New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2017.186

$LS(2)-$ Equivalence conditions of control points and application to planar Bezier curves

Muhsin Incesu[1,*],*Osman Gursoy*[2]

[1]Education Faculty, Mus Alparslan University,49100, Mus, Turkey,
[2]Educational Faculty, Maltepe University, İstanbul,Turkey

Received: 10 October 2016, Accepted: 6 April 2017
Published online: 25 August 2017.

Abstract: Having an important role in CAD and CAM systems the Bezier and B- spline curves and surfaces and NURBS modelling are based on control points belongs to these curves and surfaces. So the invariants of these curves and surfaces are the invariants of the control points of these curves and surfaces. In this study we studied the equivalence conditions of compared two different control point systems under the linear similarity transformations $LS(2)$ in R^2 according to the invariant system of these control points. Finally the equivalence conditions of two planar Bezier curves is examined.

Keywords: Linear Similarity, equivalence conditions, control points, planar Bezier curves.

1 Introduction

The developing process on invariant theory started from the late of XIX th century. It tries to examine whether the ring of G-invariant polynomial functions $R[x]^G$ has finite generators or not. This problem is given firstly in 1860's for Binary forms. In 1900 David Hilbert presented 23 amazing problems in Paris International Congress and in 14 th problem of it he expressed when the generators of the ring of G-invariant polynomial functions $R[x]^G$ is finite.

After him in 1962 M. Nagata demonstrated the ring of G-invariant polynomial functions $R[x]^G$ has finite generators in case G is linear reductive. In the studies of D. Khadjiev [1] and F. Grosshans [2], be finite conditions of generators of the ring of G-invariant polynomial functions $R[x]^G$ in case G is not linear reductive is given.

In 1946, Herman Weyl gave the complete invariant system of control points for n dimensional orthogonal group $O(n)$ in [3], after him in 1988, Dj.Khadjiev and R. Aripov generalized this invariants to all euclidean motions in [4].

Developments in the Invariant theory has affected different areas of mathematics. Until F. Klein, only certain geometries was known. In 1872, Klein showed that groups are important building blocks of geometry in his Erlangen Programme. Accordingly similarity geometry is the theory of invariants of similarity transformations' group and its certain subgroups.i.e. two elements A and B in this geometry are equivalent if and only if there exist a similarity transformation f such that $B = f(A)$ is satisfied [5] .

In mechanics the concept of similarity is mostly used in development of dimensional analysis. Dimensional analysis arose from an attempt to extend to physics some concepts like similarity, ratio, and proportion [6] , [7] . It was first

* Corresponding author e-mail: m.incesu@alparslan.edu.tr

applied by Galileo in 1638 to predict the strength of beams of given material as a function of linear dimensions [6] .

Other applications were given by Mariotte in 1679 and Newton in 1686 [6] , but it was Fourier who first stated that there are certain ”fundamental units”, in terms of which every physical quantity has certain ”dimensions” , to be written as exponents in 1822 [6] , [7] . In 20 th century Bridgman [8] , Sedov [9], and Langhaar [10] are sample contributors in this area. Sağiroğlu [15], [16], Sağiroğlu and Pekşen [17], Ören [18], Khadjiev, Ören and Pekşen [19] and Deveci and Karaduman [20] are some other contributors in this area recently.

Recently the invariants of control points have had an important role in the CAD and CAM systems. Especially Bezier and B- spline curves and surfaces and NURBS modelling base on control points belongs to these curves and surfaces. The invariants of these control points mean the invariants of curves and surfaces determined by these control points.

A similarity transformation is composed of multiplication of two transformations: a linear homothety or central dilation and an isometry transformation. So a linear similarity transformation is composed of multiplication of two transformations: a linear homothety and a linear isometry or an orthogonal transformation. Accordingly the group of linear similarity transformations is an important subgroup of group of all similarity transformations. The linear similarity transformations' group in 2 dimensional Euclidean space will be deoted by $LS(2)$. In this paper the equivalent conditions of two control points systems in terms of the rational $LS(2)-$invariants of control points is studied.

2 The rational $LS(2)-$invariants of points

Let E be a two dimensional Euclidean space then the transformation $F : E \mapsto E$ such that $\|F(x) - F(y)\| = \lambda \|x - y\|$ is called a similarity transformation if there exist a positive λ for every $X, Y \in E$.

For a positive λ the homotethy function F in two dimensional Euclidean space E is defined by $F(x) = a + \lambda (x - a)$ for every $x \in E$ where a is called the center of homotethy F.

Proposition 1. *The homotethy function is linear if and only if $\lambda = 1$ or $a = 0$. [11]*

Accordingly the linear homotethy function F, in two dimensional Euclidean space E is defined by $F(x) = \lambda x$ for every $x \in E$. Since a linear similarity transformation is composed of multiplication of two transformations: a linear homothety and a linear isometry or an orthogonal transformation, any linear similarity transformation F can be stated as

$$F(x) = \lambda g x \tag{1}$$

for every $x \in E$ where $g \in O(2)$ in two dimensional Euclidean space E [11]. Let G be a transformation group. Then, the function f is called G- invariant function, if

$$f(gx) = f(x) \tag{2}$$

for every $x \in E$ and every $g \in G$.

Definition 1. *Let G be a transformation group, H be a subgroup of G and E be a two dimensional Euclidean space. Then the function $f : E \to R$ is called proportional H-invariant function if $f(gx) = \lambda(g) f(x)$ for all $g \in H$ and for all $x \in E$, where the function $\lambda(g)$ is named the "weight" of the function f.*

Proposition 2. *Let G be a transformation group and H be a subgroup of G. In this case any H- invariant rational function f can be written as follows*

$$f(x) = \frac{P(x)}{Q(x)}, Q(x) \neq 0 \tag{3}$$

where $P(x)$ and $Q(x)$ are proportional H- invariant polynomial function such that the weights of them are equal. [4].
The group of all the orthogonal transformations defined in 2- dimensional Euclidean space E is denoted by $O(2,E)$. for shortness this group is denoted by $O(2)$. The group of all the rotations is denoted by $SO(2)$. It is clear that the group $SO(2)$ is an important subgroup of $O(2)$.

Proposition 3. *The weight of the proportional $O(2)-$invariant polynomial function $\lambda(g)$ is equal to 1 or detg for all $g \in O(2)$. The proportional $O(2)-$invariant polynomial function is called even invariant function if the weight $\lambda(g)$ is equal to 1 for all $g \in O(2)$. and the proportional $O(2)-$invariant polynomial function is called odd invariant function if the weight $\lambda(g)$ is equal to detg for all $g \in O(2)$ [3] , [4].*

Proposition 4. *Let f_1 and f_2 be even invariant functions and $\omega \in R$ be given. Then $f_1 + f_2$, $f_1.f_2$ and $\omega.f_1$ are even invariant functions [11].*

Proposition 5. *Let f_1 and f_2 be odd invariant functions and h be an even invariant function and $\omega \in R$ be given. Then $f_1 + f_2$, $f_1.h$ and $\omega.f_1$ are odd invariant functions and $f_1.f_2$ is even invariant function [11] .*

Let $x^{(1)}, x^{(2)}, ..., x^{(m)}$ be m vector variables in R^2 and $x^{(i)} = \left(x_1^{(i)}, x_2^{(i)}\right) \in R^2$ be given for $i = 1,2,...,m$. Then the matrix $\left[x^{(i)}\; x^{(j)}\right]$ means that

$$\left[x^{(i)} x^{(j)}\right] = \begin{bmatrix} x_1^{(i)} & x_1^{(j)} \\ x_2^{(i)} & x_2^{(j)} \end{bmatrix}. \tag{4}$$

Theorem 1. *Let $x^{(1)}, x^{(2)}, ..., x^{(m)}$ be m vector variables in 2- dimensional Euclidean space R^2. Then,*

(i) *Any even invariant polynomial can be expressed by the polynomial of the functions*

$$\left\langle x^{(i)}, x^{(j)} \right\rangle; i, j = 1,2,...,m.$$

(ii) *Let $\varphi\left(x^{(1)}, x^{(2)}, ..., x^{(m)}\right)$ be an even invariant polynomial with m vector variables. In this case any odd invariant polynomial can be expressed by the sum of the functions*

$$\det\left[x^{(i)} x^{(j)}\right].\varphi\left(x^{(1)}, x^{(2)}, ..., x^{(m)}\right)$$

where $i, j = 1,2,...,m$ and $i < j$ [3].

Let $R\left[x^{(1)}, x^{(2)}, ..., x^{(m)}\right]$ be a ring of polynomials for m vector variables $x^{(1)}, x^{(2)}, ..., x^{(m)}$ in 2- dimensional Euclidean space R^2 over the field R and a transformation group G be given. Then the algebra of G-invariant polynomials for m vector variables $x^{(1)}, x^{(2)}, ..., x^{(m)}$ in 2- dimensional Euclidean space R^2 over the field R is denoted by $R\left[x^{(1)}, x^{(2)}, ..., x^{(m)}\right]^G$.

Theorem 2. *Let $x^{(1)}, x^{(2)}, ..., x^{(m)}$ be m vector variables in 2- dimensional Euclidean space R^2. Then, the system of functions*

$$\left\langle x^{(i)}, x^{(j)} \right\rangle; i, j = 1,2,...,m; i \leq j \tag{5}$$

is the generator system of the algebra $R\left[x^{(1)}, x^{(2)}, ..., x^{(m)}\right]^{O(n)}$ [3].

Theorem 3. *Let $x^{(1)},x^{(2)},...,x^{(m)}$ be m vector variables in 2- dimensional Euclidean space R^2. Then,for $i,j=1,2,...,m$ the generator system of the algebra $R\left[x^{(1)},x^{(2)},...,x^{(m)}\right]^{SO(n)}$ is as follows [3].*

$$\begin{array}{l}\left\langle x^{(i)},x^{(j)}\right\rangle ; i \leq j, \\ \det\left[x^{(i)}x^{(j)}\right] ; i < j.\end{array} \tag{6}$$

Theorem 4. *Let $x^{(1)},x^{(2)},...,x^{(m)}$ be m vector variables in 2- dimensional Euclidean space R^2. If $m>2$ then, for $i,j=1,2$ and $p=3,...,m$ the generator system of the field of the $O(2)$-invariant rational functions $R\left(x^{(1)},x^{(2)},...,x^{(m)}\right)^{O(2)}$ is as follows [3].*

$$\begin{array}{l}\left\langle x^{(i)},x^{(j)}\right\rangle ; i \leq j, \\ \left\langle x^{(i)},x^{(p)}\right\rangle ;\end{array} \tag{7}$$

If $m \leq 2$ then, the generator system of the field $R\left(x^{(1)},x^{(2)},...,x^{(m)}\right)^{O(2)}$ is the same the system (5).

Theorem 5. *Let $x^{(1)},x^{(2)},...,x^{(m)}$ be m vector variables in 2- dimensional Euclidean space R^2. If $m>2$ then, for $i,j=1,2$ and $p=3,...,m$ the generator system of the field $R\left(x^{(1)},x^{(2)},...,x^{(m)}\right)^{SO(2)}$ is as follows [3].*

$$\begin{array}{l}\det\left[x^{(1)}x^{(2)}\right], \\ \left\langle x^{(i)},x^{(j)}\right\rangle ; i \leq j; i+j<4, \\ \left\langle x^{(i)},x^{(p)}\right\rangle ;\end{array} \tag{8}$$

If $m \leq 2$ then, the generator system of the field $R\left(x^{(1)},x^{(2)},...,x^{(m)}\right)^{O(2)}$ is

$$\begin{array}{l}\det\left[x^{(1)}x^{(2)}\right], \\ \left\langle x^{(i)},x^{(j)}\right\rangle ; i \leq j; i+j<4.\end{array} \tag{9}$$

Theorem 6. *Let $x^{(1)},x^{(2)},...,x^{(m)}$ be m vector variables in 2- dimensional Euclidean space R^2. For $i,j=1,2,...,m$ the generator system of the field $R\left(x^{(1)},x^{(2)},...,x^{(m)}\right)^{LS(2)}$ is as follows [11].*

$$\frac{\left\langle x^{(i)},x^{(j)}\right\rangle}{\left\langle x^{(1)},x^{(1)}\right\rangle}, i \leq j. \tag{10}$$

3 $LS(2)-$ equivalence conditions of control points

Let G be a transformation group and E be two dimensional Euclidean space. Then, the points $x,y \in E$ are called G-equivalent points if there exist a transformation $g \in G$ such that $y=gx$. If x and y are G-equivalent points then the notation $x \overset{G}{\cong} y$ is used.

Let G be a transformation group and E be two dimensional Euclidean space and two points systems $\left\{x^{(1)},x^{(2)},...,x^{(m)}\right\}$ and $\left\{y^{(1)},y^{(2)},...,y^{(m)}\right\}$ in E be given. Then, these systems are called G-equivalent if there exist a transformation $g \in G$

such that $y^{(i)} = gx^{(i)}$ for every $i \in \{1,2,...,m\}$. If these points systems $\left\{x^{(1)},x^{(2)},...,x^{(m)}\right\}$ and $\left\{y^{(1)},y^{(2)},...,y^{(m)}\right\}$ are G-equivalent systems then the notation $\left\{x^{(1)},x^{(2)},...,x^{(m)}\right\} \overset{G}{\cong} \left\{y^{(1)},y^{(2)},...,y^{(m)}\right\}$ is used.

Theorem 7. *Let $x = (x_1,x_2)$ and $y = (y_1,y_2)$ be given two vectors in R^2. Then,*

(i) *If $x = 0$ and $y \neq 0$ or $x \neq 0$ and $y = 0$ then x and y are not $LS(2)$- equivalent. ie. $x \overset{LS(2)}{\ncong} y$,*

(ii) *If $x = 0$ and $y = 0$ or $x \neq 0$ and $y \neq 0$ then $x \overset{LS(2)}{\cong} y$.*

Proof.

(i) Let $x \neq 0$ and $y = 0$ be given and $x \overset{G}{\cong} y$ be supposed. In this case there exist a transformation $h \in LS(2)$ such that $y = hx$ is satisfied. It means there exist $g \in O(2)$ and $\lambda > 0$ such that $y = \lambda gx$ is satisfied. Since the orthogonal transformations save the inner product, $\langle y,y\rangle = \lambda^2 \langle x,x\rangle$ is obtained. But because $x \neq 0$ and $\lambda > 0, \langle y,y\rangle$ must be different zero and y can not be 0 vectors. This is a contradiction. So x and y are not $LS(2)$- equivalent vectors. In case $x = 0$ and $y \neq 0$ the statement can be reduced first case since the relationship $\overset{G}{\cong}$ is an equivalence relation and has symmetry property.

(ii) Let $x = (0,0)$ and $y = (0,0)$ be given. In case $gx = 0$ for every $g \in O(2)$ and $y = \lambda gx$ can be stated since $\lambda gx = 0$ Thus from (2) $x \overset{G}{\cong} y$ is proved. Let $x \neq 0$ and $y \neq 0$ be given. Then $\langle x,x\rangle$ and $\langle y,y\rangle$ are different from zero. Thus, the positive real number λ can be chosen as $\lambda = \frac{\langle y,y\rangle}{\langle x,x\rangle}$. So $\|y\| = \|\lambda x\|$ is obtained. In this case the vectors y and λx on the same $O(2)$-orbit. It means there exist an orthogonal transformation $g \in O(2)$ such that $y = g(\lambda x) = \lambda gx$. From this $x \overset{G}{\cong} y$ is proved.

Theorem 8. *Let two points systems $\left\{x^{(1)},x^{(2)},...,x^{(m)}\right\}$ and $\left\{y^{(1)},y^{(2)},...,y^{(m)}\right\}$ in E be given. So,*

(i) *If $x^{(i)} = 0$ and $y^{(i)} \neq 0$ or $x^{(i)} \neq 0$ and $y^{(i)} = 0$ for any $i = 1,2,...,m$, then these systems are not $LS(2)$- equivalent. ie. $\left\{x^{(1)},x^{(2)},...,x^{(m)}\right\} \overset{LS(2)}{\ncong} \left\{y^{(1)},y^{(2)},...,y^{(m)}\right\}$*

(ii) *If $x^{(i)} = 0$ and $y^{(i)} = 0$ for any $i = 1,2,...,m$, then the equivalence conditions of the systems with m vector variables is reduced the equivalence conditions with $m-1$ vectors.*

(iii) *If $x^{(i)} \neq 0$ and $y^{(i)} \neq 0$ for every $i = 1,2,...,m$, then*

(a) If the rank of the matrix $\left[x^{(1)}x^{(2)}...x^{(m)}\right]$ is equal to 2 then
$\left\{x^{(1)},x^{(2)},...,x^{(m)}\right\} \overset{LS(2)}{\cong} \left\{y^{(1)},y^{(2)},...,y^{(m)}\right\}$ *if and only if* $\frac{\langle x^{(i)},x^{(j)}\rangle}{\langle x^{(1)},x^{(1)}\rangle} = \frac{\langle y^{(i)},y^{(j)}\rangle}{\langle y^{(1)},y^{(1)}\rangle}$ *where $i,j = 1,2,...,m$ and $i \leq j$.*

(b) If the rank of the matrix $\left[x^{(1)}x^{(2)}...x^{(m)}\right]$ is equal to 1 then
$\left\{x^{(1)},x^{(2)},...,x^{(m)}\right\} \overset{LS(2)}{\cong} \left\{y^{(1)},y^{(2)},...,y^{(m)}\right\}$ *if and only if* $\frac{\langle x^{(1)},x^{(j)}\rangle}{\langle x^{(1)},x^{(1)}\rangle} = \frac{\langle y^{(1)},y^{(j)}\rangle}{\langle y^{(1)},y^{(1)}\rangle}$ *where $j = 2,...,m$*

(c) If the rank of the matrix $\left[x^{(1)}x^{(2)}...x^{(m)}\right]$ is different from the rank of the matrix $\left[y^{(1)}y^{(2)}...y^{(m)}\right]$ then these system can not be $LS(2)$- equivalent.

Proof. (i) Let $x^{(i)} = 0$ and $y^{(i)} \neq 0$ be given for any $i = 1,2,...,m$ and $\left\{x^{(1)},x^{(2)},...,x^{(m)}\right\} \overset{LS(2)}{\cong} \left\{y^{(1)},y^{(2)},...,y^{(m)}\right\}$ be supposed. Then there exist a $\lambda > 0$ and an orthogonal transformation $g \in O(2)$ such that $y^{(j)} = \lambda gx^{(j)}$ for every $j = 1,2,...,m$ is satisfied. So $y^{(i)}$ must be 0 Since $x^{(i)} = 0$. This is a contradiction. Thus these systems $\left\{x^{(1)},x^{(2)},...,x^{(m)}\right\}$ and $\left\{y^{(1)},y^{(2)},...,y^{(m)}\right\}$ are not $LS(2)$- equivalent. Similarly in case $x^{(i)} \neq 0$ and $y^{(i)} = 0$ for any $i = 1,2,...,m$, then we will prove that these systems $\left\{x^{(1)},x^{(2)},...,x^{(m)}\right\}$ and $\left\{y^{(1)},y^{(2)},...,y^{(m)}\right\}$ are not $LS(2)$- equivalent. suppose that these systems are $LS(2)$- equivalent. Then, there exist a $\lambda > 0$ and $g \in O(2)$ such that $y^{(j)} = \lambda gx^{(j)}$ for every

$j=1,2,...,m$ is satisfied. From this equation for the integer i, $x^{(i)}=\frac{1}{\lambda}g^{T}y^{(i)}$ can be written. Then $x^{(i)}$ must be 0 Since $y^{(i)}=0$. This is a contradiction. So these systems $\left\{x^{(1)},x^{(2)},...,x^{(m)}\right\}$ and $\left\{y^{(1)},y^{(2)},...,y^{(m)}\right\}$ are not $LS(2)$-equivalent.

(ii) Let $x^{(i)}=0$ and $y^{(i)}=0$ for any $i=1,2,...,m$ be given and

$$\left\{x^{(1)},x^{(2)},...,x^{(m)}\right\} \overset{LS(2)}{\cong} \left\{y^{(1)},y^{(2)},...,y^{(m)}\right\}$$

be supposed. Then, $y^{(j)}=\lambda gx^{(j)}$ for every $j=1,2,...,m$ is satisfied. Excluding the i th elements of these systems $y^{(j)}=\lambda gx^{(j)}$ for every $j=1,2,...,i-1,i+1,...,m$ is satisfied. So

$$\left\{x^{(1)},...,x^{(i-1)},x^{(i+1)},...,x^{(m)}\right\} \overset{LS(2)}{\cong} \left\{y^{(1)},...,y^{(i-1)},y^{(i+1)},...,y^{(m)}\right\}$$

can be written. Conversely Let

$$\left\{x^{(1)},x^{(2)},...,x^{(i-1)},x^{(i+1)},...,x^{(m)}\right\} \overset{LS(2)}{\cong} \left\{y^{(1)},y^{(2)},...,y^{(i-1)},y^{(i+1)},...,y^{(m)}\right\}$$

be supposed. Then, there exist a $\lambda>0$ and $g\in O(2)$ such that $y^{(j)}=\lambda gx^{(j)}$ for every $j=1,2,...,i-1,i+1,...,m$ We can add ith elements of these systems since $x^{(i)}=0$ and $y^{(i)}=0$ So $y^{(j)}=\lambda gx^{(j)}$ for every $j=1,2,...,m$ is satisfied then $\left\{x^{(1)},x^{(2)},...,x^{(m)}\right\} \overset{LS(2)}{\cong} \left\{y^{(1)},y^{(2)},...,y^{(m)}\right\}$ is obtained.

(iii) (a) Let $x^{(i)}\neq 0$ and $y^{(i)}\neq 0$ for every $i=1,2,...,m$ and the rank of the matrices $\left[x^{(1)}x^{(2)}...x^{(m)}\right]$ and $\left[y^{(1)}y^{(2)}...y^{(m)}\right]$ are equal to 2 be given and $\left\{x^{(1)},x^{(2)},...,x^{(m)}\right\} \overset{LS(2)}{\cong} \left\{y^{(1)},y^{(2)},...,y^{(m)}\right\}$ be supposed. In this case there exist a $\lambda>0$ and $g\in O(2)$ such that $y^{(j)}=\lambda gx^{(j)}$ for every $j=1,2,...,m$. Then

$$\frac{\left\langle y^{(i)},y^{(j)}\right\rangle}{\left\langle y^{(1)},y^{(1)}\right\rangle}=\frac{\left\langle \lambda gx^{(i)},\lambda gx^{(j)}\right\rangle}{\left\langle \lambda gx^{(1)},\lambda gx^{(1)}\right\rangle}=\frac{\left\langle x^{(i)},x^{(j)}\right\rangle}{\left\langle x^{(1)},x^{(1)}\right\rangle}$$

is obtained since the orthogonal transformation $g\in O(2)$ save inner product.
Conversely let this equality $\frac{\left\langle x^{(i)},x^{(j)}\right\rangle}{\left\langle x^{(1)},x^{(1)}\right\rangle}=\frac{\left\langle y^{(i)},y^{(j)}\right\rangle}{\left\langle y^{(1)},y^{(1)}\right\rangle}$ be given for every $i,j=1,...,m$ and $i\leq j$. we will prove that

$$\left\{x^{(1)},x^{(2)},...,x^{(m)}\right\} \overset{LS(2)}{\cong} \left\{y^{(1)},y^{(2)},...,y^{(m)}\right\}.$$

Now from hyphothesis

$$\left\langle y^{(i)},y^{(j)}\right\rangle=\frac{\left\langle y^{(1)},y^{(1)}\right\rangle}{\left\langle x^{(1)},x^{(1)}\right\rangle}\left\langle x^{(i)},x^{(j)}\right\rangle$$

can be written. So if the multiple $\frac{\left\langle y^{(1)},y^{(1)}\right\rangle}{\left\langle x^{(1)},x^{(1)}\right\rangle}>0$ is denoted by λ^{2}, then

$$\left\langle y^{(i)},y^{(j)}\right\rangle=\left\langle \lambda x^{(i)},\lambda x^{(j)}\right\rangle \tag{11}$$

is obtained for every $i,j = 1,...,m$ and $i \leq j$. Since the vector systems $\left\{x^{(1)},x^{(2)},...,x^{(m)}\right\}$ and $\left\{y^{(1)},y^{(2)},...,y^{(m)}\right\}$ in two dimensional Euclidean space are linearly dependent and the rank of the matrix $\left[x^{(1)}x^{(2)}...x^{(m)}\right]$ is equal to 2 , it is necessary to determine which vectors in these systems are linearly independent. Suppose that $\exists k,l \in \{1,...,m\}$ such that the vectors $x^{(k)}$ and $x^{(l)}$ are linearly independent. In this case the vectors $y^{(k)}$ and $y^{(l)}$ are also linearly independent since the Gram determinant of the vectors $\lambda x^{(k)}$ and $\lambda x^{(l)}$

$$\left|Gr\left(\lambda x^{(k)},\lambda x^{(l)}\right)\right| = \begin{vmatrix} \left\langle \lambda x^{(k)},\lambda x^{(k)}\right\rangle & \left\langle \lambda x^{(k)},\lambda x^{(l)}\right\rangle \\ \left\langle \lambda x^{(l)},\lambda x^{(k)}\right\rangle & \left\langle \lambda x^{(l)},\lambda x^{(l)}\right\rangle \end{vmatrix}$$

is different from zero. So

$$\begin{aligned} x^{(i)} &= \alpha_{ik}x^{(k)} + \alpha_{il}x^{(l)}, \\ y^{(i)} &= \beta_{ik}y^{(k)} + \beta_{il}y^{(l)} \end{aligned} \tag{12}$$

for every integer $i = 1,2,...m$ are obtained. Since

$$\left[y^{(k)}y^{(l)}\right]^T \left[y^{(k)}y^{(l)}\right] = Gr\left(y^{(k)},y^{(l)}\right)$$

From (11) the equality of the matrices

$$\left[\lambda x^{(k)}\lambda x^{(l)}\right]^T \left[\lambda x^{(k)}\lambda x^{(l)}\right] = \left[y^{(k)}y^{(l)}\right]^T \left[y^{(k)}y^{(l)}\right] \tag{13}$$

is obtained. Because the vectors $\left\{x^{(k)},x^{(l)}\right\}$ and $\left\{y^{(k)},y^{(l)}\right\}$ are linearly independent the matrices $\left[\lambda x^{(k)}\lambda x^{(l)}\right]$ and $\left[y^{(k)}y^{(l)}\right]$ are regular and have inverses. So there exist a regular matrix g such that

$$\left[y^{(k)}y^{(l)}\right] = g\left[\lambda x^{(k)}\lambda x^{(l)}\right] \tag{14}$$

satisfies. If the equality (14) Substitutes to (13) and multiply both sides of (13) by $\left(\left[\lambda x^{(k)}\lambda x^{(l)}\right]^T\right)^{-1}$ at left firstly and by $\left[\lambda x^{(k)}\lambda x^{(l)}\right]^{-1}$ at right secondly, then

$$I = g^T g$$

is obtained. this means that g is orthogonal i.e. $g \in O(2)$. From (14),

$$y^{(k)} = \lambda g x^{(k)}$$

and

$$y^{(l)} = \lambda g x^{(l)}$$

are obtained. From this and (12)

$$\lambda x^{(i)} = \alpha_{ik}\lambda x^{(k)} + \alpha_{il}\lambda x^{(l)}$$

and

$$g(\lambda x^{(i)}) = \alpha_{ik}y^{(k)} + \alpha_{il}y^{(l)} \tag{15}$$

for every $i=1,2,...,m$ are obtained. From (12) and (15).

$$y^{(s)} = \beta_{sk}y^{(k)} + \beta_{sl}y^{(l)}$$
$$\lambda g x^{(s)} = \alpha_{sk}y^{(k)} + \alpha_{sl}y^{(l)}$$

can be written for $s=1,2,...,m$ The equality of these vectors $y^{(s)}$ and $\lambda g x^{(s)}$ depends on the equalities of multiples $\beta_{sk}=\alpha_{sk}$ and $\beta_{sl}=\alpha_{sl}$. let us prove these equalities. From (12)

$$\left\langle \lambda x^{(k)}, \lambda x^{(s)} \right\rangle = \alpha_{sk}\left\langle \lambda x^{(k)}, \lambda x^{(k)} \right\rangle + \alpha_{sl}\left\langle \lambda x^{(k)}, \lambda x^{(l)} \right\rangle$$

$$\left\langle \lambda x^{(l)}, \lambda x^{(s)} \right\rangle = \alpha_{sk}\left\langle \lambda x^{(l)}, \lambda x^{(k)} \right\rangle + \alpha_{sl}\left\langle \lambda x^{(l)}, \lambda x^{(l)} \right\rangle$$

and

$$\left\langle y^{(k)}, y^{(s)} \right\rangle = \beta_{sk}\left\langle y^{(k)}, x^{(k)} \right\rangle + \beta_{sl}\left\langle y^{(k)}, y^{(l)} \right\rangle$$
$$\left\langle y^{(l)}, y^{(s)} \right\rangle = \beta_{sk}\left\langle y^{(l)}, y^{(k)} \right\rangle + \beta_{sl}\left\langle y^{(l)}, y^{(l)} \right\rangle$$

can be wriiten. Solving these linear equations

$$\alpha_{sk} = \frac{\begin{vmatrix} \left\langle \lambda x^{(k)}, \lambda x^{(s)} \right\rangle & \left\langle \lambda x^{(k)}, \lambda x^{(l)} \right\rangle \\ \left\langle \lambda x^{(l)}, \lambda x^{(s)} \right\rangle & \left\langle \lambda x^{(l)}, \lambda x^{(l)} \right\rangle \end{vmatrix}}{\det Gr\left(\lambda x^{(k)}, \lambda x^{(l)}\right)}$$

$$\alpha_{sl} = \frac{\begin{vmatrix} \left\langle \lambda x^{(k)}, \lambda x^{(k)} \right\rangle & \left\langle \lambda x^{(k)}, \lambda x^{(s)} \right\rangle \\ \left\langle \lambda x^{(l)}, \lambda x^{(k)} \right\rangle & \left\langle \lambda x^{(l)}, \lambda x^{(s)} \right\rangle \end{vmatrix}}{\det Gr\left(\lambda x^{(k)}, \lambda x^{(l)}\right)}$$

$$\beta_{sk} = \frac{\begin{vmatrix} \left\langle y^{(k)}, y^{(s)} \right\rangle & \left\langle y^{(k)}, y^{(l)} \right\rangle \\ \left\langle y^{(l)}, y^{(s)} \right\rangle & \left\langle y^{(l)}, y^{(l)} \right\rangle \end{vmatrix}}{\det Gr\left(y^{(k)}, y^{(l)}\right)}$$

$$\beta_{sl} = \frac{\begin{vmatrix} \left\langle y^{(k)}, y^{(k)} \right\rangle & \left\langle y^{(k)}, y^{(s)} \right\rangle \\ \left\langle y^{(l)}, y^{(k)} \right\rangle & \left\langle y^{(l)}, y^{(s)} \right\rangle \end{vmatrix}}{\det Gr\left(y^{(k)}, y^{(l)}\right)}$$

is obtained for every $s=1,2,...m$. Using (11) $\beta_{sk}=\alpha_{sk}$ and $\beta_{sl}=\alpha_{sl}$ is obtained. So $\left\{x^{(1)}, x^{(2)}, ..., x^{(m)}\right\} \overset{LS(2)}{\cong} \left\{y^{(1)}, y^{(2)}, ..., y^{(m)}\right\}$ is proved.

(b)Let $x^{(i)} \neq 0$ and $y^{(i)} \neq 0$ for every $i=1,2,...,m$ and the rank of the matrices $\left[x^{(1)}x^{(2)}...x^{(m)}\right]$ and $\left[y^{(1)}y^{(2)}...y^{(m)}\right]$ are equal to 1 be given and $\left\{x^{(1)}, x^{(2)}, ..., x^{(m)}\right\} \overset{LS(2)}{\cong} \left\{y^{(1)}, y^{(2)}, ..., y^{(m)}\right\}$ be supposed. In this case there exist a $\lambda > 0$ and $g \in O(2)$ such that $y^{(j)} = \lambda g x^{(j)}$ for every $j=1,2,...,m$. since the rank of aboved matrices is 1, there exist one linear independent vector in each system $\left\{x^{(1)}, x^{(2)}, ..., x^{(m)}\right\}$ and $\left\{y^{(1)}, y^{(2)}, ..., y^{(m)}\right\}$. Let these linear independent vectors be denoted $x^{(s)}$ and $y^{(s)}$ for any $s \in \{1,...m\}$. So for every $i=1,...,m$

$$x^{(i)} = l_i x^{(s)} \ ; \ y^{(i)} = m_i y^{(s)} \tag{16}$$

can be written. So from (16)

$$\frac{m_j}{m_1} = \frac{\left\langle y^{(1)}, y^{(j)} \right\rangle}{\left\langle y^{(1)}, y^{(1)} \right\rangle} = \frac{\left\langle x^{(1)}, x^{(j)} \right\rangle}{\left\langle x^{(1)}, x^{(1)} \right\rangle} = \frac{l_j}{l_1}$$

is obtained for every $j = 2, \ldots, m$.

Conversely let this equality $\frac{\left\langle x^{(1)}, x^{(j)} \right\rangle}{\left\langle x^{(1)}, x^{(1)} \right\rangle} = \frac{\left\langle y^{(1)}, y^{(j)} \right\rangle}{\left\langle y^{(1)}, y^{(1)} \right\rangle}$ be given for every $j = 2, \ldots, m$. we will prove that

$$\left\{ x^{(1)}, x^{(2)}, \ldots, x^{(m)} \right\} \overset{LS(2)}{\cong} \left\{ y^{(1)}, y^{(2)}, \ldots, y^{(m)} \right\}.$$

From theorem 3.1 $x^{(s)} \overset{LS(2)}{\cong} y^{(s)}$ and then there exist a $\lambda > 0$ and $g \in O(2)$ such that $y^{(s)} = \lambda g x^{(s)}$ then from hyphothesis and (16)

$$m_j = \frac{m_1}{l_1} l_j$$

is obtained where $\frac{m_1}{l_1} = \sqrt{\frac{\left\langle y^{(1)}, y^{(1)} \right\rangle}{\left\langle x^{(1)}, x^{(1)} \right\rangle}} > 0$. If the real number $\frac{m_1}{l_1}\lambda$ is denoted by $\overline{\lambda}$, then $\overline{\lambda} > 0$. So for every $j = 1, \ldots, m$;

$$y^{(j)} = m_j y^{(s)} = \frac{m_1}{l_1} l_j (\lambda g x^{(s)}) = \overline{\lambda} g x^{(j)}$$

can be written. So

$$\left\{ x^{(1)}, x^{(2)}, \ldots, x^{(m)} \right\} \overset{LS(2)}{\cong} \left\{ y^{(1)}, y^{(2)}, \ldots, y^{(m)} \right\}.$$

is proved.

(c) Let $x^{(i)} \neq 0$ and $y^{(i)} \neq 0$ for every $i = 1, 2, \ldots, m$ and the ranks of the matrices $\left[x^{(1)} x^{(2)} \ldots x^{(m)} \right]$ and $\left[y^{(1)} y^{(2)} \ldots y^{(m)} \right]$ be different from each other. firstly each rank of given matrices must be different from zero. Otherwise it means all of the vectors in these systems are zero. it is mentioned above. Just let $rank \left[x^{(1)} x^{(2)} \ldots x^{(m)} \right] = 1$ and $rank \left[x^{(1)} x^{(2)} \ldots x^{(m)} \right] = 2$ be given. In this case there exist the integers $i, j, s \in \{1, 2, \ldots, m\}$ such that the vector $x^{(i)}$ is linearly independent in the system $\left\{ x^{(1)}, x^{(2)}, \ldots, x^{(m)} \right\}$ and these vectors $y^{(j)}, y^{(s)}$ are linearly independent in the system $\left\{ y^{(1)}, y^{(2)}, \ldots, y^{(m)} \right\}$. So

$$x^{(k)} = a_k x^{(i)}$$

$$y^{(k)} = b_{kj} y^{(j)} + b_{ks} y^{(s)}$$

can be written for every $k = 1, 2, \ldots, m$. It is clear that $a_i = b_{jj} = b_{ss} = 1$ and $b_{js} = b_{sj} = 0$. Let $\left\{ x^{(1)}, x^{(2)}, \ldots, x^{(m)} \right\} \overset{LS(2)}{\cong} \left\{ y^{(1)}, y^{(2)}, \ldots, y^{(m)} \right\}$ be supposed. In this case there exist a $\lambda > 0$ and $g \in O(2)$ such that $y^{(k)} = \lambda g x^{(k)}$ for every $k = 1, \ldots m$. Then it follows

$$y^{(k)} = \lambda g \left(a_k x^{(i)} \right) = a_k y^{(i)}$$

for every $k=1,\ldots m$. It means each vectors $y^{(1)},y^{(2)},\ldots,y^{(m)}$ can be expressed by the vector $x^{(i)}$. This is a contradiction and so $\left\{x^{(1)},x^{(2)},\ldots,x^{(m)}\right\} \overset{LS(2)}{\ncong} \left\{y^{(1)},y^{(2)},\ldots,y^{(m)}\right\}$.

Let $rank\left[x^{(1)}x^{(2)}\ldots x^{(m)}\right]=2$ and rank $\left[x^{(1)}x^{(2)}\ldots x^{(m)}\right]=1$ be given. In this case there exist the integers $i,j,s\in\{1,2,\ldots,m\}$ such that these vectors $x^{(i)},x^{(j)}$ are linearly independent in the system $\left\{x^{(1)},x^{(2)},\ldots,x^{(m)}\right\}$ and the vector $y^{(s)}$ is linearly independent in the system $\left\{y^{(1)},y^{(2)},\ldots,y^{(m)}\right\}$ be given. So,

$$y^{(k)}=a_k y^{(s)}$$

can be written for every $k=1,\ldots m$. Let $\left\{x^{(1)},x^{(2)},\ldots,x^{(m)}\right\} \overset{LS(2)}{\cong} \left\{y^{(1)},y^{(2)},\ldots,y^{(m)}\right\}$ be supposed. In this case there exist a $\lambda>0$ and $g\in O(2)$ such that $y^{(k)}=\lambda g x^{(k)}$ for every $k=1,\ldots m$. This statement can be written as matrix form as

$$\left[y^{(1)}y^{(2)}\ldots y^{(m)}\right]=g\left[\lambda x^{(1)}\lambda x^{(2)}\ldots\lambda x^{(m)}\right]$$

It follows

$$\left[\lambda x^{(1)}\lambda x^{(2)}\ldots\lambda x^{(m)}\right]=g^T\left[y^{(1)}y^{(2)}\ldots y^{(m)}\right]$$

It means for every $k=1,\ldots,m$

$$\lambda x^{(k)}=g^T\left(a_k y^{(s)}\right)=a_k\lambda x^{(s)}$$

can be written. And every vectors $\left\{\lambda x^{(1)},\lambda x^{(2)},\ldots,\lambda x^{(m)}\right\}$ can be stated by the vector $\lambda x^{(s)}$ and it is a contradiction. So the assumption is wrong. As a result of this

$$\left\{x^{(1)},x^{(2)},\ldots,x^{(m)}\right\} \overset{LS(2)}{\ncong} \left\{y^{(1)},y^{(2)},\ldots,y^{(m)}\right\}$$

4 Application to planar Bezier curves

Bezier Curves are polynomial curves stated by control points. Recently the invariants of control points have had an important role in the systems CAD and CAM. Especially Bezier and B- spline curves and surfaces and NURBS modelling based on control points belongs to these curves and surfaces. It is extremely important to determine whether or not the surface modellings have durable, precision, and applicable outputs in the geometries in view of the reliability of the imagings of the mechanism. Therefore it is necessary that the invariant properties of modelling of mechanism is able to be known, so that the realized modellings introduce durable and precision, and reliable results under transformation groups. In this sense, knowing of complete of invariants of Bezier and B-spline curves and surfaces which gives most stable solutions within CAD systems [14] is important.

4.1 Linear bezier curves

A linear Bezier curve $X(t)$ with control points b_0,b_1 is defined by

$$X(t)=(1-t)b_0+tb_1$$

where $t\in[0,1]$ [12] .

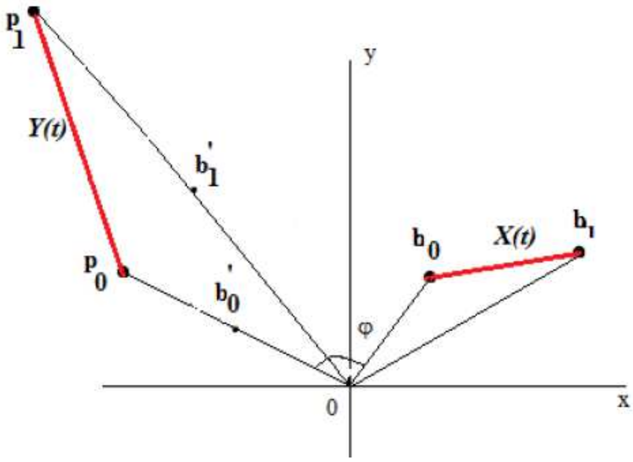

Fig. 1: $LS(2)-$ equivalent linear Bezier curves $X(t)$ and $Y(t)$.

Theorem 9. *Let X,Y be given two linear Bezier curves with control points b_0,b_1 and p_0,p_1 in R^2 respectively. So,*

(i) *If any $b_i=(0,0)$ and any $p_j=(0,0)$ for $i,j=0,1$ then these curves are always $LS(2)-$equivalent.*

(ii) *If any $b_i=(0,0)$ and each $p_j\neq(0,0)$ or each $b_i\neq(0,0)$ and any $p_j=(0,0)$ for $i,j=0,1$ then these curves are not $LS(2)-$equivalent.*

(iii) *If every $b_i\neq(0,0)$ and $p_j\neq(0,0)$ for $i,j=0,1$ then these Bezier curves are $LS(2)-$equivalent if and only if $\frac{\langle b_0,b_1\rangle}{\langle b_0,b_0\rangle}=\frac{\langle p_0,p_1\rangle}{\langle p_0,p_0\rangle}$ and $\frac{\langle b_1,b_1\rangle}{\langle b_0,b_0\rangle}=\frac{\langle p_1,p_1\rangle}{\langle p_0,p_0\rangle}$.*

Proof. This theorem is a conclusion of Theorem 2. In first case the ranks of matrice of control points of each Bezier curves X and Y are equal to 1. In second case the ranks of matrice of control points of each Bezier curves X and Y are different from each other. In third case the ranks of matrice of control points of each Bezier curves X and Y are equal to 2.

4.2 Quadratic Bezier curves

A quadratic Bezier curve $X(t)$ with control points b_0,b_1,b_2 is defined by

$$X(t)=(1-t)^2b_0+2(1-t)tb_1+t^2b_2$$

where $t\in[0,1]$ [12] .

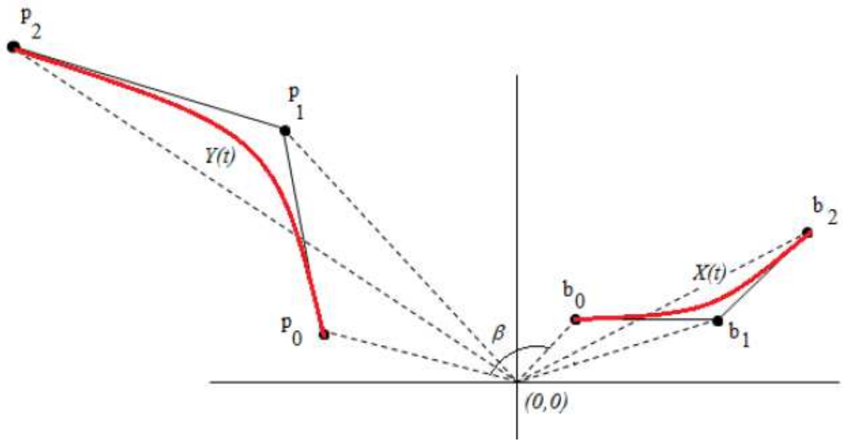

Fig. 2: $LS(2)-$ equivalent quadratic Bezier curves $X(t)$ and $Y(t)$.

Theorem 10. *Let X, Y be given two quadratic Bezier curves with control points b_0, b_1, b_2 and p_0, p_1, p_2 in R^2 respectively. So,*

(i) *If any $b_i = (0,0)$ and $p_i = (0,0)$ for $i = 0,1,2$ then these curves are $LS(2)-$equivalent if and only if the condition of third case of theorem 4.1 is satisfied excluding b_i and p_i.*

(ii) *If any $b_i = (0,0)$ and each $p_j \neq (0,0)$ or each $b_i \neq (0,0)$ and any $p_j = (0,0)$ for $i, j = 0,1,2$ then these curves are not $LS(2)-$equivalent.*

(iii) *If every $b_i \neq (0,0)$ and $p_j \neq (0,0)$ for $i, j = 0,1,2$ then these Bezier curves are $LS(2)-$equivalent if and only if* $\frac{\langle b_0,b_1\rangle}{\langle b_0,b_0\rangle} = \frac{\langle p_0,p_1\rangle}{\langle p_0,p_0\rangle}$, $\frac{\langle b_1,b_1\rangle}{\langle b_0,b_0\rangle} = \frac{\langle p_1,p_1\rangle}{\langle p_0,p_0\rangle}$, $\frac{\langle b_0,b_2\rangle}{\langle b_0,b_0\rangle} = \frac{\langle p_0,p_2\rangle}{\langle p_0,p_0\rangle}$, $\frac{\langle b_1,b_2\rangle}{\langle b_0,b_0\rangle} = \frac{\langle p_1,p_2\rangle}{\langle p_0,p_0\rangle}$, $\frac{\langle b_2,b_2\rangle}{\langle b_0,b_0\rangle} = \frac{\langle p_2,p_2\rangle}{\langle p_0,p_0\rangle}$.

Proof. This theorem is also a conclusion of Theorem 3.2. Since the Bezier Curves are quadratic in this theorem, the ranks of matrice of control points of each Bezier curves X and Y are equal to 2.

4.3 Cubic Bezier curves

A Cubic Bezier Curve $X(t)$ with control points b_0, b_1, b_2, b_3 is defined by

$$X(t) = (1-t)^3 b_0 + 3(1-t)^2 t b_1 + 3(1-t)t^2 b_2 + t^3 b_3$$

where $t \in [0,1]$ [12] .

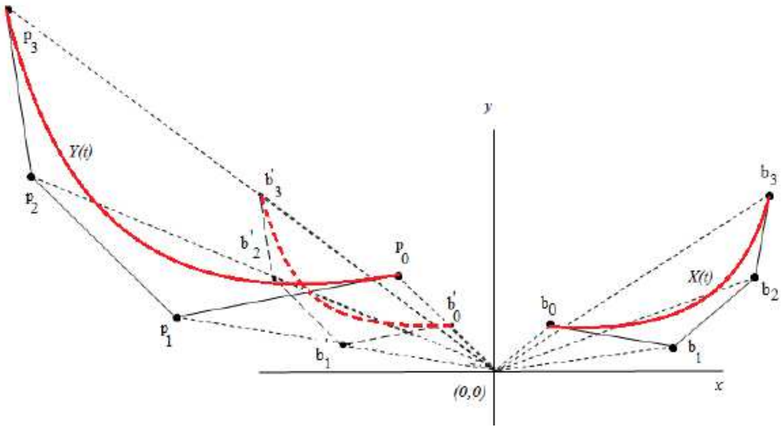

Fig. 3: $LS(2)-$ equivalent cubic Bezier curves $X(t)$ and $Y(t)$.

Theorem 11. *Let X,Y be given two cubic Bezier curves with control points b_0,b_1,b_2,b_3 and p_0,p_1,p_2,p_3 in R^2 respectively. So,*

(i) *If any $b_i = (0,0)$ and $p_i = (0,0)$ for $i,j = 0,1,2,3$ then these curves are $LS(2)-$equivalent if and only if the condition of third case of theorem 11 is satisfied excluding b_i and p_i.*

(ii) *If any $b_i = (0,0)$ and each $p_j \neq (0,0)$ or each $b_i \neq (0,0)$ and any $p_j = (0,0)$ for $i,j = 0,1,2,3$ then these curves are not $LS(2)-$equivalent.*

(iii) *If every $b_i \neq (0,0)$ and $p_j \neq (0,0)$ for $i,j = 0,1,2,3$ then these Bezier curves are $LS(2)-$equivalent if and only if*

$$\frac{\langle b_i, b_j\rangle}{\langle b_0, b_0\rangle} = \frac{\langle p_i, p_j\rangle}{\langle p_0, p_0\rangle}$$

is satisfied. where $i,j = 0,1,2,3$ and $i \leq j$.

Proof. This theorem is also a conclusion of Theorem 8. Since the Bezier Curves are cubic in this theorem, the ranks of matrice of control points of each Bezier curves X and Y are also equal to 2.

4.4 General Bezier curves

A general Bezier curve $X(t)$ of degree n with control points $b_0,b_1,b_2,...,b_n$ is defined by

$$X(t) = \sum_{i=0}^{n} B_i^n(t) b_i$$

where $t \in [0,1]$ and $B_i^n(t)$ are Bernstein basis polynomials of degree n defined by

$$B_i^n(t) = \binom{n}{i} (1-t)^{n-i} t^i$$

[13]

Theorem 12. *Let X,Y be given two general Bezier curves of degree n with control points $b_0,b_1,b_2,...,b_n$ and $p_0,p_1,p_2,...,p_n$ in R^2 respectively. So,*

(i) *If any $b_i = (0,0)$ and any $p_i = (0,0)$ for $i = 0,1,2,...,n$ then these curves are $LS(2)-$equivalent if and only if the condition of third case of theorem 11 is satisfied excluding b_i and p_i.*

(ii) *If any $b_i = (0,0)$ and each $p_j \neq (0,0)$ or each $b_i \neq (0,0)$ and any $p_j = (0,0)$ for $i, j = 0,1,2,...,n$ then these curves are not $LS(2)-$equivalent.*

(iii) *If every $b_i \neq (0,0)$ and $p_j \neq (0,0)$ for $i, j = 0,1,2,...,n$ then these Bezier curves are $LS(2)-$equivalent if and only if*

$$\frac{\langle b_i, b_j \rangle}{\langle b_0, b_0 \rangle} = \frac{\langle p_i, p_j \rangle}{\langle p_0, p_0 \rangle}$$

is satisfied. where $i, j = 0,1,2,...,n$ and $i \leq j$.

Proof. This theorem is also a conclusion of Theorem 3.2. The ranks of matrice of control points of each Bezier curves X and Y are also equal to 2.

Acknowledgement

The author acknowledges the financial support by the BAP project of Karadeniz Tachnical University Research Fund - 2004.111.003.1, 2004.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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