series $\sum_{n=1}^{\infty} \widetilde{\varphi}_n n e^{-4\pi^2 n^2 t}$ is uniformly convergent by Weierstrass-M test.

Moreover, the solution of ordinary differential equation (17) is obtained as follows,

$$V_0(t) = \int_0^t P(\xi)d\xi + V_0(0) \tag{27}$$

where $P(t) = g(t) - \frac{g'(t)}{12} - \frac{g''(t)}{720} - \frac{g'''(t)}{30240}$. These results allow us to rewrite the ordinary differential equation (18) as follows:

$$V_n'(t) + 4\pi^2 n^2 V_n(t) = R_n(t)$$

where $R_n(t) = -4\pi n \widetilde{\varphi}_n e^{-4\pi^2 n^2 t} - \frac{g'''(t)}{(2\pi n)^6}$ which has the solution

$$V_n(t) = e^{-4\pi^2 n^2 t}\left(V_n(0) + \int_0^t e^{4\pi^2 n^2 \xi} R_n(\xi)d\xi\right). \tag{28}$$

Consequently, from definition (11), the inverse problem (1-3) has a unique solution $\{u(x,t), g(t)\}$.

3 Examples

Let's consider the inverse problem of determining unknown functions $g(t)$ and $u(x,t)$ in (1- 3) with initial condition $\varphi(x) = \sin(2\pi x)$, under over-determination $k(t) = \int_0^1 u(x,t)dx = 0$. The coefficients of the series expansion of initial condition $\varphi(x)$ in the basis (8) are determined as follows.

$$\varphi_0 = \int_0^1 \sin(2\pi x)x dx = -\frac{1}{2\pi}$$

$$\varphi_n = \int_0^1 \sin(2\pi x)x\cos(2\pi nx)dx = \frac{1-n^2}{2\pi(n^4 - n^2 + 1)}$$

$$\widetilde{\varphi}_n = \int_0^1 \sin(2\pi x)\sin(2\pi nx)dx = \begin{cases} \frac{1}{2}, & n = 1 \\ 0, & n \geq 2 \end{cases}$$

Thus unknown function $g(t)$ is obtained from equation (26) as follows:

$$g(t) = 2\pi e^{-4\pi^2 t}.$$

The components of the series expansion of $v(x,t)$ are obtained from equations (23), (27) and (28) as follows:

$$V_0 = \frac{-2\pi^6 + 21\pi^4 - 315\pi^2 - 945}{1890\pi e^{4\pi^2 t}} + \frac{\pi}{3}$$

$$V_n = \begin{cases} \frac{-733}{2\pi} e^{-4\pi^2 t}, & n = 1 \\ e^{-4\pi^2 n^2 t}\left[\frac{1-n^2}{2\pi(n^4-n^2+1)} - \frac{1}{2\pi n^2} - \frac{6}{\pi n^4} - \frac{360}{\pi n^6} + \frac{2\pi}{n^6}\left(\frac{e^{4\pi^2 n^2 t}}{4\pi^2 n^2 e^{4\pi^2 t} - 4\pi^2 e^{4\pi^2 t}} - \frac{1}{4\pi^2 n^2}\right)\right], & n \geq 2 \end{cases}$$

$$\widetilde{V}_n = \begin{cases} \frac{1}{2} e^{-4\pi^2 t}, & n = 1 \\ 0, & n \geq 2 \end{cases}$$

Thus from equation (16), we have following the series expansion

$$v(x,t) = \frac{-2\pi^6 + 21\pi^4 - 315\pi^2 - 945}{945\pi e^{4\pi^2 t}} + \frac{2\pi}{3} - \frac{1466}{\pi} e^{-4\pi^2 t}\cos(2\pi x) + 2e^{-4\pi^2 t}(1-x)\sin(2\pi x) + 4\sum_{n=2}^{\infty} V_n \cos(2\pi n x)$$

and the unknown function $u(x,t)$ is obtained from definition (11) as follows:

$$u(x,t) = v(x,t) + 2\pi e^{-4\pi^2 t}(x^2 - x)(1 - \frac{1}{3}\pi^2 x(x-1) + \frac{1}{45}x(x-1)(2x^2 - 2x - 1))$$

After some arrangements, we obtain the solution $u(x,t)$ as follows:

$$u(x,t) = e^{-4\pi^2 t}\sin(2\pi x).$$

Conclusion

The system (1-3) can be transform to the system (5-7) by equation (11) easily. It is shown that, the system constructed by the equation (11) satisfies the conditions $(A1)$, $(A2)$, $(A3)$ if the system (5-7) satisfies the conditions (10). Thus, we determine the pair of functions $\{g(t), u(x,t)\}$, i.e.,the boundary condition and the solution of diffusion equation by employing Fourier method under an over-determination condition of integral type. As we proved in theorem 2, this inverse problem has a unique solution and the solutions are obtained explicitly by using the biorthogonal systems which are obtained from non-selfadjoint spectral problem.

In the future, we plan to apply this method to various inverse problems.

References

[1] D. Mantzavinos, A.S. Fokas, The unified method for the heat equation: I. non-separable boundary conditions and non-local constraints in one dimension, Euro. J. Appl. Math. 24, (2013), 857–886.

[2] E. K. Lenzi, H. V. Ribeiro, J. Martins, M. K. Lenzi, G. G. Lenzi, S. Specchia; Non-Markovian diffusion equation and diffusion in a porous catalyst, Chemical Engineering Journal 172, (2011), 1083–1087.

[3] G. Freiling, V.A. Yurko, Inverse problems for Sturm–Liouville differential operators with a constant delay, Appl. Math. Lett. 25, (2012),1999–2004.

[4] G. Özkum, A. Demir, S. Erman, E. Korkmaz, B. Özgür, On the Inverse Problem of the Fractional Heat-Like Partial Differential Equations: Determination of the Source Function, Advances in Mathematical Physics 2013,(2013), 1–8.

[5] G. Wei, X. Wei, A generalization of three spectra theorem for inverse Sturm-Liouville problems, App. Math. Lett. 35, (2014) 41-45.

[6] J. Cannon, Determination of an unknown heat source from overspecified boundary data, SIAM J. Numer. Anal. 2, (1968), 275–286.

[7] J. R. Cannon, S. P. Esteva, J. V. D. Hoek, A Galerkin procedure for the diffusion subject to the specification of mass, SIAM J. Numer. Anal. 24, No. 3,(1987),499–515.

[8] J. R. Cannon, Y. Lin, S. Wang, Determination of a control parameter in a parabolic partial differential equation, J. Austral. Math. Soc. Ser. B. 33,(1991),149–163.

[9] K.M. Levere, H. Kunze, D. Torre, A collage-based approach to solving inverse problems for second-order nonlinear parabolic PDEs, J. Math. Anal. Appl. 406 ,no. 1,(2013) , 120–133.

[10] L.O.C. Pérez, J.P. Zubell, On the Inverse Problem for Scattering of Electromagnetic Radiation by a Periodic Structure, Studies in Applied Mathematics 111,(2003), 115–166.

[11] M.C. Drignei, Uniqueness of solutions to inverse Sturm–Liouville problems with $L^2(0,a)$ image potential using three spectra, Adv. in Appl. Math. 42, (2009), 471–482.

[12] M. Kirane, S.A. Malik, and M. A. Al-Gwaiz, An inverse source problem for a two dimensional time fractional diffusion equation with nonlocal boundary conditions, Math. Meth. Appl. Sci. 36, (2013), 1056–1069.

[13] M. Kirane, S.A. Malik, Determination of an unknown source term and the temperature distribution for the linear heat equation involving fractional derivative in time, Appl. Math.Comp. 218, (2011), 163–170.

[14] M. Mierzwiczak, J.A. Kolodziej, Application of the method of fundamental solutions and radial basis functions for inverse transient heat source problem , Comput. Phys. Commun. 12, (2010), 2035–2043.

[15] P. DuChateau, An introduction to inverse problem in partial differential equations for engineers, physicists and mathematicians, In Proceedings of workshop on parameter identification and inverse problems in hydrology, geology and ecology. Edited by J. Gottlieb, P. DuChateau, Kluwer Academic Publishers, The Netherlands,(1995), 3–38.

[16] P.J. Bendeich, J.M. Barry, W. Payten, Determination of specific heat with a simple inverse approach, Appl. Math. Modelling. 27, (2003), 337–344.

[17] R. Zolfaghari, A. Shidfar, Reconstructing an unknown time-dependent function in the boundary conditions of a parabolic PDE, Appl. Math. Comput. 226, (2014), 238–249.

[18] S. Wang, M. Zhao, X. Li, Radial anomalous diffusion in an annulus, Physica A. 390, (2011), 3397–3403.

[19] T.S. Aleroev, M. Kirane, S.A. Malik, Determination of a source term for a time fractional diffusion equation with an integral type over-determining condition, E.J.D.E. 270,(2013), 1–16.

[20] V. Yurko, An inverse problem for Sturm–Liouville differential operators on A-graphs, Appl. Math. Lett. 23,(2010),875–879.

[21] Y.M. Kwon, W.K.Park, Analysis of subspace migrations in limited-view inverse scattering problems, Appl. Math. Lett. 26, (2013), 1107–1113.

[22] Z.C. Deng, L. Yang, N. Chen, Uniqueness and stability of the minimizer for a binary functional arising in an inverse heat conduction problem, J. Math. Anal. Appl. 382, (2011), no. 1 474–486.

[23] Z. Sun, Inverse boundary value problems for a class of semilinear elliptic equations, Adv. in Appl. Math. 32, (2004), 791–800.