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Solution of integral and Integro-Differential equations system using Hybrid orthonormal Bernstein and block-pulse functions

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Abstract: This article introduces a numerical method based on an $M(n+1)$ set of general, hybrid orthonormal Bernstein functions coupled with Block-Pulse Functions(HOBB) on the interval [0,1] for approximating solutions of a Coupled System of linear and non linear Volterra integral and Integro-Differential equations. This method reduces a Coupled System of Volterra integral and Integro-Differential equations to a system of algebraic equations. Three numerical examples are illustrated by this method.

Keywords: Orthonormal Bernstein functions, Block-pulse functions, a Coupled System of linear and non linear Volterra , Volterra Integro-Differential equations, integration of the cross product, product matrix, coefficient matrix.

1 Introduction

Systems of integral equations, linear or nonlinear, appear in scientific applications in engineering, physics, chemistry and populations growth models[1-5]. Studies of systems of integral equations have attracted much concern in applied sciences. Volterra studied the hereditary influences when he was examining a population growth model. The research resulted in a specific topic, where both differential and integral operators appeared together in the same equation. This first new type of equation is named as Volterra integro- differential equation, given in the form,

$$y^{i}(x) = f(x) + \int_{0}^{x} k(x,t)\,y(t)\,dt, \tag{1}$$

and the second new type of equation is named as Fredholm integro - differential equation, given in the form,

$$y^{i}(x) = f(x) + \int_{0}^{1} k(x,t)\,y(t)\,dt, \tag{2}$$

where $k(x,t)$ a function of two variables x and t, is called the kernel. In this paper, /we will study systems of Volttera integro-differential equations given by

$$\frac{dy_i(x)}{dx} = f_j(x) + \sum_{j=1}^{n} p_{ij}(x)\,y_j(x) + \int_{0}^{x} \sum_{j=1}^{n} k_{ij}(x,t)\,y_j(t)\,dt, \tag{3}$$

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and systems of Fredholm integro-differential equations given by

$$\frac{dy_i(x)}{dx} = f_j(x) + \sum_{j=1}^{n} p_{ij}(x) y_j(x) + \int_0^1 \sum_{j=1}^{n} k_{ij}(x,t) y_j(t)\, dt, \tag{4}$$

where

$$y_i(0) = \alpha_i \, , i = 1, \dots, \; n.$$

The functions $f_j(x)$ are given real valued functions and unknown functions $y_1(x), y_2(x), \dots\dots, \; y_i(x)$ will be determined. Many physical phenomena may be modeled by a system of integro-differential equations. Lots of work has been done on nonlinear integro-differential equations using pulse functions and Legendre polynomials [6-10], as well as a recent work using this technique to solve higher dimensional problem [11]. Also different methods were used to approximate its solutions such as Chebyshev wavelets method, Galerkin method or the modified decomposition method, see [12-15]. We suppose that system (3,4) have a unique solution. The necessary and sufficient conditions for existence and uniqueness of the solution of system (3,4) could be found in [16].

This paper is organized as follows. In Section 2, we introduce HOBB functions and its properties. In Section 3, we apply these sets of HOBB functions for approximating the solution of system of linear volterra integral equations, In Section 4, we apply these sets of HOBB functions for approximating the solution of system of non linear integro-differential equations. Numerical results are reported in Section 5. Finally, Section 6 concludes the paper.

2 HOBB functions and some of their properties

Hybrid Orthonormal Bernstein and Block-Pulse Functions $\text{HOBB}_{i,j}(x)$, $i = 1, 2, \dots M$, $j = 0, 1, 2, \dots n$where $i$is the order for Block-Pulse functions, j is the order for Orthonormal Bernstein polynomials and $x$is the normalized time, is defined on the interval [0,1) as follows

$$\text{HOBB}_{i,j}(x) = \begin{cases} B_{j,n}(Mx - i + 1), \frac{i-1}{M} \le x < \frac{i}{M} \\ 0, otherwise \end{cases} \tag{5}$$

A set of Block-Pulse functions $b_i(x)$,$i = 1, \; 2, \; \dots M$on the interval [0,1) is defined as

$$b_i(x) = \begin{cases} 1, \frac{i-1}{M} \le x < \frac{i}{M} \\ 0, elsewhere \end{cases} \tag{6}$$

The Block-Pulse functions on [0,1) are disjoint, so for $i, j = 1, 2, \dots M$,we have $b_i(x)\, b_j(x) = \delta_{ij}\, b_i(x)$also these functions have the property of orthogonality on [0,1).

$\text{HOBB}_{i,j}(x)$ is the combination of orthonormal Bernstein polynomials and Block-Pulse functions which are both complete and orthogonal, then the set of orthonormal Bernstein and Block-Pulse functions is a complete orthogonal system in $L^2[0, 1)$.

2.1 Function expansion

Any function $u(x) \in L^2[0, 1)$ can be expanded in a hybrid orthonormal Bernstein and Block-Pulse functions

$$u(x) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} c_{ij} \text{HOBB}_{ij}(x), \tag{7}$$

where the hybrid orthonormal Bernstein and Block-Pulse coefficients are given by

$$c_{ij} = \frac{(u(x), \mathrm{HOBB}_{ij}(x))}{(\mathrm{HOBB}_{ij}(x), \mathrm{HOBB}_{ij}(x))}$$

for $i = 1,2,...,\infty,\ j = 1,2,...,\infty$, such that $(\,.\,,\,.\,)$ denotes the inner product. Usually, the series expansion Eq. (7) contains an infinite number of terms for a smooth $u(x)$. If $u(x)$ is piecewise constant or may be approximated as piecewise constant, then the sum in Eq. (7) may be terminated after nm terms, that is

$$u(x) \cong \sum_{i=1}^{M}\sum_{j=0}^{n} c_{ij}\mathrm{HOBB}_{ij}(x) = C^T\mathrm{HOBB}(x) \tag{8}$$

where

$$C = [c_{1,0}, c_{1,1}, ..., c_{M,n}]^T,$$

$$\mathrm{HOBB}(x) = [\mathrm{HOBB}_{1,0}, \mathrm{HOBB}_{1,1}, ..., \mathrm{HOBB}_{M,n}]^T$$

We can also approximate the function $k(x,t) \in L^2[0,\,1) \times [0,\,1)$ as follows $k(x,t) \approx \mathrm{HOBB}^T(x)K\,\mathrm{HOBB}(t)$, where K is an $M(n+1) \times M(n+1)$ matrix that we can obtain as,

$$K_{ij} = \frac{(\mathrm{HOBB}_i(x), (k(x,t), \mathrm{HOBB}_j(t))}{(\mathrm{HOBB}_i(x), \mathrm{HOBB}_i(x))(\mathrm{HOBB}_j(t), \mathrm{HOBB}_j(t))} \tag{9}$$

for $i, j = 1,\,2,\,...,\,Mn$.

2.2 Operational matrix of integration

The integration of the vector $\mathrm{HOBB}(x)$defined in Eq. (5) is given by

$$\int_0^x \mathrm{HOBB}(\tau')\,d(\tau') \approx P\,\mathrm{HOBB}(x), \tag{10}$$

where$P$is the$M(n+1) \times M(n+1)$ operational matrix for integration is given in [17-18] as,

$$\mathrm{P} = \begin{bmatrix} \mathrm{H} & \mathrm{G} & G & \cdots & G \\ 0 & \mathrm{H} & G & \cdots & G \\ 0 & 0 & H & \cdots & G \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & H \end{bmatrix} \tag{11}$$

that H and $G$are $M \times M$matrices that have the following shapes

$$\mathrm{G} = \frac{1}{\mathrm{M(n+1)}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{12}$$

and H is the operational matrix of integration and can be obtained as

$$\mathrm{H} = \frac{1}{2\mathrm{M}(\mathrm{n}+1)} \begin{bmatrix} \frac{1}{35} & \frac{263}{105} & \frac{263}{105} & \frac{71}{35} \\ \frac{-3}{35} & \frac{17}{35} & \frac{87}{35} & \frac{67}{35} \\ \frac{3}{35} & \frac{-17}{35} & \frac{53}{35} & \frac{73}{35} \\ \frac{-1}{35} & \frac{17}{105} & \frac{-53}{105} & \frac{69}{35} \end{bmatrix}. \quad (13)$$

2.3 The integration of the cross product

The integration of the cross product of two HOBB function vectors in Eq. (5) can be obtained as

$$D = \int_0^1 \mathrm{HOBB}(x)\,\mathrm{HOBB}^T(x)\, d(x) \approx \begin{bmatrix} \mathrm{L} & 0 \cdots & 0 \\ 0 & L \cdots & 0 \\ \cdots & \cdots \ddots & \cdots \\ 0 & 0 \cdots & L \end{bmatrix} \quad (14)$$

where L is an $M \times (n+1)$diagonal matrix given by

$$\mathrm{L} = \frac{1}{\mathrm{M}(n+\mathrm{M})} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{5} & \frac{1}{20} \\ \frac{1}{2} & \frac{3}{5} & \frac{9}{20} & \frac{1}{5} \\ \frac{1}{5} & \frac{9}{20} & \frac{3}{5} & \frac{1}{2} \\ \frac{1}{20} & \frac{1}{5} & \frac{1}{2} & 1 \end{bmatrix}. \quad (15)$$

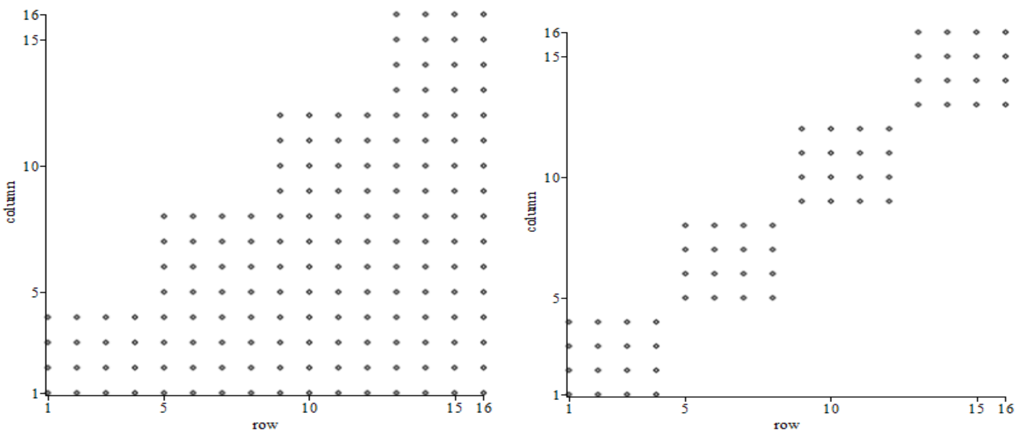

Fig. 1: Patterns of the matrices D (right) and P (left).

The efficacy of matrix D is used for converting the Fredholm part of integral equations to an algebraic equation. Because of its diagonal shape it can increase the calculating speed. Fig. 1 shows the pattern of matrix D and P when $M = 4$ and $n = 3$.

2.4 Multiplication of HOBB functions

It is always necessary to evaluate the product of HOBB(x) andHOBB$^T(x)$, that is called the product matrix of HOBB functions. Let

$$M(x) \cong \text{HOBB}(x)\,\text{HOBB}^T(x) \tag{16}$$

where $M(x)$is $(M(n+1)\times M(n+1)$matrix. Multiplying the matrix $M(x)$by vector C we obtain

$$M(x)C = \tilde{C}\,\text{HOBB}(x) \tag{17}$$

where $\tilde{C}$ is $(M(n+1)\times M(n+1)$matrix and called the coefficient matrix. To illustrate the calculation procedure in Eq. (16), we consider that $M=4, n=3$. [19] we have

$$\tilde{C} = \begin{bmatrix} \tilde{C}_0\ 0\ 0\ 0 \\ 0\ \tilde{C}_1\ 0\ 0 \\ 0\ 0\ \tilde{C}_2\ 0 \\ 0\ 0\ 0\ \tilde{C}_3 \end{bmatrix}$$

where C_i , $i=0,1,2,3$ are 4×4 matrices given

$$\tilde{C}_i = \begin{bmatrix} \frac{13}{4}\mathbf{c}_{1i}+\frac{2}{21}\mathbf{c}_{2i} & \frac{23}{24}\mathbf{c}_{1i}+\frac{5}{14}\mathbf{c}_{2i} & \frac{5}{21}\mathbf{c}_{1i}+\frac{3}{14}\mathbf{c}_{2i} & \frac{-1}{21}\mathbf{c}_{1i}+\frac{1}{21}\mathbf{c}_{2i} \\ -\frac{2}{105}\mathbf{c}_{3i}-\frac{1}{210}\mathbf{c}_{4i} & +\frac{6}{35}\mathbf{c}_{3i}+\frac{2}{105}\mathbf{c}_{4i} & -\frac{3}{70}\mathbf{c}_{3i}+\frac{2}{105}\mathbf{c}_{4i} & +\frac{1}{210}\mathbf{c}_{3i}-\frac{1}{210}\mathbf{c}_{4i} \\ \frac{13}{4}\mathbf{c}_{1i}+\frac{2}{21}\mathbf{c}_{2i} & \frac{23}{24}\mathbf{c}_{1i}+\frac{5}{14}\mathbf{c}_{2i} & \frac{5}{21}\mathbf{c}_{1i}+\frac{3}{14}\mathbf{c}_{2i} & \frac{-1}{21}\mathbf{c}_{1i}+\frac{1}{21}\mathbf{c}_{2i} \\ -\frac{2}{105}\mathbf{c}_{3i}-\frac{1}{210}\mathbf{c}_{4i} & +\frac{6}{35}\mathbf{c}_{3i}+\frac{2}{105}\mathbf{c}_{4i} & -\frac{3}{70}\mathbf{c}_{3i}+\frac{2}{105}\mathbf{c}_{4i} & +\frac{1}{210}\mathbf{c}_{3i}-\frac{1}{210}\mathbf{c}_{4i} \\ \frac{13}{4}\mathbf{c}_{1i}+\frac{2}{21}\mathbf{c}_{2i} & \frac{23}{24}\mathbf{c}_{1i}+\frac{5}{14}\mathbf{c}_{2i} & \frac{5}{21}\mathbf{c}_{1i}+\frac{3}{14}\mathbf{c}_{2i} & \frac{-1}{21}\mathbf{c}_{1i}+\frac{1}{21}\mathbf{c}_{2i} \\ -\frac{2}{105}\mathbf{c}_{3i}-\frac{1}{210}\mathbf{c}_{4i} & +\frac{6}{35}\mathbf{c}_{3i}+\frac{2}{105}\mathbf{c}_{4i} & -\frac{3}{70}\mathbf{c}_{3i}+\frac{2}{105}\mathbf{c}_{4i} & +\frac{1}{210}\mathbf{c}_{3i}-\frac{1}{210}\mathbf{c}_{4i} \\ \frac{13}{4}\mathbf{c}_{1i}+\frac{2}{21}\mathbf{c}_{2i} & \frac{23}{24}\mathbf{c}_{1i}+\frac{5}{14}\mathbf{c}_{2i} & \frac{5}{21}\mathbf{c}_{1i}+\frac{3}{14}\mathbf{c}_{2i} & \frac{-1}{21}\mathbf{c}_{1i}+\frac{1}{21}\mathbf{c}_{2i} \\ -\frac{2}{105}\mathbf{c}_{3i}-\frac{1}{210}\mathbf{c}_{4i} & +\frac{6}{35}\mathbf{c}_{3i}+\frac{2}{105}\mathbf{c}_{4i} & -\frac{3}{70}\mathbf{c}_{3i}+\frac{2}{105}\mathbf{c}_{4i} & +\frac{1}{210}\mathbf{c}_{3i}-\frac{1}{210}\mathbf{c}_{4i} \end{bmatrix}$$

Let R is $(M(n+1)\times M(n+1)$ matrix. Multiplying the matrix R by vector HOBB(x) and multiplying the matrix HOBB(x) by the resulted matrix RHOBB(x) we obtain

$$\text{HOBB}^T(x)\,R\,\text{HOBB}(x) = \tilde{R}\,\text{HOBB}(x) \tag{18}$$

where $\tilde{R}$ is $(1\times M(n+1)$ matrix and called the coefficient matrix With the powerful properties of Eq. (16) We can achieve $\tilde{R}$ by a way like $\tilde{C}$ we can convert the Volterra part of integral and Integro-Differential equations System equations to an algebraic equation. Fig. 2. patterns of the matrices R (right) and C (left).

3 Numerical solution of system of linear volterra integral equations using HOBB functions

Let's consider

$$\begin{aligned} u(x) &= f(x) + q(x)v(x) + \lambda_1 \int_0^x (k_1(x,t)\,u(t) + k_2(x,t)\,v(t))dt \\ v(x) &= g(x) + h(x)u(x) + \lambda_2 \int_0^x (k_3(x,t)\,(u(t)) + k_4(x,t)\,v(t))dt \end{aligned} \tag{19}$$

where $k_1(x,t)$, $k_2(x,t) \in L_2(\,[0,\,1]\times[0,\,1]\,)$ and $h(x)$, $g(x)\in L_2([0,\,1])$.

The unknown functions $u(x)$, $v(x)$can be expanded as

$$u(x) \approx U^T\text{HOBB}(x), v(x) \approx V^T\text{HOBB}(x) \tag{20}$$

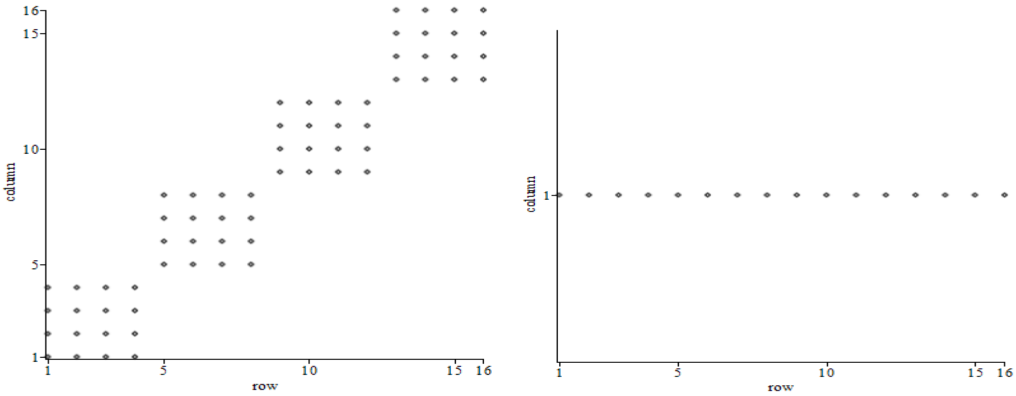

Fig. 2: Patterns of the matrices R (right) and C (left).

where U is the unknown M(n+1) vector and HOBB$(x)$is given by Eq. (5). Likewise

$$k_1(x,t),\ k_2(x,t),\ k_3(x,t),\ k_4(x,t),\ f(x),\ g(x),\ q(x),\ h(x)$$

are also expanded into the hybrid functions

$$\begin{array}{ll} k_1(x,t) \approx \mathrm{HOBB}^T(x)\,K_1\,\mathrm{HOBB}(t), & k_2(x,t) \approx \mathrm{HOBB}^T(x)\,K_2\mathrm{HOBB}(t), \\ k_3(x,t) \approx \mathrm{HOBB}^T(x)\,K_3\,\mathrm{HOBB}(t), & k_4(x,t) \approx \mathrm{HOBB}^T(x)\,K_4\mathrm{HOBB}(t), \\ f(x) \approx F^T\mathrm{HOBB}(x), g(x) \approx G^T\mathrm{HOBB}(x), & q(x) \approx Q^T\mathrm{HOBB}(x), h(x) \approx H^T\mathrm{HOBB}(x). \end{array} \tag{21}$$

After substituting the approximate equations (20) – (21) into (19) we get

$$\begin{aligned} U^T\mathrm{HOBB}(x) &= Q^T\mathrm{HOBB}(x)\,\mathrm{HOBB}^T(x)\,V + \lambda_1\,\mathrm{HOBB}^T(x)\,K_1\int_0^x \mathrm{HOBB}(t)\,\mathrm{HOBB}^T(t)\,U\,dt \\ &+ \lambda_1\,\mathrm{HOBB}^T(x)\,K_2\int_0^x \mathrm{HOBB}(t)\,\mathrm{HOBB}^T(t)\,V\,dt + F^T\mathrm{HOBB}(x) \\ V^T\mathrm{HOBB}(x) &= H^T\mathrm{HOBB}(x)\,\mathrm{HOBB}^T(x)\,U + \lambda_2\,\mathrm{HOBB}^T(x)\,K_3\int_0^x \mathrm{HOBB}(t)\,\mathrm{HOBB}^T(t)\,U\,dt \\ &+ \lambda_2\,\mathrm{HOBB}^T(x)\,K_4\int_0^x \mathrm{HOBB}(t)\,\mathrm{HOBB}^T(t)\,V\,dt + G^T\mathrm{HOBB}(x), \end{aligned} \tag{22}$$

where

$$\begin{aligned} \int_0^x \mathrm{HOBB}(t)\,\mathrm{HOBB}^T(t)\,U\,dt &= \int_0^x \tilde{U}\,\mathrm{HOBB}(t)\,dt = \tilde{U}\,P\,\mathrm{HOBB}(x) \\ \int_0^x \mathrm{HOBB}(t)\,\mathrm{HOBB}^T(t)\,V\,dt &= \int_0^x \tilde{V}\,\mathrm{HOBB}(t)\,dt = \tilde{V}\,P\,\mathrm{HOBB}(x), \end{aligned} \tag{23}$$

using Eqs. (17) and operational matrix P, we get

$$\begin{aligned} U^T \mathrm{HOBB}(x) &= F^T \mathrm{HOBB}(x) + Q^T \mathrm{HOBB}(x)\,\mathrm{HOBB}^T(x)\,V + \lambda_1[\mathrm{HOBB}^T(x)\,K_1\tilde{U}\,P\,\mathrm{HOBB}(x) \\ &\quad + \mathrm{HOBB}^T(x)\,K_2\tilde{V}\,P\,\mathrm{HOBB}(x)] \\ V^T \mathrm{HOBB}(x) &= G^T \mathrm{HOBB}(x) + H^T \mathrm{HOBB}(x)\,\mathrm{HOBB}^T(x)\,U + \lambda_2[\mathrm{HOBB}^T(x)\,K_3\tilde{U}\,P\,\mathrm{HOBB}(x) \\ &\quad + \mathrm{HOBB}^T(x)\,K_4\tilde{V}P\,\mathrm{HOBB}(x)]. \end{aligned} \tag{24}$$

Therefore

$$\begin{aligned} U^T \mathrm{HOBB}(x) &= F^T \mathrm{HOBB}(x) + Q^T\tilde{V}\,\mathrm{HOBB}(x) + \lambda_1[\mathrm{HOBB}^T(x)\,K_1\tilde{U}\,P\,\mathrm{HOBB}(x) \\ &\quad + \mathrm{HOBB}^T(x)\,K_2\tilde{V}\,P\,\mathrm{HOBB}(x)], \\ V^T \mathrm{HOBB}(x) &= G^T \mathrm{HOBB}(x) + H^T\tilde{U}\,\mathrm{HOBB}(x) + \lambda_2[\mathrm{HOBB}^T(x)\,K_3\tilde{U}\,P\,\mathrm{HOBB}(x) \\ &\quad + \mathrm{HOBB}^T(x)\,K_4\tilde{V}\,P\,\mathrm{HOBB}(x)]. \end{aligned} \tag{25}$$

If we approximate

$$\begin{aligned} &\mathrm{HOBB}^T(x)\,K_1\tilde{U}\,P\,\mathrm{HOBB}(x) \approx \tilde{R}_1\,\mathrm{HOBB}(x), \mathrm{HOBB}^T(x)\,K_2\tilde{V}\,P\,\mathrm{HOBB}(x) \approx \tilde{R}_2\,\mathrm{HOBB}(x) \\ &\mathrm{HOBB}^T(x)\,K_3\tilde{U}\,P\,\mathrm{HOBB}(x) \approx \tilde{R}_3\,\mathrm{HOBB}(x), \mathrm{HOBB}^T(x)\,K_4\tilde{V}\,P\,\mathrm{HOBB}(x) \approx \tilde{R}_4\,\mathrm{HOBB}(x), \end{aligned}$$

We can achieve $\tilde{R}$ by a way like $\tilde{C}$ and we see that each element of $\tilde{R}$ is obtained by the sum of column elements of $K_1\tilde{U}\,P$

$$\begin{aligned} U^T \mathrm{HOBB}(x) &\approx F^T \mathrm{HOBB}(x) + Q^T\tilde{V}\,\mathrm{HOBB}(x) + \lambda_1(\tilde{R}_1\,\mathrm{HOBB}(x) + \tilde{R}_2\,\mathrm{HOBB}(x)), \\ V^T \mathrm{HOBB}(x) &\approx G^T \mathrm{HOBB}(x) + H^T\tilde{U}\,\mathrm{HOBB}(x) + \lambda_2(\tilde{R}_3\,\mathrm{HOBB}(x) + \tilde{R}_4\,\mathrm{HOBB}(x)). \end{aligned}$$

Therefore

$$\begin{aligned} U &\approx F + \tilde{V}^T Q + \lambda_1(\tilde{R}_1^T + \tilde{R}_2^T) \\ V &\approx G + \tilde{U}^T H + \lambda_2(\tilde{R}_3^T + \tilde{R}_4). \end{aligned}$$

After replacing $\approx$ with $=$, we have a linear system that can be solved with Guass method for the unknown vectors U, V then by the use of $u(x) \approx U^T \mathrm{HOBB}(x)$,$v(x) \approx V^T \mathrm{HOBB}(x)$ the approximated solution is given.

4 Numerical solution of a system of non linear integro-differential equations using HOBBfunctions

In [20], a system of integro-differential equations was approximated using the modified decomposition method, and in [21], a similar system was approximated using the approximation method. We now consider a system of Volterra integro-differential equations of the form

$$\begin{aligned} u^{/}(x) &= f(x) + w(x)\,u(x) + q(x)\,v(x) + \lambda_1 \int_0^x (k_1(x,t)\,(u(t))^p + k_2(x,t)\,(v(t))^r dt \\ v^{/}(x) &= g(x) + l(x)\,u(x) + h(x)\,v(x) + \lambda_2 \int_0^x (k_3(x,t)\,(u(t))^l + k_4(x,t)\,(v(t)))^f\,dt \end{aligned} \tag{26}$$

where $k_1(x,t)$,$k_2(x,t)$,$k_3(x,t)$, $k_4(x,t) \in L_2([0,1] \times [0,1])$ and

$$h(x),\ g(x) \in L_2([0,1]), \quad u(x) \approx U^T \mathrm{HOBB}(x), \quad v(x) \approx V^T \mathrm{HOBB}(x),$$

where U,V are the unknown $M(n+1)-$vector and HOBB(x) is given by Eq. (5). Likewise, $k_1(x,t),\ k_1(x,t),\ k_1(x,t),\ f(x),\ g(x),\ h(x),\ q(x)$ are also expanded into the HOBB functions

$$\begin{aligned}
&k_1(x,t) \approx \mathrm{HOBB}^T(x) K_1\, \mathrm{HOBB}(t), \quad k_2(x,t) \approx \mathrm{HOBB}^T(x) K_2\, \mathrm{HOBB}(t),\\
&k_3(x,t) \approx \mathrm{HOBB}^T(x) K_3\, \mathrm{HOBB}(t), \quad k_4(x,t) \approx \mathrm{HOBB}^T(x) K_4\, \mathrm{HOBB}(t),\\
&w(x) \approx W^T \mathrm{HOBB}(x), \quad l(x) \approx L^T \mathrm{HOBB}(x),\\
&f(x) \approx F^T \mathrm{HOBB}(x), \quad g(x) \approx G^T \mathrm{HOBB}(x),\\
&h(x) \approx H^T \mathrm{HOBB}(x), \quad q(x) \approx Q^T \mathrm{HOBB}(x),
\end{aligned} \tag{27}$$

where $K_1,\ K_2,\ K_3,\ K_4$ are $M(n+1)\times M(n+1)$ matrices and F is an $M(n+1)-$vector. We approximate $u^{/}(x)$ as follows

$$u^{/}(x) \approx U^{/^T}\, \mathrm{HOBB}(x), \quad v^{/}(x) \approx V^{/^T}\, \mathrm{HOBB}(x) \tag{28}$$

which $U^{/}, V^{/}$ will be evaluated in terms of U, V.

$$u(x) = \int_0^x u^{/}(t)dt + u(0).$$

If we expand $u(0), v(0)$ with HOBB basis i.e. $u(0) = U_0\, \mathrm{HOBB}(x),\ v(0) = V_0\, \mathrm{HOBB}(x)$ then U_0, V_0 is obtained as follows.

$$\begin{aligned}
U_0 &= \Big[\underbrace{\overbrace{u(0), u(0), \dots, u(0)}^{M},\ \overbrace{u(0), u(0), \dots, u(0)}^{M},\ \overbrace{u(0), u(0), \dots, u(0)}^{M}}_{M(n+1)}\Big]\\
V_0 &= \Big[\underbrace{\overbrace{v(0), v(0), \dots, v(0)}^{M},\ \overbrace{v(0), v(0), \dots, v(0)}^{M},\ \overbrace{v(0), v(0), \dots, v(0)}^{M}}_{M(n+1)}\Big].
\end{aligned} \tag{29}$$

$$\begin{aligned}
U^T \mathrm{HOBB}(x) &\cong \int_0^x (U^{/})^T\, \mathrm{HOBB}(t)dt + U_0{}^T \mathrm{HOBB}(x)\\
&\cong (U^{/})^T \int_0^x \mathrm{HOBB}(t)dt + U_0{}^T \mathrm{HOBB}(x)\\
&\cong (U^{/})^T P\, \mathrm{HOBB}(x) + U_0{}^T \mathrm{HOBB}(x)\\
&\cong \left((U^{/})^T P + U_0{}^T\right) \mathrm{HOBB}(x),
\end{aligned}$$

and we have

$$\begin{aligned}
U^T &\cong U^{/^T} P + U_0{}^T\\
U &\cong U^{/} P + U_0,\\
V &\cong V^{/} P + V_0.
\end{aligned} \tag{30}$$

Therefore,

$$U^{/} \cong (P^T)^{-1}(U - U_0), \quad V^{/} \cong (P^T)^{-1}(V - V_0). \tag{31}$$

Functions $u^q(x), v^q(x)$ can be expanded into the HOBB functions as

$$
\begin{aligned}
u^2(x) &= [U^T \text{HOBB}(x)]^2 = U^T \text{HOBB}(x)\,\text{HOBB}(x)^T U = \text{HOBB}(x)^T \tilde{U} U, &(32)\\
v^3(x) &= V^T \text{HOBB}(x)\,[V^T \text{HOBB}(x)]^2 = V^T \text{HOBB}(x)\,\text{HOBB}(x)^T \tilde{V} V \\
&= \text{HOBB}(x)^T \tilde{V}\tilde{V} V = \text{HOBB}(x)^T (\tilde{V})^2 V, &(33)\\
u^q(x) &= \text{HOBB}(x)^T (\tilde{U})^{q-1} U,\;\; v^q(x) = \text{HOBB}(x)^T (\tilde{V})^{q-1} V. &(34)
\end{aligned}
$$

After substituting the approximate equations (27)–(34) into (26) we get

$$
\begin{aligned}
U^{/T}\text{HOBB}(x) \approx{}& F^T \text{HOBB}(x) + W^T \text{HOBB}(x)\,\text{HOBB}^T(x)\,U + Q^T \text{HOBB}(x)\,\text{HOBB}^T(x)\,V \\
&+ \lambda_1 \text{HOBB}^T(x)\,K_1 \int_0^x \text{HOBB}(t)\,\text{HOBB}^T(t)\,(\tilde{U})^{P-1} U\,dt \\
&+ \lambda_1 \text{HOBB}^T(x)\,K_2 \int_0^x \text{HOBB}(t)\,\text{HOBB}^T(t)\,(\tilde{V})^{r-1} V\,dt
\end{aligned}
$$

$$
\begin{aligned}
V^{/T}\text{HOBB}(x) \approx{}& G^T \text{HOBB}(x) + L^T \text{HOBB}(x)\,\text{HOBB}^T(x)\,U + H^T \text{HOBB}(x)\,\text{HOBB}^T(x)\,V \\
&+ \lambda_2 \text{HOBB}^T(x)\,K_3 \int_0^x \text{HOBB}(t)\,\text{HOBB}^T(t)\,(\tilde{U})^{l-1} U\,dt \qquad (35)\\
&+ \lambda_2 \text{HOBB}^T(x)\,K_3 \int_0^x \text{HOBB}(t)\text{HOBB}^T(t)(\tilde{V})^{f-1} V\,dt,
\end{aligned}
$$

where

$$
\begin{aligned}
\int_0^x \text{HOBB}(t)\,\text{HOBB}^T(t)\,(\tilde{U})^{P-1} U\,dt &= \int_0^x ((\tilde{U})^{\tilde{P}-1} U)\,\text{HOBB}(t)\,dt = ((\tilde{U})^{\tilde{P}-1} U)\,P\,\text{HOBB}(x)\\
\int_0^x \text{HOBB}(t)\,\text{HOBB}^T(t)\,(\tilde{V})^{P-1} V\,dt &= \int_0^x ((\tilde{V})^{\tilde{P}-1} V)\,\text{HOBB}(t)\,dt = ((\tilde{V})^{\tilde{P}-1} V)\,P\,\text{HOBB}(x),
\end{aligned}
$$

making use of Eqs. (17) and operational matrix P, we get

$$
\begin{aligned}
U^{/T}\text{HOBB}(x) \approx{}& F^T \text{HOBB}(x) + W^T \tilde{U}\,\text{HOBB}(x) + Q^T \tilde{V}\text{HOBB}(x) + \lambda_1 \text{HOBB}^T(x)\,K_1\,((\tilde{U})^{\tilde{P}-1} U)\,P\,\text{HOBB}(x) \\
&\lambda_1 \text{HOBB}^T(x)\,K_2\,((\tilde{V})^{\tilde{P}-1} V)\,P\,\text{HOBB}(x) \qquad (36)\\
V^{/T}\text{HOBB}(x) \approx{}& G^T \text{HOBB}(x) + L^T \tilde{U}\,OBH(x) + H^T \tilde{V}\,\text{HOBB}(x) + \lambda_2\,OBH^T(x)\,K_3\,((\tilde{U})^{\tilde{P}-1} U)\,P\,\text{HOBB}(x) \\
&+ \lambda_2 \text{HOBB}^T(x)\,K_4\,((\tilde{V})^{\tilde{P}-1} V)\,P\,\text{HOBB}(x).
\end{aligned}
$$

If we approximate

$$\begin{aligned}
&\mathrm{HOBB}^T(x)\,K_1((\tilde{U})^{P-1}\,U)\;P\,\mathrm{HOBB}(x) \approx \tilde{R}_1\,\mathrm{HOBB}(x),\\
&\mathrm{HOBB}^T(x)\,K_2((\tilde{V})^{P-1}\,V)\,P\,\mathrm{HOBB}(x) \approx \tilde{R}_2\,\mathrm{HOBB}(x),\\
&\mathrm{HOBB}^T(x)\,K_3((\tilde{U})^{P-1}\,U)\,P\,\mathrm{HOBB}(x) \approx \tilde{R}_3\,\mathrm{HOBB}(x),\\
&\mathrm{HOBB}^T(x)\,K_4((\tilde{V})^{P-1}\,V)\,P\,\mathrm{HOBB}(x) \approx \tilde{R}_4\,\mathrm{HOBB}(x).
\end{aligned} \tag{37}$$

We can achieve $(\tilde{R})$ by a way like $\tilde{C}$, and we see that for element of $\tilde{R}_1$ is obtained by the sum of column elements of $K_1((\tilde{U})^{P-1}\,U)\;P$ with respect to coefficient $\tilde{R}$ in Eq. (18) at each column. By using this property and omitting hybrid vector functions in Eq. (36), we will have

$$\begin{aligned}
U^{/T} &\approx F^T + W^T\,\tilde{U} + Q^T\,\tilde{V} + \lambda_1\,(\tilde{R}_1) + \lambda_1\,(\tilde{R}_2)\\
V^{/T} &\approx G^T + L^T\,\tilde{U} + H^T\,\tilde{V} + \lambda_2\,(\tilde{R}_3) + \lambda_2(\tilde{R}_4),
\end{aligned}$$

another equivalent form is

$$\begin{aligned}
U^{/} &\approx F + \tilde{U}^T W + \tilde{V}^T Q + \lambda_1\,(\tilde{R}_1)^T + \lambda_1\,(\tilde{R}_2)^T\\
V^{/} &\approx G + \tilde{U}^T L + \tilde{V}^T H + \lambda_2\,(\tilde{R}_3)^T + \lambda_2\,(\tilde{R}_4)^T,
\end{aligned} \tag{38}$$

multiplying matrix P on both sides of Eq. (38) and applying Eq. (30) in Eq. (38) we get

$$\begin{aligned}
(U - U_0) &\approx P^T F + P^T\,\tilde{U}^T W + P^T\,\tilde{V}^T Q + \lambda_1 P^T(\tilde{R}_1)^T + \lambda_1 P^T(\tilde{R}_2)^T\\
(V - V_0) &\approx P^T G + P^T\,\tilde{U}^T L + P^T\,\tilde{V}^T H + \lambda_2 P^T(\tilde{R}_3)^T + \lambda_2\,P^T(\tilde{R}_4)^T.
\end{aligned}$$

After replacing $\approx$ with $=$, we have a coupled nonlinear system that can be solved with Newton's method for the unknown vectors U, V then by the use of $u(x) \approx U^T\mathrm{HOBB}(x)$,$v(x) \approx V^T\mathrm{HOBB}(x)$ the approximated solutions are given.

5 Numerical examples

In this section we implemented our method on three different examples. Our results achieved by a proper value for M (this feather is experimental) and different values for n. The results are tabulated in three tables, in these tables the exact solutions are compared with hybrid function solutions. It is noticed that our method has quite acceptable results but it is clear for lower values of n we have less accuracy in some end points of the interval that by increasing n, the results become better.

Example 1. Consider the following system of

$$u(x) = (-x^2 - \frac{2}{3}x^4) + x\,v(x) + \int_0^x (2x\,u(t) + 2\,v(t))dt$$

$$v(x) = (x - \frac{x^2}{4} + \frac{2}{3}x^3 + \frac{1}{2}x^4) - \frac{1}{2}(x + x^2)\,u(x) - \frac{1}{2}\int_0^x (u(t) - v(t))dt$$

The exact solution is $u(x) = x^2, \quad v(x) = x$.

By applying the HOBB method and solving the resulted linear system, the following results would be achieved. The elements of vector functions U and V can be obtained as follows

$$U = [1.584893036\times 10^{-13}, -5.806913601\times 10^{-13}, 0.02083333333, 0.06249999999, 0.06249999999, 0.1041666667, 0.1666666667, 0.2500000000, 0.2500000000, 0.3333333333, 0.4375000000, 0.4375000000, 0.5624999999, 0.5624999999, 0.6874999997, 0.8333333333, 1.000000000]$$

$$V = [-1.746906809\times 10^{-14}, 0.08333333333, 0.1666666667, 0.25, 0.25, 0.3333333333, 0.4166666667, 0.5000000000, 0.5000000000, 0.5833333334, 0.6666666667, 0.7499999999, 0.7499999998, 0.8333333330, 0.9166666669, 1]$$

Table 1 shows some values of the solutions and absolute errors.

Table 1: The comparison between exact solutions and approximate solutions for HOBB functions at $M = 4,\ n = 3$.

	HOBB solution		The Exact Solution		Absolute error	
x	u_{HOBB}	v_{HOBB}	u_{exact}	v_{exact}	$e(u_{\text{HOBB}})$	$e(v_{\text{HOBB}})$
0.1	0.0099999999999	0.1	0.01	0.1	2×10^{-12}	1×10^{-11}
0.2	0.0399999999999	0.2	0.04	0.2	1×10^{-11}	0
0.3	0.09	0.3	0.09	0.3	0	0
0.4	0.16	0.4	0.16	0.4	1×10^{-11}	0
0.5	0.25	0.5	0.25	0.5	0	0
0.6	0.36	0.6	0.36	0.6	1×10^{-11}	1×10^{-11}
0.7	0.4899999999999	0.69999999999999	0.49	0.7	1×10^{-10}	1.1×10^{-10}
0.8	0.6399999999999	0.79999999999999	0.64	0.8	2×10^{-10}	1.7×10^{-10}
0.9	0.8099999999999	0.9	0.81	0.9	1×10^{-10}	0

Example 2. [20] Consider the following nonlinear system of two integro-differential equations

$$u(x) = 2x + \frac{1}{3}x^3 - \frac{1}{2}x^2 + \int_0^x (-u^2(t) + v(t))dt,$$

$$v^{/}(x) = -1 + \int_0^x (u(t) - v(t))dt,$$

with the exact solutions

$$u(x) = x,\ v(x) = x - 1.$$

By applying the HOBB method and solving the resulted nonlinear system, the following results would be achieved. The elements of vector functions U and V can be obtained as follows

$$
\begin{aligned}
U =&[1.797326462\times 10^{-12}, 0.08333333335, 0.1666666667, 0.2500000000,\\
&0.2500000000, 0.3333333333, 0.4166666666, 0.5000000000, 0.5000000000,\\
&0.5833333333, 0.6666666663, 0.7500000000, 0.7500010751, 0.8334015084,\\
&0.9167007631, 1.000032639]\\
V =&[-1.000000000, -0.9166666670, -0.8333333329, -0.7499999999, -0.7500000001,\\
&-0.6666666669, -0.5833333331, -0.4999999999, -0.5000000001, -0.4166666668,\\
&-0.3333333332, -0.2500000000, -0.2491081394, -0.1667350542, -0.08337160066,\\
&-0.00003916014580]
\end{aligned}
$$

Table 2 shows some values of the solutions and absolute errors.

Table 2: The comparison between exact solutions and approximate solutions for HOBB functions at $M = 4,\ n = 3$.

	HOBB solution		The Exact Solution		Absolute error	
x	u_{HOBB}	v_{HOBB}	u_{exact}	v_{exact}	$e(u_{\text{HOBB}})$	$e(v_{\text{HOBB}})$
0.1	0.1	-0.89999999999	0.1	-0.9	1×10^{-11}	1×10^{-11}
0.2	0.2	-0.79999999999	0.2	-0.8	0	1×10^{-11}
0.3	0.3	-0.7	0.3	-0.7	0	1×10^{-11}
0.4	0.4	-0.6	0.4	-0.6	0	1×10^{-11}
0.5	0.5	-0.5	0.5	-0.5	0	1×10^{-11}
0.6	0.59999999999	-0.4	0.6	-0.4	1×10^{-10}	1×10^{-11}
0.7	0.69999999999	-0.29999999999	0.7	-0.3	1×10^{-10}	1×10^{-10}
0.8	0.80000030261	-0.19995736152	0.8	-0.2	3.02×10^{-6}	4.26×10^{-6}
0.9	0.90000414828	-0.0999987606	.9	-0.1	4.14×10^{-6}	1.23×10^{-6}

Example 3. [22] Consider the following system of integro-differential equations with the exact solutions $u(x) = \cosh(x)$ and $v(x) = \sinh(x)$

$$\frac{d\,u(x)}{d\,x} = -x^3 - 6x - 1 + u(x) + (7-2x)\,v(x) + \int_0^x ((x+t)\,u(t) + (t-x)^3 v(t))\,dt,\ x_1(0) = 1.$$

$$\frac{d\,v(x)}{d\,x} = -3x^2 + x - 6 + (7-2x)\,u(x) + v(x) + \int_0^x ((x-t)^3\,u(t) + (x+t)v(t))\,dt,\ x_2(0) = 0.$$

By applying the HOBB method and solving the resulted nonlinear system, the following results would be achieved. The elements of vector functions U and V can be obtained as follows

$$
\begin{aligned}
U =&[0.9999975956, 1.000013307, 1.010375450, 1.031410679, 1.031410527, 1.052478184,\\
&1.084215587, 1.127623284, 1.127622972, 1.171066120, 1.226172644, 1.294679694,\\
&1.294679205, 1.363226936, 1.445164306, 1.543073606].\\
V =&[-3.038043356\times 10^{-7}, 0.08333475445, 0.1666631192, 0.2526119952, 0.2526113784,\\
&0.3385682089, 0.4271317611, 0.5210942590, 0.5210935851, 0.6150724739,\\
&0.7144352440, 0.8223144151, 0.8223136440, 0.9302186492, 1.046622817, 1.175195156].
\end{aligned}
$$

Table 3 shows some values of the solutions and absolute errors.

Table 3: The comparison between exact solutions and approximate solutions for HOBB functions at $M = 4,\ n = 3$.

x	HOBB solution		The Exact Solution		Absolute error	
	u_{HOBB}	v_{HOBB}	u	v	$e(u_{\text{HOBB}})$	$e(v_{\text{HOBB}})$
0.1	1.005003642	0.1001666943	1.0050041	0.10016675	5.25×10^{-7}	5.56×10^{-8}
0.2	1.020067699	0.2013361133	1.0200667	0.20133600	9.42×10^{-9}	1.10×10^{-7}
0.3	1.045339495	0.3045206211	1.0453385	0.30452029	9.81×10^{-7}	3.27×10^{-7}
0.4	1.081071754	0.4107520531	1.0810723	0.41075232	6.18×10^{-7}	2.72×10^{-7}
0.5	1.127622972	0.5210935851	1.1276259	0.52109530	2.91×10^{-6}	1.72×10^{-6}
0.6	1.185464347	0.6366529960	1.1854652	0.63665358	8.70×10^{-7}	5.87×10^{-7}
0.7	1.255169630	0.7585838204	1.2551690	0.75858370	6.24×10^{-7}	1.18×10^{-7}
0.8	1.337435258	0.8881058986	1.3374349	0.88810598	3.12×10^{-7}	8.35×10^{-8}
0.9	1.433083706	1.026514255	1.4330863	1.0265167	2.67×10^{-6}	2.47×10^{-6}

6 Conclusion

In this research, we have presented the Hybrid Orthonormal Bernstein and Block-Pulse Functions operational matrices of integration D, operational matrix Pproduct matrix $G$and coefficient matrix $\tilde{C}$which are sparse matrices, are used to converting integral and Integro-Differential equations System to system of equations that can be solved by known iterative methods. By making use of these operational matrices, the problem has been reduced to solve a set of algebraic equations that can simply appeared in matrix form. The solution obtained using the suggested method shows that this approach can solve integral and Integro-Differential equations System effectively. Although we do not claim this method shows superiority over other methods from the viewpoint of accuracy, it seems that this method is more practical, quite good accurate and has lower calculation.

Illustrative examples have been discussed to demonstrate the validity and applicability of the technique and the results have been compared with the exact solution.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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