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On initial-boundary value problem for nonlinear integro-differential equations with variable exponents of nonlinearity

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Abstract: Some nonlinear parabolic integro-differential equations with variable exponents of the nonlinearity are considered. The initial-boundary value problem for these equations is investigated and the existence theorem for the problem is proved.

Keywords: Nonlinear parabolic equation, integro-differential equation, generalized Lebesgue space, variable exponents of nonlinearity.

1 Introduction

Let $n \in \mathbb{N}$ and $T > 0$ be fixed numbers, $\Omega \subset \mathbb{R}^n$ be a bounded domain with the boundary $\partial\Omega$, $Q_{0,T} := \Omega \times (0,T)$, $\Sigma_{0,T} := \partial\Omega \times (0,T)$, $\Omega_\tau := \{(x,t) \mid x \in \Omega,\ t = \tau\}$, $\tau \in \mathbb{R}$. We seek a weak solution $u : Q_{0,T} \to \mathbb{R}^1$ of the problem

$$|u|^{r-2}u_t - a\Delta u + g(x,t)|u|^{q(x)-2}u + \phi(Eu) = f(x,t), \quad (x,t) \in Q_{0,T}, \tag{1}$$

$$u|_{\Sigma_{0,T}} = 0, \qquad u|_{t=0} = u_0(x), \quad x \in \Omega, \tag{2}$$

where $a > 0$ and $r > 1$ are some numbers, $\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$ is the Laplacian,

$$(Eu)(x,t) := \int_\Omega \mathfrak{Z}(x,t,z)\big(\widetilde{u}(x+z,t) - \widetilde{u}(x,t)\big)\, dz, \quad (x,t) \in Q_{0,T}, \tag{3}$$

$g, q, \phi, \mathfrak{Z}, f, u_0$ are some functions, and $\widetilde{u}$ is zero extension of u from $Q_{0,T}$ into $(\mathbb{R}^n \setminus \Omega) \times (0,T)$.

The equations of type (1) have been widely used in many applications. For example (see [4]), let us consider a circuit, which consists of a resistance, a condenser, and a generator of the jump impulses. It is well known that the supply $U(t)$ satisfies the stochastic differential equation

$$dU(t) = -\frac{1}{RC}U(t)\, dt + \sigma\, dW(t) + dQ(t), \quad t \in [0,T], \qquad U(0) = U_0, \tag{4}$$

where R is a value of the resistance, C is a capacitance of the condenser, σ is a intensity of the loss of the signal, U_0 is a start supply, $T = RC$, $\{W(t)\}_{t\in[0,T]}$ is the standard Brownian motion, and $Q(t)$ is the compound Poisson process. Let

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$u(x,t)$, $x\in\mathbb{R}$, be a density of the stochastic process $U(t)$, $t\in[0,T]$. If some additional conditions are satisfied, then (see [4, p. 4]) u is a solution to the Cauchy problem for the Kolmogorov-Feller equation

$$u_t-\frac{\sigma^2}{2}u_{xx}-\frac{1}{RC}(xu)_x-\lambda\int\limits_{\mathbb{R}}\psi(\xi,t)\big(u(x+\xi,t)-u(x,t)\big)\,d\xi=0,\quad (x,t)\in\mathbb{R}\times(0,T),\quad u(x,0)=u_0(x),\quad x\in\mathbb{R},\quad (5)$$

where u_0 is a density of U_0, λ is a intensity of the jump creation, ψ is a density of the jump size.

If $n=1$, then equation (1) is a nonlinear modification of (5) in bounded domain $Q_{0,T}$ instead of the unbounded strip $\mathbb{R}\times(0,T)$. Equation (5) also arises in the Merton and Kou models of the option pricing (see [15], [31], and [34]). The Cauchy problem for the equations of type (5) is considered in [9], [16], [21], and [41].

The number r and the function q (see (1)) are called exponents of the nonlinearity of double nonlinear parabolic equation (1). Since q is a function, here we have a variable exponent of the nonlinearity. The double nonlinear parabolic equations with variable exponents of the nonlinearity without the integral terms are considered in [2], [7]. In [6], [17], [18], [19], [39], and [40], the authors investigate the problems for nonlinear parabolic equations with constant exponents of the nonlinearity and with the integral terms which differ from term (3).

In [38], J.P. Pinasco proved the existence of the local solution and the nonexistence of the global solution to the initial-boundary value problem for the equation

$$u_t-\Delta u-|u|^{\alpha(x)}=\int\limits_{\Omega}|u(z,t)|^{\beta(z)}\,dz,\qquad (6)$$

where $\alpha(x)>1$ and $\beta(y)>1$.

In this paper we extend our investigations which is started in [13] for the double nonlinear equation

$$u_t-a\Delta(|u|^{\gamma-2}u)+g(x,t)|u|^{q(x)-2}u+\phi(Eu)=f(x,t),\quad (x,t)\in Q_{0,T}.\qquad (7)$$

We recall that if $\gamma\in[2,3)$, then the existence of the weak solution to problem (7), (2) is proved in [13]. The Cauchy problem for the equations of type (7) is considered in [20] if $q(x)\equiv 2$, $\phi(s)=s$, and $\gamma=3$.

2 Notation and statement of main result

Let $(\cdot,\cdot)_H$ be a scalar product of some Hilbert space H, $||\cdot||_B\equiv||\cdot;B||$ a norm of some Banach space B, B^* a dual space, and $\langle\cdot,\cdot\rangle_B$ a scalar product between B^* and B. For the Banach spaces X and Y the notation $X\circlearrowleft Y$ means the continuous embedding; the notation $X\,\bar{\circlearrowleft}\,Y$ means the continuous and densely embedding; the notation $X\overset{K}{\subset}Y$ means the compact embedding.

Suppose that $m,N\in\mathbb{N}$, $p\in[1,\infty]$, $Q=\Omega$ or $Q=Q_{0,T}$, $\mathscr{M}(Q)$ is a set of all measurable functions $v:Q\to\mathbb{R}$ (see [24, p. 120]), $\mathrm{Lip}(Q)$ is a set of all Lipschitz-continuous functions $v:Q\to\mathbb{R}$ (see [32, p. 29]), $C^m(Q)$ and $C_0^\infty(Q)$ are determined from [1, p. 9], $L^p(Q)$ is the Lebesgue space (see [1, p. 22, 24]), $W^{m,p}(Q)$ and $W_0^{m,p}(Q)$ are Sobolev spaces (see [1, p. 45]), $H^m(Q):=W^{m,2}(Q)$, $H_0^m(Q):=W_0^{m,2}(Q)$, $C([0,T];X)$ and $C^m([0,T];X)$ are determined from [27, p. 147], $L^p(0,T;X)$ is determined from [27, p. 155], $W^{m,p}(0,T;X)$ is determined from [25, p. 286],

$H^m(0,T;X) := W^{m,2}(0,T;X)$, and

$$\mathscr{B}_+(\mathsf{Q}) := \{\delta \in L^\infty(\mathsf{Q}) \mid \operatorname*{ess\,inf}_{y\in\mathsf{Q}} \delta(y) > 0\}.$$

For every $\delta \in \mathscr{B}_+(\mathsf{Q})$, by definition, put

$$\delta_0 := \operatorname*{ess\,inf}_{y\in\mathsf{Q}} \delta(y), \quad \delta^0 := \operatorname*{ess\,sup}_{y\in\mathsf{Q}} \delta(y), \qquad S_\delta(s) := \max\{s^{\delta_0}, s^{\delta^0}\}, \quad s \geq 0, \tag{8}$$

$$\delta'(y) := \frac{\delta(y)}{\delta(y)-1} \quad \text{for a.e.} \quad y \in \mathsf{Q} \quad \left(\text{note that} \quad \frac{1}{\delta(y)} + \frac{1}{\delta'(y)} = 1 \quad \text{and} \quad \delta' \in \mathscr{B}_+(\mathsf{Q})\right), \tag{9}$$

$$\rho_\delta(v;\mathsf{Q}) := \int_{\mathsf{Q}} |v(y)|^{\delta(y)}\, dy, \quad v \in \mathscr{M}(\mathsf{Q}). \tag{10}$$

For every function $u \in L^1(Q_{0,T}) = L^1(0,T;L^1(\Omega))$ we have $u(\cdot,t) \in L^1(\Omega)$, $t \in (0,T)$. For the sake of convenience we shall write $u(t)$ instead of $u(\cdot,t)$.

Assume that $\delta \in \mathscr{B}_+(\mathsf{Q})$ and $\delta_0 > 1$. The set

$$L^{\delta(y)}(\mathsf{Q}) := \{v \in \mathscr{M}(\mathsf{Q}) \mid \rho_\delta(v;\mathsf{Q}) < +\infty\}$$

with the Luxemburg norm

$$\|v; L^{\delta(y)}(\mathsf{Q})\| := \inf\{\lambda > 0 \mid \rho_\delta(v/\lambda;\mathsf{Q}) \leq 1\}$$

is called a generalized Lebesgue space. It is well known that $L^{\delta(y)}(\mathsf{Q})$ is the Banach space which is reflexive and separable (see [30, p. 599, 600, 604]). The generalized Lebesgue space was introduced in [36]. Its properties were widely studied in [2], [10], [22], [26], and [30].

We shall need the following assumptions:

- **(A):** $a > 0$, $\frac{3}{2} < r \leq 2$;
- **(Q):** $q \in \mathscr{B}_+(\Omega)$, $1 < q_0 \leq q^0 \leq 2$;
- **(G):** $g \in \mathscr{B}_+(Q_{0,T})$;
- **(E):** $\mathfrak{Z} \in L^\infty(Q_{0,T} \times \Omega)$;
- **(Φ):** $\phi \in \operatorname{Lip}(\mathbb{R})$, $|\phi(\xi)| \leq \phi^*|\xi|$ for every $\xi \in \mathbb{R}$, where $\phi^* \in [0,+\infty)$;
- **(F):** $f \in L^{r'}(Q_{0,T})$, where $\frac{1}{r} + \frac{1}{r'} = 1$;
- **(U):** $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$.

By definition, put

$$\gamma := \frac{4}{4-r} \quad \left(\text{note that} \quad \gamma \in \left(\frac{8}{5}, 2\right] \quad \text{iff} \quad r \in \left(\frac{3}{2}, 2\right]\right), \tag{11}$$

$$(Gu)(x,t) := g(x,t)|u(x,t)|^{q(x)-2}u(x,t), \quad (x,t) \in Q_{0,T}, \quad u \in \mathscr{M}(Q_{0,T}), \tag{12}$$

$$(\mathscr{R}u)(x,t) := \frac{1}{r-1}|u(x,t)|^{r-2}u(x,t), \quad (x,t) \in Q_{0,T}, \quad u \in \mathscr{M}(Q_{0,T}). \tag{13}$$

Notice that if $u \in C^1(Q_{0,T})$ and $u \neq 0$, then

$$(\mathscr{R}u)_t = |u|^{r-2}u_t. \tag{14}$$

Then we formally rewrite equation (1) as

$$(\mathscr{R}u)_t - a\Delta u + Gu + \phi(Eu) = f(x,t), \quad (x,t) \in Q_{0,T}. \tag{1'}$$

Definition 1. *A real-valued function* $u \in L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^\gamma(\Omega))$ *is called a weak solution of problem (1)-(2) if* $\mathscr{R}u, Gu, Eu \in L^2(Q_{0,T})$, *u satisfies (2), and for every* $v \in H_0^1(Q_{0,T})$ *we have*

$$\int\limits_{Q_{0,T}} \Big[-\mathscr{R}u\, v_t + a(\nabla u,\nabla v) + Gu\, v + \phi(Eu)\, v\Big]\, dxdt = \int\limits_{Q_{0,T}} fv\, dxdt. \tag{15}$$

Here $\nabla v := (v_{x_1},\dots,v_{x_n})$, $(\cdot,\cdot)$ is a scalar product in $\mathbb{R}^n$.

Note that the integral operator $E : L^2(Q_{0,T}) \to L^2(Q_{0,T})$ (see (3)) is a linear bounded operator and for every $u \in L^2(Q_{0,T})$ we have

$$||Eu;L^2(Q_{0,T})|| \le E^*||u;L^2(Q_{0,T})||, \tag{16}$$

where $E^* > 0$ is independent of u. Indeed, using the Cauchy-Bunyakowski-Schwarz inequality (see Lemma 6.1 [27, Chapter 1, §6]), we get

$$\int\limits_{Q_{0,T}} |(Eu)(x,t)|^2\, dxdt = \int\limits_{Q_{0,T}} \Big|\int\limits_{\Omega} \mathfrak{Z}(x,t,z)\Big(u(x+z,t)-u(x,t)\Big)\, dz\Big|^2\, dxdt$$
$$\le \int\limits_{Q_{0,T}} \Big(\int\limits_{\Omega} |\mathfrak{Z}(x,t,z)|^2\, dz\Big)\Big(\int\limits_{\Omega} |u(x+z,t)-u(x,t)|^2\, dz\Big)\, dxdt$$
$$\le C_1 \int\limits_{Q_{0,T}} \int\limits_{\Omega} (|u(x+z,t)|^2 + |u(x,t)|^2)\, dz\, dxdt \le C_2 \int\limits_{Q_{0,T}} |u(x,t)|^2\, dxdt$$

and so (16) holds.

Assume that the following condition is satisfied:

(Z): $a > \phi^* E^* M_\Omega$, where a is defined by (1), ϕ^* is determined from **(Φ)**, E^* is defined by (16), M_Ω is determined from the Friedrichs inequality (see Lemma 1.26 [27, Chapter 2, §1])

$$\int\limits_{\Omega} |w|^2\, dx \le M_\Omega \int\limits_{\Omega} \sum_{i=1}^{n} |w_{x_i}|^2\, dx, \quad w \in H_0^1(\Omega). \tag{17}$$

Notice that M_Ω depends on n and does not depend on u.

Theorem 1. *Suppose that* $\partial\Omega \in C^4$, *conditions* **(A)-(U)** *hold, and* **(Z)** *is satisfied. If* $g_t \in L^\infty(Q_{0,T})$ *and if* $\phi \in L^\infty(\mathbb{R})$, *then problem (1)-(2) has a weak solution u such that* $u \in W^{1,\gamma}(Q_{0,T})$ *and* $|u|^{\frac{r}{2}-1}u \in H^1(0,T;L^2(\Omega))$.

3 Auxiliary facts

3.1 Functional spaces and some operators

First we recall some properties of the generalized Lebesgue spases.

Proposition 1. *([37, p. 31]). If* $\delta \in \mathscr{B}_+(Q)$ *and* $\delta_0 > 1$, *then for every* $\eta > 0$ *there exists a number* $Y_\delta(\eta) > 0$ *such that for every* $a,b \ge 0$ *and for a.e.* $y \in Q$ *the generalized Young inequality (the Young inequality, if* $\delta \equiv const$*)*

$$ab \le \eta\, a^{\delta(y)} + Y_q(\eta)\, b^{\delta'(y)} \tag{18}$$

holds. In addition, $Y_\delta(\eta)$ *depends on* δ_0, δ^0 *and it is independent of* y, $Y_2(\eta) = \frac{1}{4\eta}$, $Y_2(\frac{1}{2}) = \frac{1}{2}$, $Y_\delta(+0) = +\infty$, *and* $Y_\delta(+\infty) = 0$.

Proposition 2. *Assume that* $\delta \in \mathscr{B}_+(Q)$ *and* $\delta_0 > 1$. *Then the following statements are fulfilled:*

(i) *([30, p. 600]) if* $\delta(y) \geq r(y) \geq 1$ *for a.e.* $y \in Q$, *then* $L^{\delta(y)}(Q) \circlearrowleft L^{r(y)}(Q)$ *and*

$$||v;L^{r(y)}(Q)|| \leq (1+\operatorname{mes} Q)||v;L^{\delta(y)}(Q)||, \quad v \in L^{\delta(y)}(Q);$$

(ii) *([26, p. 431]) for every* $u \in L^{\delta(y)}(Q)$ *and* $v \in L^{\delta'(y)}(Q)$ *we get* $uv \in L^1(Q)$ *and the following generalized Hölder inequality is true*

$$\int_Q |u(y)v(y)|\,dy \leq 2\,||u;L^{\delta(y)}(Q)|| \cdot ||v;L^{\delta'(y)}(Q)||. \tag{19}$$

Proposition 3. ([11, p. 168]). *Suppose that* $\delta \in \mathscr{B}_+(Q)$, $\delta_0 > 1$, S_δ *is defined by (8), and* ρ_δ *is defined by (10). Then for every* $v \in \mathscr{M}(Q)$ *the following statements are fulfilled:*

(i) $||v;L^{\delta(y)}(Q)|| \leq S_{1/\delta}(\rho_\delta(v;Q))$ *if* $\rho_\delta(v;Q) < +\infty$;
(ii) $\rho_\delta(v;Q) \leq S_\delta(||v;L^{\delta(y)}(Q)||)$ *if* $||v;L^{\delta(y)}(Q)|| < +\infty$.

Proposition 4. (Lemma 1 [7, p. 714]). *Suppose that* $\delta \in \mathscr{B}_+(Q)$, $\delta_0 > 1$, *and* $\{u^m\}_{m\in\mathbb{N}} \subset L^{\delta(y)}(Q)$. *If* $u^m \underset{m\to\infty}{\longrightarrow} u$ *weakly in* $L^{\delta(y)}(Q)$ *and if* $u^m \underset{m\to\infty}{\longrightarrow} v$ *almost everywhere in* Q, *then* $u = v$.

Let $\Delta^0 v := v$, $\Delta^1 v := \Delta v$, $\Delta^r v := \Delta(\Delta^{r-1} v)$, and

$$H^{2r}_\Delta(\Omega) := \{v \in H^{2r}(\Omega) \mid v|_{\partial\Omega} = \Delta v|_{\partial\Omega} = \ldots = \Delta^{r-1} v|_{\partial\Omega} = 0\}, \quad r \in \mathbb{N}. \tag{20}$$

Take a number $r \in \mathbb{N}$. It is easy to verify that $H^{2r}_\Delta(\Omega)$ is the reflexive Hilbert space with respect to the scalar product

$$(u,v)_{H^{2r}_\Delta(\Omega)} := \int_\Omega \Delta^r u(x)\,\Delta^r v(x)\,dx, \quad u,v \in H^{2r}_\Delta(\Omega).$$

If $\partial\Omega \subset C^1$, then the following integration by parts formulae is true

$$\int_\Omega v\,\Delta^r u\,dx = \int_\Omega u\,\Delta^r v\,dx, \quad u,v \in H^{2r}_\Delta(\Omega). \tag{21}$$

Proposition 5. (Lemma 3 [35, p. 229]). *If* $\partial\Omega \subset C^{2r}$, *then there exists a constant* $C_3 > 0$ *such that*

$$||v;H^{2r}(\Omega)|| \leq C_3||\Delta^r v;L^2(\Omega)|| \tag{22}$$

holds for all $v \in H^{2r}_\Delta(\Omega)$.

Proposition 5 and definition (20) imply that

$$H^{2r}_\Delta(\Omega) \circlearrowleft H^{2r}(\Omega), \qquad H^{2r}_\Delta(\Omega) \bar{\circlearrowleft} L^2(\Omega) \bar{\circlearrowleft} [H^{2r}_\Delta(\Omega)]^*. \tag{23}$$

Let $\{w^j\}_{j\in\mathbb{N}}$ is an orthonormal basis for the space $L^2(\Omega)$ that consists of all eigenfunctions of the problem

$$-\Delta w = \lambda w \ \text{ in } \ \Omega, \qquad w|_{\partial\Omega} = 0, \qquad w \in H^1_0(\Omega), \tag{24}$$

and $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{R}_+$ is the set of the corresponding eigenvalues. It is easy to verify that the functions $\{w^j\}_{j\in\mathbb{N}}$ satisfy the equalities

$$(-1)^r \Delta^r w^j = \lambda_j^r w^j \quad \text{in} \quad \Omega, \qquad w^j|_{\partial\Omega} = \Delta w^j|_{\partial\Omega} = \ldots = \Delta^{r-1} w^j|_{\partial\Omega} = 0. \tag{25}$$

The following statements are needed for the sequel.

Proposition 6. (Theorem 8 [35, p. 230]). *If $\partial\Omega \subset C^{2r}$, then the set $\{w^j\}_{j\in\mathbb{N}}$ is a basis for the space $H_\Delta^{2r}(\Omega)$.*

Remark 1 *Let us consider the eigenvalue problem*

$$-v_{tt} - v_{x_1x_1} - v_{x_2x_2} - \ldots - v_{x_nx_n} = \zeta v \quad in \quad Q_{0,T}, \qquad v|_{\partial\Omega\times(0,T)} = 0, \quad v|_{t=0} = 0, \quad v_t|_{t=T} = 0. \tag{26}$$

Separating of variables $v(x,t) = w(x)\theta(t)$ yields

$$-w(x)\theta''(t) - w_{x_1x_1}(x)\theta(t) - w_{x_2x_2}(x)\theta(t) - \ldots - w_{x_nx_n}(x)\theta(t) = \zeta w(x)\theta(t),$$

$$-\frac{\theta''(t)}{\theta(t)} - \frac{\Delta w(x)}{w(x)} = \zeta,$$

and so $\zeta = \sigma + \lambda$, $\theta''(t) + \sigma\theta(t) = 0$ for $t \in (0,T)$, and $\Delta w(x) + \lambda w(x) = 0$ for $x \in \Omega$. Hence, instead of (26) we obtain the uncouple system of the following problem:

(i) *problem (24);*
(ii) *the problem*

$$\theta''(t) + \sigma\theta(t) = 0 \quad for \quad t \in (0,T), \quad \theta(0) = \theta'(T) = 0. \tag{27}$$

The eigenvalues and the corresponding eigenfunctions of problem (27) have the following form:

$$\sigma^k = \Big(\frac{\pi(2k-1)}{2T}\Big)^2, \quad k \in \mathbb{N}, \qquad \theta^k(t) = \sin\Big(\frac{\pi(2k-1)}{2T}t\Big), \quad t \in (0,T), \quad k \in \mathbb{N}.$$

Whence, the eigenvalues and the corresponding eigenfunctions of problem (26) have the following form:

$$\zeta = \lambda^j + \sigma^k, \qquad v(x,t) = w^j(x)\sin\Big(\frac{\pi(2k-1)}{2T}t\Big), \quad (x,t) \in Q_{0,T}, \quad j,k \in \mathbb{N}.$$

If we renumber it, then we get the set $\{v^m\}_{m\in\mathbb{N}}$ of the eigenfunctions and the set $\{\zeta_m\}_{m\in\mathbb{N}}$ of the corresponding eigenvalues $\zeta_m = \widehat{\lambda_m} + \widehat{\sigma_m}$ ($m \in \mathbb{N}$) of problem (26). Similarly to [42, Section 11.1] we prove that $\{v^m\}_{m\in\mathbb{N}}$ is a complete set in $L^2(Q_{0,T})$.

For problem (24) we have Proposition 6. In particular, if $r \in \mathbb{N}$ and if the condition $\partial\Omega \in C^{2r}$ holds, we obtain $\{w^j\}_{j\in\mathbb{N}} \subset H_\Delta^{2r}(\Omega) \subset H^{2r}(\Omega)$ and so $\{v^m\}_{m\in\mathbb{N}} \subset C^\infty([0,T]; H^{2r}(\Omega)) \subset H^{2r}(Q_{0,T})$. Moreover, the positive numbers $\{\widehat{\lambda_m}\}_{m\in\mathbb{N}}$ and $\{\widehat{\sigma_m}\}_{m\in\mathbb{N}}$ satisfy the equalities

$$\widehat{\lambda_m} v^m = -\Delta v^m \quad in \quad Q_{0,T}, \quad \widehat{\sigma_m} v^m = -v_{tt}^m \quad in \quad Q_{0,T}, \quad m \in \mathbb{N}. \tag{28}$$

Lemma 1. *If $p \in [1,+\infty)$ and if $z, z_t \in L^p(Q_{0,T})$, then the following inequalities are true:*

$$\int_\Omega |z(x,\tau)|^p\, dx \le C_4\Big(\int_\Omega |z(x,0)|^p\, dx + \int_{Q_{0,\tau}} |z_t(x,t)|^p\, dxdt\Big), \quad \tau \in [0,T], \tag{29}$$

$$\int\limits_{\Omega} |z(x,\tau)|^p\,dx \le C_4\Big(\int\limits_{\Omega} |z(x,T)|^p\,dx + \int\limits_{Q_{\tau,T}} |z_t(x,t)|^p\,dxdt\Big),\quad \tau\in[0,T], \tag{30}$$

$$\int\limits_{\Omega} |z(x,\tau)|^p\,dx \le C_4 \int\limits_{Q_{0,T}} \Big[|z(x,t)|^p + |z_t(x,t)|^p\Big]\,dxdt,\quad \tau\in[0,T], \tag{31}$$

where $C_4 > 0$ is independent of z, τ.

Proof. The proof of estimate (31) is found in Lemma 6 [13]. In the same way we prove (29) and (30). □

Lemma 2. *Suppose that the Nemytskij operators G and $\mathscr{R}$ are defined by (12) and (13) respectively. If $q,r\in\mathscr{B}_+(\Omega)$, $q_0 > 1$, $r_0 > 1$, and $g\in L^\infty(Q_{0,T})$, then the operators $G : L^{q(x)}(Q_{0,T}) \to L^{q'(x)}(Q_{0,T})$ and $\mathscr{R} : L^{r(x)}(Q_{0,T}) \to L^{r'(x)}(Q_{0,T})$ are bounded and continuous.*

We shall omit the proof because it is analogous to the proof which was given in [30, p. 613].

Proposition 7. *(the Aubin theorem, see [3] and [5, p. 393]). If $s,h > 1$ are fixed numbers, $\mathscr{W}$, $\mathscr{L}$, $\mathscr{B}$ are the Banach spaces, and $\mathscr{W} \overset{K}{\subset} \mathscr{L} \circlearrowleft \mathscr{B}$, then $\{u\in L^s(0,T;\mathscr{W}) \mid u_t \in L^h(0,T;\mathscr{B})\} \overset{K}{\subset} [L^s(0,T;\mathscr{L})\cap C([0,T];\mathscr{B})]$.*

Proposition 8. *(the Vishyk lemma, see Lemma 4.3 [33, p. 66] and [23, p. 60]). Suppose that $m\in\mathbb{N}$, the vector valued function $P := (P_1,\dots,P_m) : \mathbb{R}^m \to \mathbb{R}^m$ is continuous, and there exists a number $\rho > 0$ such that $(P(z),z)_{\mathbb{R}^m} \ge 0$ holds for all $z\in\mathbb{R}^m$ such that $|z| = \rho$. Then there exists a vector $z^m\in\mathbb{R}^m$ such that $|z^m|\le\rho$ and $P(z^m) = 0$.*

3.2 Differentiability of nonlinear expressions

The following statements are needed for the sequel.

Theorem 2. *Suppose that $r\in\mathscr{B}_+(\Omega)$. Then the following statements are fulfilled:*

(1) *If $r_0 > 1$, then the equality*

$$\left(|u|^{r(x)}\right)_t = r(x)|u|^{r(x)-2}u\,u_t \tag{32}$$

is true if one of the following alternatives hold:

(i) *$u\in C^1(\overline{Q_{0,T}})$ (here we have $|u|^{r(x)}, \left(|u|^{r(x)}\right)_t \in L^\infty(Q_{0,T})$);*

(ii) *$u, u_t \in L^{p(x)}(Q_{0,T})$ and $p(x)\ge r(x)$ for a.e. $x\in\Omega$ (here we have $|u|^{r(x)}, \left(|u|^{r(x)}\right)_t \in L^{\frac{p(x)}{r(x)}}(Q_{0,T})$).*

(2) *If $r_0 > 2$, then the equality*

$$\left(|u|^{r(x)-2}u\right)_t = (r(x)-1)|u|^{r(x)-2}u_t \tag{33}$$

is true if one of the following alternatives hold:

(i) *$u\in C^1(\overline{Q_{0,T}})$ (here we have $|u|^{r(x)-2}u, \left(|u|^{r(x)-2}u\right)_t \in L^\infty(Q_{0,T})$);*

(ii) *$u, u_t \in L^{p(x)}(Q_{0,T})$ and $p(x)\ge r(x)-1$ for a.e. $x\in\Omega$ (here $|u|^{r(x)-2}u, \left(|u|^{r(x)-2}u\right)_t \in L^{\frac{p(x)}{r(x)-1}}(Q_{0,T})$).*

This Theorem 2 coincides with Theorem 3 [13] (see also Remark 1 [13]). Then the proof is omitted.

Proposition 9. (Theorem 2 [25, p. 286]). *If X is the Banach space and if $p\in[1,\infty]$, then $W^{1,p}(0,T;X) \circlearrowleft C([0,T];X)$ and the following integration by parts formulae is true*

$$\int\limits_s^\tau u_t(t)\,dt = u(\tau) - u(s),\qquad 0\le s<\tau\le T,\quad u\in W^{1,p}(0,T;X). \tag{34}$$

Lemma 3. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded $C^{0,1}$-domain (see [27, p. 48]). Then the integration by parts formulae*

$$\int_{Q_{s,\tau}} w_t z\,dxdt = \int_{\Omega_t} wz\,dx\Big|_{t=s}^{t=\tau} - \int_{Q_{s,\tau}} wz_t\,dxdt, \quad 0 \le s < \tau \le T, \tag{35}$$

holds if one of the following alternatives hold:

(i) $w \in L^{q(x)}(Q_{0,T})$, *where* $q \in \mathscr{B}_+(\Omega)$, $q_0 > 1$, $w_t \in L^1(Q_{0,T})$, $z \in L^\infty(Q_{0,T})$, *and* $z_t \in L^{q'(x)}(Q_{0,T})$;
(ii) $w, w_t \in L^1(Q_{0,T})$ *and* $z, z_t \in L^\infty(Q_{0,T})$.

Proof. (i). Take $W := \{w \in L^{q(x)}(Q_{0,T}) \mid w_t \in L^1(Q_{0,T})\}$ and $Z := \{z \in L^\infty(Q_{0,T}) \mid z_t \in L^{q'(x)}(Q_{0,T})\}$. If $\varphi \in C^1([0,T])$ and if $z \in Z$, then $\varphi z \in W^{1,1}(0,T; L^{\frac{q^0}{q^0-1}}(\Omega))$. By (34) with $u = \varphi(t)z(x,t)$, we get

$$\int_s^\tau \varphi_t(t)z(x,t)\,dt = \varphi(\tau)z(x,\tau) - \varphi(s)z(x,s) - \int_s^\tau \varphi(t)z_t(x,t)\,dt, \quad x \in \Omega. \tag{36}$$

Take a function $v \in C^1(\overline{\Omega})$. By (36), we obtain

$$\int_{Q_{s,\tau}} \varphi_t vz\,dxdt = \int_{\Omega_t} \varphi vz\,dx\Big|_{t=s}^{t=\tau} - \int_{Q_{s,\tau}} \varphi vz_t\,dxdt. \tag{37}$$

Clearly, $C^1([0,T]; C^1(\overline{\Omega})) \mathring{\subset} W \mathring{\subset} W^{1,1}(0,T; L^1(\Omega))$. Then the set

$$\Big\{\sum_{i=1}^m \varphi_i(t)v_i(x) \;\Big|\; m \in \mathbb{N}, \;\; \varphi_1,\dots,\varphi_m \in C^1([0,T]), \;\; v_1,\dots,v_m \in C^1(\overline{\Omega})\Big\}$$

is dense in W and (37) yields (35).

We shall omit the proof of *(ii)* because it is analogous to the previous one. □

Lemma 4. *Suppose that $\alpha \in \mathscr{B}_+(\mathrm{Q})$,*

$$\psi_{\alpha(y)}(s) := \begin{cases} s^{\alpha(y)} & \text{if } s > 0, \\ 0 & \text{if } s \le 0, \end{cases} \quad y \in \mathrm{Q}, \tag{38}$$

$p, \delta \in \mathscr{B}_+(\mathrm{Q})$, $p_0, \delta_0 > 1$, $p(y) \ge \alpha(y)$ for a.e. $y \in \mathrm{Q}$, and $\delta(y) \le \frac{p(y)}{\alpha(y)}$ for a.e. $y \in \mathrm{Q}$. Then for every $u \in L^{p(y)}(\mathrm{Q})$ we have that $\psi_{\alpha(y)}(u) \in L^{\frac{p(y)}{\alpha(y)}}(\mathrm{Q})$,

$$\rho_{p/\alpha}(\psi_{\alpha(y)}(u); \mathrm{Q}) \le \rho_p(u; \mathrm{Q}), \tag{39}$$

$$||\psi_{\alpha(y)}(u); L^{\delta(y)}(\mathrm{Q})|| \le C_5 S_{\alpha/p}\big(\rho_p(u; \mathrm{Q})\big), \tag{40}$$

where $C_5 > 0$ is independent of u.

Proof. Clearly, $\frac{p(y)}{\alpha(y)} \ge 1$ for a.e. $y \in \mathrm{Q}$, $|\psi_{\alpha(y)}(u)|^{\frac{p(y)}{\alpha(y)}} \le |u|^{p(y)} \in L^1(\mathrm{Q})$. Then by [29, p. 297], we obtain $\psi_{\alpha(y)}(u) \in L^{\frac{p(y)}{\alpha(y)}}(\mathrm{Q})$. Moreover, (39) and

$$||\psi_{\alpha(y)}(u); L^{\delta(y)}(\mathrm{Q})|| \le C_6 ||\psi_{\alpha(y)}(u); L^{p(y)/\alpha(y)}(\mathrm{Q})|| \le C_6 S_{\alpha/p}\Big(\rho_{p/\alpha}(\psi_{\alpha(y)}(u); \mathrm{Q})\Big)$$

hold. This inequality and (39) imply (40). □

Lemma 5. *Suppose that $\beta \in \mathscr{B}_+(\Omega)$, $\psi_{\beta(x)}$ is determined from (38) if we replace $\alpha(y)$ by $\beta(x)$, and*

$$\chi_k(s) := \begin{cases} 1 \text{ if } s > \dfrac{1}{k}, \\ 0 \text{ if } s \leq \dfrac{1}{k}, \end{cases} \quad k \in \mathbb{N}. \tag{41}$$

If $u \in C^1(\overline{Q_{0,T}})$ and if $v, v_t \in L^1(Q_{0,T})$, then

$$\lim_{k \to +\infty} \int_{Q_{0,T}} \chi_k(u)\,\beta(x)\,\psi_{\beta(x)-1}(u)u_t\,v\,dxdt = \int_{\Omega_t} \psi_{\beta(x)}(u)\,v\,dx\Big|_{t=0}^{t=T} - \int_{Q_{0,T}} \psi_{\beta(x)}(u)\,v_t\,dxdt. \tag{42}$$

Proof. By definition, put

$$\psi_{\beta(x),k}(s) := \begin{cases} k^{\beta(x)} \text{ if } s \geq k, \\ s^{\beta(x)} \text{ if } \dfrac{1}{k} < s < k, \\ \dfrac{1}{k^{\beta(x)}} \text{ if } s \leq \dfrac{1}{k}, \end{cases} \qquad \tilde{\xi}_{\beta(x),k}(s) := \begin{cases} \beta(x)\,s^{\beta(x)-1} \text{ if } \dfrac{1}{k} < s < k, \\ 0 \qquad \text{if } s \leq \dfrac{1}{k} \text{ and } s \geq k, \end{cases} \quad k \in \mathbb{N}, \quad k \geq 2, \quad x \in \Omega.$$

Clearly, $\psi_{\beta(x),k}(s) \underset{k\to\infty}{\longrightarrow} \psi_{\beta(x)}(s)$ if $s \in \mathbb{R}$ and $x \in \Omega$. In addition, for $k \in \mathbb{N}$ ($k \geq 2$) and $x \in \Omega$ the function $s \mapsto \psi_{\beta(x),k}(s)$ has the Lipschitz property in $\mathbb{R}$ and it is not differentiable only in the point $s = \frac{1}{k}$ and $s = k$. Moreover, $\frac{\partial}{\partial s}\,\psi_{\beta(x),k}(s) = \tilde{\xi}_{\beta(x),k}(s)$ if $s \neq \frac{1}{k}$ and $s \neq k$. Whence, by Lemma 4 [13], we obtain

$$\left(\psi_{\beta(x),k}(u)\right)_t = \tilde{\xi}_{\beta(x),k}(u)u_t \quad \text{almost everywhere in} \quad Q_{0,T}. \tag{43}$$

Thus, $\psi_{\beta(x),k}(u), \left(\psi_{\beta(x),k}(u)\right)_t \in L^\infty(Q_{0,T})$. Using case *(ii)* of Lemma 3 with $z = \psi_{\beta(x),k}(u)$ and $w = v$, we get (see (35))

$$\int_{Q_{0,T}} \left(\psi_{\beta(x),k}(u)\right)_t v\,dxdt = \int_{\Omega} \psi_{\beta(x),k}(u)\,v\,dx\Big|_{t=0}^{t=T} - \int_{Q_{0,T}} \psi_{\beta(x),k}(u)\,v_t\,dxdt. \tag{44}$$

Let $M := \max\{|u(x,t)| \,:\, (x,t) \in \overline{Q_{0,T}}\}$, $k_0 \in \mathbb{N}$, $k_0 \geq \max\{2, M\}$. Since $|u| \leq M \leq k_0 \leq k$, from (43) we have

$$\left(\psi_{\beta(x),k}(u)\right)_t = \tilde{\xi}_{\beta(x),k}(u)u_t = \chi_k(u)\,\beta(x)\,\psi_{\beta(x)-1}(u)u_t,$$

where $k \geq k_0$. By the inequality $|\psi_{\beta(x),k}(u(x,t))| \leq M^{\beta(x)}$ $\forall\ (x,t) \in \overline{Q_{0,T}}$ and the Lebesgue Dominate Convergence Theorem, we obtain

$$\lim_{k \to +\infty} \int_{\Omega_t} \psi_{\beta(x),k}(u)\,v\,dx = \int_{\Omega_t} \psi_{\beta(x)}(u)\,v\,dx \quad \text{if} \quad t = 0 \quad \text{and} \quad t = T,$$

$$\lim_{k \to +\infty} \int_{Q_{0,T}} \psi_{\beta(x),k}(u)\,v_t\,dxdt = \int_{Q_{0,T}} \psi_{\beta(x)}(u)\,v_t\,dxdt.$$

Therefore, (42) follows from (44). □

Theorem 3. *Suppose that $\alpha \in \mathscr{B}_+(\Omega)$, $\frac{1}{2} < \alpha_0 \leq \alpha(x) \leq \alpha^0$ for a.e. $x \in \Omega$, and $\psi_{\alpha(x)}$ is defined by (38) if we replace $\alpha(y)$ by $\alpha(x)$. Then the following statements are fulfilled:*

(1) *if* $u \in C^2(\overline{Q_{0,T}})$, *then* $\psi_{\alpha(x)}(u) \in L^\infty(Q_{0,T})$, $\big(\psi_{\alpha(x)}(u)\big)_t \in L^2(Q_{0,T})$, *and*

$$\big(\psi_{\alpha(x)}(u)\big)_t = \alpha(x)\,\psi_{\alpha(x)-1}(u)\,u_t; \tag{45}$$

(2) *if* $u, u_t, u_{tt} \in L^{p(x)}(Q_{0,T})$, $p \in L^\infty_+(\Omega)$, *and* $p(x) \geq 2\alpha(x)$ *for a.e.* $x \in \Omega$, *then* $\psi_{\alpha(x)}(u) \in W^{1,2}(0,T;L^2(\Omega))$, *formulae (45) holds, and*

$$||\psi_{\alpha(x)}(u);L^2(Q_{0,T})|| \leq C_7 S_{\alpha/p}\big(\rho_p(u;Q_{0,T})\big), \tag{46}$$

$$||\big(\psi_{\alpha(x)}(u)\big)_t;L^2(Q_{0,T})|| \leq C_8 S_{\alpha/p}\big(\rho_p(u;Q_{0,T}) + \rho_p(u_t;Q_{0,T}) + \rho_p(u_{tt};Q_{0,T})\big), \tag{47}$$

where $C_7, C_8 > 0$ *are independent of* u.

Proof. First let us prove Case 1. Take a function $u \in C^2(\overline{Q_{0,T}})$. Since $u_t, u_{tt} \in L^1(Q_{0,T})$, using (42) with $\beta = 2\alpha - 1 > 0$ and $v = u_t$, we obtain

$$\lim_{k\to+\infty} \int\limits_{Q_{0,T}} f_k\,dxdt = \int\limits_{\Omega_t} \psi_{2\alpha(x)-1}(u)u_t\,dx\Big|_{t=0}^{t=T} - \int\limits_{Q_{0,T}} \psi_{2\alpha(x)-1}(u)u_{tt}\,dxdt, \tag{48}$$

where χ_k is defined by (41), $f_k = \chi_k(u)(2\alpha(x)-1)\psi_{2\alpha(x)-2}(u)|u_t|^2$, $k \in \mathbb{N}$. Clearly,

$$f_k(x,t) \underset{k\to\infty}{\longrightarrow} f(x,t) \quad \text{if} \quad (x,t) \in Q_{0,T}, \tag{49}$$

where $f = (2\alpha(x)-1)\psi_{2\alpha(x)-2}(u)|u_t|^2 = (2\alpha(x)-1)|\psi_{\alpha(x)-1}(u)u_t|^2$. The existence of the limit in (48) implies that there exists a constant $C_9 > 0$ such that for every $k \in \mathbb{N}$ we have the estimate

$$\int\limits_{Q_{0,T}} f_k\,dxdt \leq C_9.$$

Clearly, $f_{k_1} \leq f_{k_2}$ if $k_1 \leq k_2$. Then the Levi Monotone Convergence Theorem (see Theorem 4.1 [8, p. 90]) and (49) yield that $f \in L^1(Q_{0,T})$ and so $\psi_{\alpha(x)-1}(u)u_t \in L^2(Q_{0,T})$. In addition,

$$\int\limits_{Q_{0,T}} f_k\,dxdt \underset{k\to\infty}{\longrightarrow} \int\limits_{Q_{0,T}} f\,dxdt. \tag{50}$$

Hence, by (48), we get

$$\int\limits_{Q_{0,T}} (2\alpha(x)-1)|\psi_{\alpha(x)-1}(u)u_t|^2\,dxdt = \int\limits_{\Omega_t} \psi_{2\alpha(x)-1}(u)u_t\,dx\Big|_{t=0}^{t=T} - \int\limits_{Q_{0,T}} \psi_{2\alpha(x)-1}(u)u_{tt}\,dxdt. \tag{51}$$

By (50), we obtain

$$||\chi_k(u)\sqrt{2\alpha-1}\,\psi_{\alpha(x)-1}(u)u_t;L^2(Q_{0,T})|| \underset{k\to\infty}{\longrightarrow} ||\sqrt{2\alpha-1}\,\psi_{\alpha(x)-1}(u)u_t;L^2(Q_{0,T})||. \tag{52}$$

Whence, using (49), we prove the following convergence

$$\chi_k(u)\psi_{\alpha(x)-1}(u)u_t \underset{k\to\infty}{\longrightarrow} \psi_{\alpha(x)-1}(u)u_t \quad \text{strongly in} \quad L^2(Q_{0,T}). \tag{53}$$

Take $v \in L^2(Q_{0,T})$ such that $v_t \in L^1(Q_{0,T})$. Using (53) and (42) with $\beta = \alpha$, we obtain

$$\int\limits_{Q_{0,T}} \alpha(x)\,\psi_{\alpha(x)-1}(u)u_t\,v\,dxdt = \int\limits_{\Omega_t} \psi_{\alpha(x)}(u)\,v\,dx\Big|_{t=0}^{t=T} - \int\limits_{Q_{0,T}} \psi_{\alpha(x)}(u)\,v_t\,dxdt. \tag{54}$$

Taking in (54) function $v \in C_0^\infty(Q_{0,T})$, we get

$$\int\limits_{Q_{0,T}} \alpha(x)\psi_{\alpha(x)-1}(u)u_t\,v\,dxdt = -\int\limits_{Q_{0,T}} \psi_{\alpha(x)}(u)\,v_t\,dxdt.$$

Therefore, (45) holds. Thus, $(\psi_{\alpha(x)}(u))_t \in L^2(Q_{0,T})$. Moreover, $\psi_{\alpha(x)}(u) \in L^\infty(Q_{0,T})$.

Further let us prove Case 2. We start from proof of (46) and (47) if $u \in C^2(\overline{Q_{0,T}})$. Since $p \geq 2\alpha$ (and so $2 \leq \frac{p}{\alpha}$), estimate (40) with $\mathrm{Q} = Q_{0,T}$ and $q \equiv 2$ yields that (46) holds. Using (45) (we already proved it), by (51), we obtain

$$\int\limits_{Q_{0,T}} \left|\left(\psi_{\alpha(x)}(u)\right)_t\right|^2 dxdt \leq \frac{|\alpha^0|^2}{2\alpha_0 - 1}\Big(\int\limits_{\Omega_0} \psi_{2\alpha(x)-1}(u)|u_t|\,dx + \int\limits_{\Omega_T} \psi_{2\alpha(x)-1}(u)|u_t|\,dx + \int\limits_{Q_{0,T}} \psi_{2\alpha(x)-1}(u)|u_{tt}|\,dxdt\Big). \tag{55}$$

On the other hand, for every $r \in \mathscr{B}_+(\Omega)$ $(r_0 > 1)$ the estimate

$$J_1 := \int\limits_{\Omega_0} \psi_{2\alpha(x)-1}(u)|u_t|\,dx \leq ||\psi_{2\alpha(x)-1}(u(0)); L^{r'(x)}(\Omega)|| \cdot ||u_t(0); L^{r(x)}(\Omega)||$$

holds. Since $p \geq 2\alpha > 1$, for $r = 2\alpha$ the following inequalities are true: $1 < r \leq p$, $2\alpha - 1 < p$, and $1 < r' = \frac{r}{r-1} = \frac{r}{2\alpha-1} \leq \frac{p}{2\alpha-1}$. Then estimate (40) with $\mathrm{Q} = \Omega$ and $q = r'$ implies that

$$||\psi_{2\alpha(x)-1}(u(0)); L^{r'(x)}(\Omega)|| \leq C_{10} S_{(2\alpha-1)/p}(\rho_p(u(0);\Omega))$$

holds. In addition, by Proposition 3, we obtain

$$||u_t(0); L^{r(x)}(\Omega)|| \leq C_{11}||u_t(0); L^{p(x)}(\Omega)|| \leq C_{11} S_{1/p}(\rho_p(u_t(0);\Omega)).$$

Therefore, by (31) with $p = p(x)$ and by the equality $S_a(z)S_b(z) = S_{a+b}(z)$ $(a,b,z \geq 0)$, we get

$$\begin{aligned} J_1 &\leq C_{12} S_{(2\alpha-1)/p}\Big(\int\limits_{\Omega} |u(0)|^{p(x)}\,dx\Big) S_{1/p}\Big(\int\limits_{\Omega} |u_t(0)|^{p(x)}\,dx\Big) \\ &\leq C_{13} S_{2\alpha/p}\Big(\int\limits_{Q_{0,T}} \Big[|u(x,t)|^{p(x)} + |u_t(x,t)|^{p(x)} + |u_{tt}(x,t)|^{p(x)}\Big]\,dxdt\Big), \end{aligned}$$

where $C_{13} > 0$ is independent of u. Continuing in the same way, by (55), we obtain (47).

Finally let us take $u, u_t, u_{tt} \in L^{p(x)}(Q_{0,T})$, $p \geq 2\alpha$, and $\{u^\ell\}_{\ell\in\mathbb{N}} \subset C^2(\overline{Q_{0,T}})$, where

$$u^\ell \underset{\ell\to\infty}{\longrightarrow} u, \quad u_t^\ell \underset{\ell\to\infty}{\longrightarrow} u_t, \quad u_{tt}^\ell \underset{\ell\to\infty}{\longrightarrow} u_{tt} \quad \text{strongly in} \quad L^{p(x)}(Q_{0,T}).$$

Estimates (46) and (47) imply that the sequence $\{\psi_{\alpha(x)}(u^\ell)\}_{\ell\in\mathbb{N}}$ is bounded in $W^{1,2}(0,T;L^2(\Omega))$. Then there exists a subsequence $\{u^{\ell_k}\}_{k\in\mathbb{N}}\subset\{u^\ell\}_{\ell\in\mathbb{N}}$ such that

$$\psi_{\alpha(x)}(u^{\ell_k})\underset{k\to\infty}{\longrightarrow}\chi \quad \text{weakly in} \quad W^{1,2}(0,T;L^2(\Omega))$$

and so $\alpha\,\psi_{\alpha(x)-1}(u^{\ell_k})u_t^{\ell_k}=\big(\psi_{\alpha(x)}(u^{\ell_k})\big)_t\underset{k\to\infty}{\longrightarrow}\zeta$ weakly in $L^2(Q_{0,T})$. By choosing the sequence $\{u^\ell\}_{\ell\in\mathbb{N}}$, there exists a subsequence (we call it $\{u^{\ell_k}\}_{k\in\mathbb{N}}$ again) such that

$$u^{\ell_k}\underset{k\to\infty}{\longrightarrow}u,\quad u_t^{\ell_k}\underset{k\to\infty}{\longrightarrow}u_t \quad \text{almost everywhere in} \quad Q_{0,T},$$

$$\alpha(x)\psi_{\alpha(x)-1}(u^{\ell_k})u_t^{\ell_k}\underset{k\to\infty}{\longrightarrow}\alpha(x)\psi_{\alpha(x)-1}(u)u_t \quad \text{almost everywhere in} \quad Q_{0,T}.$$

Therefore $\chi=\psi_{\alpha(x)}(u)$ and $\zeta=\alpha\,\psi_{\alpha(x)-1}(u)u_t$. Thus, (54) with $v\in W^{1,2}(0,T;L^2(\Omega))$ holds and so (45) is true.

Since $\{u^\ell\}_{\ell\in\mathbb{N}}\subset C^2(\overline{Q_{0,T}})$ inequality (46) and (47) with $u=u^\ell$ are true, i.e.

$$||\psi_{\alpha(x)}(u^{\ell_k});L^2(Q_{0,T})||\le C_7S_{\alpha/p}\big(\rho_p(u^{\ell_k};Q_{0,T})\big),$$

$$||\big(\psi_{\alpha(x)}(u^{\ell_k})\big)_t;L^2(Q_{0,T})||\le C_8S_{\alpha/p}\big(\rho_p(u^{\ell_k};Q_{0,T})+\rho_p(u_t^{\ell_k};Q_{0,T})+\rho_p(u_{tt}^{\ell_k};Q_{0,T})\big),$$

where $C_7,C_8>0$ are independent of u^{ℓ_k},ℓ_k. Thus, using Lemma 5.3 [27, p. 20], we obtain (46) and (47). □

Notice that the case $\alpha(x)\in(0,\frac{1}{2}]$ (see Theorem 3) is considered in [12].

Theorem 4. *Assume that $r\in\mathscr{B}_+(\Omega)$. Then the following statements are fulfilled:*
(1) *If $\frac{1}{2}<r_0\le r^0\le 1$, then the equality (32) is true if one of the following alternatives hold:*

(i) *$u\in C^2(\overline{Q_{0,T}})$ (here we have $|u|^{r(x)}\in L^\infty(Q_{0,T})$ and $\big(|u|^{r(x)}\big)_t\in L^2(Q_{0,T})$);*
(ii) *$u,u_t,u_{tt}\in L^{p(x)}(Q_{0,T})$ and $p(x)\ge 2r(x)$ for a.e. $x\in\Omega$ (here we have $|u|^{r(x)}\in H^1(0,T;L^2(\Omega))$ and*

$$||\,|u|^{r(x)};H^1(0,T;L^2(\Omega))||\le C_{14}S_{r/p}\big(\rho_p(u;Q_{0,T})+\rho_p(u_t;Q_{0,T})+\rho_p(u_{tt};Q_{0,T})\big), \tag{56}$$

where $C_{14}>0$ is independent of u).

(2) *If $\frac{3}{2}<r_0\le r^0\le 2$, then the equality (33) is true if one of the following alternatives hold:*

(i) *$u\in C^2(\overline{Q_{0,T}})$ (here we have $|u|^{r(x)-2}u\in L^\infty(Q_{0,T})$ and $\big(|u|^{r(x)-2}u\big)_t\in L^2(Q_{0,T})$);*
(ii) *$u,u_t,u_{tt}\in L^{p(x)}(Q_{0,T})$ and $p(x)\ge 2(r(x)-1)$ for a.e. $x\in\Omega$ (here we have $|u|^{r(x)-2}u\in H^1(0,T;L^2(\Omega))$ and*

$$||\,|u|^{r(x)-2}u;H^1(0,T;L^2(\Omega))||\le C_{15}S_{(r-1)/p}\big(\rho_p(u;Q_{0,T})+\rho_p(u_t;Q_{0,T})+\rho_p(u_{tt};Q_{0,T})\big), \tag{57}$$

where $C_{15}>0$ is independent of u.

Proof. The proof of Theorem 4 follows from Theorem 3 if we recall that

$$|s|^{r(x)}=\psi_{r(x)}(s)+\psi_{r(x)}(-s),\quad |s|^{r(x)-2}s=\psi_{r(x)-1}(s)-\psi_{r(x)-1}(-s),\quad x\in\Omega,\quad s\in\mathbb{R}.$$

Notice that we omit the notation ψ in the right-hand sides of (32) and (33). Thus the right-hand sides of (32) and (33) equal zero on $\{(x,t)\in Q_{0,T}\mid u(x,t)=0\}$. □

If we take $p(x)\equiv const$ and $r(x)\equiv const$, we rewrite Theorem 4 as follows.

Theorem 5. *Assume that $r > 0$. Then the following statements are fulfilled:*

(1) *If $\frac{1}{2} < r \leq 1$, then the equality*

$$(|u|^r)_t = r\,|u|^{r-2} u\, u_t \tag{58}$$

is true if one of the following alternatives hold:

(i) $u \in C^2(\overline{Q_{0,T}})$ *(here we have $|u|^r \in C(\overline{Q_{0,T}})$ and $(|u|^r)_t \in L^2(Q_{0,T})$);*
(ii) $u \in W^{2,p}(0,T;L^p(\Omega))$ *and $p \geq 2r$ (here we have $|u|^r \in H^1(0,T;L^2(\Omega))$ and*

$$||\,|u|^r;H^1(0,T;L^2(\Omega))|| \leq C_{16}\big(||u;L^p(Q_{0,T})||^r + ||u_t;L^p(Q_{0,T})||^r + ||u_{tt};L^p(Q_{0,T})||^r\big), \tag{59}$$

where $C_{16} > 0$ is independent of u.

(2) *If $\frac{3}{2} < r \leq 2$, then the equality*

$$(|u|^{r-2}u)_t = (r-1)|u|^{r-2}u_t \tag{60}$$

is true if one of the following alternatives hold:

(i) $u \in C^2(\overline{Q_{0,T}})$ *(here we have $|u|^{r-2}u \in C(\overline{Q_{0,T}})$ and $(|u|^{r-2}u)_t \in L^2(Q_{0,T})$);*
(ii) $u \in W^{2,p}(0,T;L^p(\Omega))$ *and $p \geq 2(r-1)$ (here we have $|u|^{r-2}u \in H^1(0,T;L^2(\Omega))$ and*

$$||\,|u|^{r-2}u;H^1(0,T;L^2(\Omega))|| \leq C_{17}\big(||u;L^p(Q_{0,T})||^{r-1} + ||u_t;L^p(Q_{0,T})||^{r-1} + ||u_{tt};L^p(Q_{0,T})||^{r-1}\big), \tag{61}$$

where $C_{17} > 0$ is independent of u.

3.3 Boundary value problem for some elliptic equation

The following Dirichlet-Neumann boundary value problem is needed for the sequel

$$-\varepsilon\, u^\varepsilon_{tt} + (\mathscr{R}u^\varepsilon)_t - a\,\Delta u^\varepsilon + Gu^\varepsilon + \phi(Eu^\varepsilon) = f(x,t), \quad (x,t) \in Q_{0,T}, \tag{62}$$

$$u^\varepsilon|_{\Sigma_{0,T}} = 0, \qquad u^\varepsilon|_{t=0} = u_0, \qquad u^\varepsilon_t|_{t=T} = 0, \tag{63}$$

where $\varepsilon > 0$. By definition, put $U_0(Q_{0,T}) := \{v \in H^1(Q_{0,T}) \mid v|_{\Sigma_{0,T}} = 0,\ v|_{t=0} = 0\}$,

$$D(x,t) := \left(1 - \frac{t}{T}\right)^2 u_0(x), \quad (x,t) \in Q_{0,T}. \tag{64}$$

Clearly, (64) and condition **(U)** yield that $D \in H^2(Q_{0,T})$,

$$D|_{\Sigma_{0,T}} = 0, \quad D|_{t=0} = u_0, \quad D|_{t=T} = 0, \quad D_t|_{t=T} = 0. \tag{65}$$

Definition 2. *A real-valued function $u^\varepsilon \in H^1(Q_{0,T})$ is called a weak solution of problem (62)-(63) if u satisfies (63_1), (63_2) (i.e. $u^\varepsilon - D \in U_0(Q_{0,T})$, where D is defined by (64)), $\mathscr{R}u^\varepsilon, Gu^\varepsilon, Eu^\varepsilon \in L^2(Q_{0,T})$, and for every $v \in U_0(Q_{0,T})$ we have*

$$\int\limits_{Q_{0,T}} \Big[\varepsilon u^\varepsilon_t v_t - \mathscr{R}u^\varepsilon\, v_t + a(\nabla u^\varepsilon, \nabla v) + Gu^\varepsilon\, v + \phi(Eu^\varepsilon)v\Big]\, dxdt + \int\limits_{\Omega} \mathscr{R}u^\varepsilon(T)v(T)\, dx = \int\limits_{Q_{0,T}} fv\, dxdt. \tag{66}$$

For functions $w,v: Q_{0,T} \to \mathbb{R}^1$ by definition, put

$$\mathscr{L}_\varepsilon(w,v) := \int\limits_{Q_{0,T}} \Big[\varepsilon w_t v_t + |w|^{r-2} w_t v + a(\nabla w, \nabla v) + Gw\, v + \phi(Ew)v\Big]\, dxdt. \tag{67}$$

Lemma 6. *If $r \in (1,2)$, conditions **(G)-(Φ)** hold, and **(Z)** is satisfied, then there exists a constant $\alpha_0 > 0$ such that for every $u \in H^1(Q_{0,T})$ and $\varepsilon > 0$ the estimate*

$$\mathscr{L}_\varepsilon(u,u) \ge \frac{1}{r}\int\limits_\Omega |u(x,t)|^r\, dx\Big|_{t=0}^{t=T} + \int\limits_{Q_{0,T}} \Big[\varepsilon|u_t|^2 + \alpha_0(|\nabla u|^2 + |u|^2 + |u|^r) + g_0|u|^{q(x)}\Big]\, dxdt - C_{18} \tag{68}$$

holds, where $C_{18} > 0$ is independent of u, ε.

Proof. By (67), it follows that

$$\mathscr{L}_\varepsilon(u,u) = \int\limits_{Q_{0,T}} \Big[\varepsilon|u_t|^2 + |u|^{r-2}u_t u + a|\nabla u|^2 + g|u|^{q(x)} + \phi(Eu)u\Big]\, dxdt. \tag{69}$$

Since $u \in H^1(Q_{0,T})$ and $2 \ge r > 1$, Case 1.ii of Theorem 2 with $p(x) \equiv 2$ and $r(x) \equiv r$ (see also Proposition 9) implies that

$$|u|^r \in W^{1,\frac{2}{r}}(0,T;L^{\frac{2}{r}}(\Omega)), \quad (|u|^r)_t = r|u|^{r-2}u\,u_t, \quad \int\limits_{Q_{0,T}} (|u|^r)_t\, dxdt = \int\limits_\Omega |u(x,t)|^r\, dx\Big|_{t=0}^{t=T}. \tag{70}$$

Taking into account condition **(Φ)**, the Cauchy-Bunyakowski-Schwarz inequality, (16), and (17), we obtain

$$\Big|\int\limits_{Q_{0,T}} \phi(Eu)u\, dxdt\Big| \le \phi^*||Eu;L^2(Q_{0,T})||\cdot||u;L^2(Q_{0,T})|| \le \phi^*E^*||u;L^2(Q_{0,T})||^2 \le \phi^*E^*M_\Omega \int\limits_{Q_{0,T}} |\nabla u|^2\, dxdt.$$

Using the Young inequality with the exponents $\frac{2}{r}, \frac{2}{2-r} > 1$ and inequality (17), we get

$$\int\limits_\Omega |u(t)|^r\, dx \le \int\limits_\Omega |u(t)|^2\, dx + C_{19} \le M_\Omega \int\limits_\Omega |\nabla u(t)|^2\, dx + C_{19}, \quad t \in [0,T],$$

where $C_{19} > 0$ is independent of u, t. Then for $\alpha := a - \phi^*E^*M_\Omega > 0$ we have

$$\int\limits_{Q_{0,T}} \Big[a|\nabla u|^2 + \phi(Eu)u\Big]\, dxdt \ge \alpha \int\limits_{Q_{0,T}} |\nabla u|^2\, dxdt = \int\limits_{Q_{0,T}} \Big(\frac{\alpha}{3} + \frac{\alpha}{3} + \frac{\alpha}{3}\Big)|\nabla u|^2\, dxdt \ge$$

$$\ge \int\limits_{Q_{0,T}} \Big[\frac{\alpha}{3}|\nabla u|^2 + \frac{\alpha}{3M_\Omega}|u|^2 + \frac{\alpha}{3M_\Omega}|u|^r\Big]\, dxdt - C_{20}.$$

Taking into account (69) and the transformations above, we obtain (68). □

Theorem 6. *Suppose that $\partial\Omega \in C^4$ and the following conditions are satisfied:*

- ***(A):*** $a > 0$, $\frac{3}{2} < r \le 2$;
- ***(Q):*** $q \in \mathscr{B}_+(\Omega)$, $1 < q_0 \le q^0 \le 2$;
- ***(G):*** $g \in \mathscr{B}_+(Q_{0,T})$;
- ***(E):*** $\mathfrak{Z} \in L^\infty(Q_{0,T} \times \Omega)$;
- ***(Φ):*** $\phi \in \mathrm{Lip}(\mathbb{R})$, $|\phi(\xi)| \le \phi^*|\xi|$ *for every* $\xi \in \mathbb{R}$, *where* $\phi^* \in [0,+\infty)$;
- ***(F):*** $f \in L^{r'}(Q_{0,T})$, *where* $\frac{1}{r} + \frac{1}{r'} = 1$;

(U): $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$;

(Z): $a > \phi^* E^* M_\Omega$, *where* E^* *is defined by (16),* M_Ω *is determined from (17).*

Then problem (62)-(63) has a weak solution u^ε *such that* $u^\varepsilon \in H^2(Q_{0,T}) \cap H^1(0,T;H_0^1(\Omega))$ *and* $\mathscr{R}u^\varepsilon \in H^1(0,T;L^2(\Omega))$.

Proof. The Case $r = 2$ is trivial. Let us assume that $r \in (\frac{3}{2}, 2)$. The solution will be constructed via Galerkin's method.

Step 1. Let $\{v^m\}_{m\in\mathbb{N}}$ be determined from Remark 1, D defined by (64), $\varepsilon > 0$ a fixed number, $\mathscr{L}_\varepsilon$ defined by (67),

$$u^{\varepsilon,m}(x,t) := D(x,t) + z^{\varepsilon,m}(x,t), \quad z^{\varepsilon,m}(x,t) := \sum_{\mu=1}^{m} \varphi_\mu^{\varepsilon,m} v^\mu(x,t), \quad (x,t) \in Q_{0,T}, \quad m \in \mathbb{N}, \tag{71}$$

and $\varphi := (\varphi_1^{\varepsilon,m}, \ldots, \varphi_m^{\varepsilon,m}) \in \mathbb{R}^m$. For the sake of convenience we have omitted index ε in $u^{\varepsilon,m}, \varphi_1^{\varepsilon,m}, \ldots, \varphi_m^{\varepsilon,m}$, and $z^{\varepsilon,m}$. Notice also that

$$z^m|_{\Sigma_{0,T}} = 0, \qquad z^m|_{t=0} = 0, \qquad z_t^m|_{t=T} = 0, \tag{72}$$

$$u^m|_{\Sigma_{0,T}} = 0, \qquad u^m|_{t=0} = u_0, \qquad u_t^m|_{t=T} = 0. \tag{73}$$

Since $r < 2$, we have $2 \geq 2(r-1)$. In addition, the inequality $r > \frac{3}{2}$ holds. Since $\partial\Omega \in C^2$, Remark 1 implies that $\{v^m\}_{m\in\mathbb{N}} \subset H^2(Q_{0,T})$. Then $\{z^m\}_{m\in\mathbb{N}}, \{u^m\}_{m\in\mathbb{N}} \subset H^2(Q_{0,T})$ and Case 2.ii of Theorem 5 with $p = 2$ yields that

$$\mathscr{R}u^m \in H^1(0,T;L^2(\Omega)), \qquad (\mathscr{R}u^m)_t = |u^m|^{r-2} u_t^m. \tag{74}$$

Assume that the unknown vector φ (see (71)) satisfies the following equalities:

$$\int_{Q_{0,T}} \Big[\varepsilon u_t^m v_t^\mu + |u^m|^{r-2} u_t^m v^\mu + a(\nabla u^m, \nabla v^\mu) + G u^m\, v^\mu + \phi(E u^m) v^\mu \Big]\, dxdt = \int_{Q_{0,T}} f v^\mu \, dxdt, \tag{75}$$

where $\mu = \overline{1,m}$. Let us prove that this φ exists. Equality (75) we rewrite to read

$$P(\varphi) = 0, \tag{76}$$

where $P := (P_1, \ldots, P_m)$, $P_\mu(\varphi) := \mathscr{L}_\varepsilon(u^m, v^\mu) - \int_{Q_{0,T}} f v^\mu \, dxdt$, $\mu = \overline{1,m}$. Clearly,

$$(P(\varphi), \varphi)_{\mathbb{R}^m} = \sum_{\mu=1}^{m} \Big(\mathscr{L}_\varepsilon(u^m, v^\mu) - \int_{Q_{0,T}} f v^\mu \, dxdt \Big) \varphi_\mu^m = \mathscr{L}_\varepsilon(u^m, z^m) - \int_{Q_{0,T}} f z^m \, dxdt$$

$$= \mathscr{L}_\varepsilon(u^m, u^m - D) - \int_{Q_{0,T}} f(u^m - D)\, dxdt = \int_{Q_{0,T}} fD \, dxdt - \int_{Q_{0,T}} f u^m \, dxdt + \mathscr{L}_\varepsilon(u^m, u^m) - \mathscr{L}_\varepsilon(u^m, D). \tag{77}$$

Using Young's inequality, we obtain

$$|fD| \leq \frac{1}{2}|f|^2 + \frac{1}{2}|D|^2, \qquad |f u^m| \leq \varkappa_1 |u^m|^2 + \frac{1}{4\varkappa_1}|f|^2, \quad \varkappa_1 > 0.$$

By (68) and (73_2), we get

$$\mathscr{L}_\varepsilon(u^m, u^m) \geq \frac{1}{r}\int_\Omega |u^m(T)|^r \, dx + \int_{Q_{0,T}} \Big[\varepsilon |u_t^m|^2 + \alpha_0 |\nabla u^m|^2 + \alpha_0 |u^m|^2 + \alpha_0 |u^m|^r + g_0 |u^m|^{q(x)} \Big]\, dxdt - \frac{1}{r}\int_\Omega |u_0|^r \, dx - C_{21},$$

where $C_{21} > 0$ is independent of m, ε.

Now let us consider the expression

$$\mathscr{L}_\varepsilon(u^m, D) = \int\limits_{Q_{0,T}} \Big[\varepsilon u_t^m D_t + |u^m|^{r-2} u_t^m D + a(\nabla u^m, \nabla D) + G u^m D + \phi(E u^m) D\Big]\, dxdt.$$

Using Young's inequality, we obtain

$$|u_t^m D_t| \le \frac{1}{2}|u_t^m|^2 + \frac{1}{2}|D_t|^2, \qquad |a(\nabla u^m, \nabla D)| \le a\varkappa_2 |\nabla u^m|^2 + \frac{a}{4\varkappa_2}|\nabla D|^2, \quad \varkappa_2 > 0.$$

Taking into account (74), integration by parts formulae, (65), (73), and the Young inequality with $r', r > 1$, we obtain

$$\int\limits_{Q_{0,T}} |u^m|^{r-2} u_t^m D\, dxdt = \int\limits_{Q_{0,T}} (\mathscr{R} u^m)_t D\, dxdt = \int\limits_{\Omega} \mathscr{R} u^m\, D\, dx \Big|_{t=0}^{t=T} - \int\limits_{Q_{0,T}} \mathscr{R} u^m\, D_t\, dxdt$$

$$\le -\int\limits_{\Omega} \mathscr{R} u_0\, u_0\, dx + \frac{1}{r-1}\int\limits_{Q_{0,T}} |u^m|^{r-1}|D_t|\, dxdt \le \int\limits_{Q_{0,T}} \Big[\varkappa_3 |u^m|^r + C_{22}(\varkappa_3)|D_t|^r\Big]\, dxdt, \quad \varkappa_3 > 0.$$

By generalized Young's inequality with $q'(x), q(x) > 1$, it follows that

$$|G u^m D| \le g^0 |u^m|^{q(x)-1}|D| \le \varkappa_4 |u^m|^{q(x)} + C_{23}(\varkappa_4)|D|^{q(x)}, \quad \varkappa_4 > 0.$$

Taking into account the Cauchy-Bunyakowski-Schwarz inequality, the Young inequality with $r', r > 1$, and (16), we get

$$\Big|\int\limits_{Q_{0,T}} \phi(E u^m) D\, dxdt\Big| \le \phi^* ||E u^m; L^2(Q_{0,T})|| \cdot ||D; L^2(Q_{0,T})||$$

$$\le \phi^* E^* ||u^m; L^2(Q_{0,T})|| \cdot ||D; L^2(Q_{0,T})|| \le \varkappa_5 \int\limits_{Q_{0,T}} |u^m|^2\, dxdt + C_{24}(\varkappa_5)\int\limits_{Q_{0,T}} |D|^2\, dxdt, \quad \varkappa_5 > 0.$$

Here $C_{22}, \dots, C_{24} > 0$ are independent of m.

By **(U)** and **(F)**, it follows that $\int_\Omega |u_0|^r\, dx + \int_{Q_{0,T}} [|f|^2 + |D|^2 + |D|^{q(x)} + |D_t|^r + |D_t|^2 + |\nabla D|^2]\, dxdt \le C_{25}$. Therefore, (77) yields that

$$\mathscr{L}_\varepsilon(u^m, z^m) - \int\limits_{Q_{0,T}} f z^m\, dxdt \ge \frac{1}{r}\int\limits_{\Omega} |u^m(T)|^r\, dx + \int\limits_{Q_{0,T}} \Big[\Big(\varepsilon - \frac{\varepsilon}{2}\Big)|u_t^m|^2 + (\alpha_0 - a\varkappa_2)|\nabla u^m|^2$$

$$+ (\alpha_0 - \varkappa_1 - \varkappa_5)|u^m|^2 + (\alpha_0 - \varkappa_3)|u^m|^r + (g_0 - \varkappa_4)|u^m|^{q(x)}\Big]\, dxdt - C_{26}(\varkappa_1, \dots, \varkappa_5), \tag{78}$$

where $C_{26} > 0$ is independent of m. Choosing $\varkappa_1, \dots, \varkappa_5 > 0$ small enough, we obtain the existence of the nonnegative constants C_{27} and C_{28} such that (see (77))

$$(P(\varphi), \varphi)_{\mathbb{R}^m} \ge C_{27}\int\limits_{Q_{0,T}} \Big[|\nabla u^m|^2 + |u^m|^2\Big]\, dxdt - C_{28} \underset{|\varphi| \to +\infty}{\longrightarrow} +\infty.$$

Thus, the Vishyk Lemma (see Proposition 8) implies that there exist a solution $\varphi_1^m, \dots, \varphi_m^m$ of system (76), i.e. (75).

Step 2. Multiplying (75) by φ_μ^m and summing in $\mu = \overline{1,m}$, we obtain

$$\mathscr{L}_\varepsilon(u^m,z^m) = \int\limits_{Q_{0,T}} fz^m \, dxdt.$$

Then $0 = \mathscr{L}_\varepsilon(u^m,z^m) - \int_{Q_{0,T}} fz^m \, dxdt$ and, using (78), we get

$$\varepsilon \int\limits_{Q_{0,T}} |u_t^m|^2 \, dxdt + \int\limits_{\Omega} |u^m(T)|^r \, dx + \int\limits_{Q_{0,T}} \left[|\nabla u^m|^2 + |u^m|^2 + |u^m|^r + |u^m|^{q(x)} \right] dxdt \le C_{29}. \tag{79}$$

By (79) and Proposition 3, we have

$$||u^m; L^{q(x)}(Q_{0,T})|| \le C_{30}. \tag{80}$$

Taking into account Lemma 2, (79), and (80), we obtain

$$||Gu^m; L^{q'(x)}(Q_{0,T})|| \le C_{31}, \qquad ||\mathscr{R}u^m; L^{r'}(Q_{0,T})|| \le C_{32}, \tag{81}$$

$$||\mathscr{R}u^m(T); L^{r'}(\Omega)|| \le C_{33}. \tag{82}$$

Here $C_{29}, \ldots, C_{33} > 0$ are independent of m, ε.

Estimates (79)-(82) yield that there exists a subsequence $\{u^{m_j}\}_{j\in\mathbb{N}} \subset \{u^m\}_{m\in\mathbb{N}}$ such that

$$u^{m_j} \underset{j\to\infty}{\longrightarrow} u \quad \text{weakly in} \quad H^1(Q_{0,T}) \cap L^r(Q_{0,T}) \cap L^{q(x)}(Q_{0,T}), \tag{83}$$

$$Gu^{m_j} \underset{j\to\infty}{\longrightarrow} \widetilde{\chi}_1 \quad \text{weakly in} \quad L^{q'(x)}(Q_{0,T}), \tag{84}$$

$$\mathscr{R}u^{m_j} \underset{j\to\infty}{\longrightarrow} \widetilde{\chi}_2 \quad \text{weakly in} \quad L^{r'}(Q_{0,T}), \tag{85}$$

$$\mathscr{R}u^{m_j}(T) \underset{j\to\infty}{\longrightarrow} \widetilde{\chi}_3 \quad \text{weakly in} \quad L^{r'}(\Omega). \tag{86}$$

By (83), the Rellich-Kondrachov theorem (see Lemma 1.28 [27, p. 47]), and Lemma 1.18 [27, p. 39], it follows that there exists a subsequence (we call it $\{u^{m_j}\}_{j\in\mathbb{N}}$ again) such that

$$u^{m_j} \underset{j\to\infty}{\longrightarrow} u \quad \text{strongly in} \quad L^2(Q_{0,T}) \quad \text{and almost everywhere in} \quad Q_{0,T}. \tag{87}$$

By the Aubin theorem (see Proposition 7), we get

$$u^{m_j} \underset{j\to\infty}{\longrightarrow} u \quad \text{in} \quad C([0,T]; L^2(\Omega)). \tag{88}$$

Therefore, since E is continuous and ϕ satisfies the Lipschitz condition, we obtain

$$\phi(Eu^{m_j}) \underset{j\to\infty}{\longrightarrow} \phi(Eu) \quad \text{strongly in} \quad L^2(Q_{0,T}). \tag{89}$$

In addition,

$$u^{m_j}(T) \underset{j\to\infty}{\longrightarrow} u(T) \quad \text{strongly in} \quad L^2(\Omega) \quad \text{and almost everywhere in} \quad \Omega. \tag{90}$$

By (87), it follows that $Gu^{m_j} \underset{j\to\infty}{\longrightarrow} Gu$ almost everywhere in $Q_{0,T}$. Hence, Proposition 4 yields that $\widetilde{\chi}_1 = Gu$. Continuing in the same way, we see that $\widetilde{\chi}_2 = \mathscr{R}u$ and $\widetilde{\chi}_3 = \mathscr{R}u(T)$.

Using (75) with $m = m_j$, (74), and the condition $v^\mu|_{t=0} = 0$, we obtain

$$\int\limits_{Q_{0,T}} \Big[\varepsilon u_t^{m_j} v_t^\mu - \mathscr{R}u^{m_j} v_t^\mu + a(\nabla u^{m_j}, \nabla v^\mu) + Gu^{m_j}\, v^\mu + \phi(Eu^{m_j})v^\mu\Big]\, dxdt + \int\limits_\Omega \mathscr{R}u^{m_j}(T)v^\mu(T)\, dx = \int\limits_{Q_{0,T}} f v^\mu\, dxdt, \quad (91)$$

where $\mu = \overline{1, m_j}$. Letting $j \to \infty$, taking into account the properties of $\{v^\mu\}_{\mu\in\mathbb{N}}$ and (91), we get (66). Inequalities $r' > 2$, $q'(x) \geq 2$ for a.e. $x \in \Omega$, (16), and (81) imply that $\mathscr{R}u, Gu, Eu \in L^2(Q_{0,T})$. Thus, u is a weak solution to problem (62)-(63).

Step 3. Now we shall prove the additional estimates. By definition, put

$$\mathscr{N}w := Gw + \phi(Ew), \quad w \in L^2(Q_{0,T}). \quad (92)$$

By inequalities $r' > 2$, $q'(x) \geq 2$ for a.e. $x \in \Omega$, (16), and (81), it follows that

$$\int\limits_{Q_{0,T}} \Big[|\mathscr{R}u^m|^2 + |\mathscr{N}u^m|^2\Big]\, dxdt \leq C_{34}, \quad (93)$$

where $C_{34} > 0$ is independent of m, ε.

We recall that (74) holds, $\frac{3}{2} < r \leq 2$, and $u^m \in H^2(Q_{0,T})$. Whence, using (61) with $p = 2$ (note that $p \geq 2(r-1)$), we obtain

$$\big|\big|\big(|u^m|^{r-2}u^m\big)_t; L^2(Q_{0,T})\big|\big| \leq C_{35}(||u^m; L^2(Q_{0,T})||^{r-1} + ||u_t^m; L^2(Q_{0,T})||^{r-1} + ||u_{tt}^m; L^2(Q_{0,T})||^{r-1}), \quad (94)$$

where $C_{35} > 0$ is independent of m. By (30) and (73_3), it follows that

$$||u_t^m; L^2(Q_{0,T})||^2 \leq C_{36} \int\limits_0^T \Big[\int\limits_\Omega |u_t^m(x,T)|^2\, dx + ||u_{tt}^m; L^2(Q_{0,T})||^2\Big]\, dt = C_{37}||u_{tt}^m; L^2(Q_{0,T})||^2.$$

Thus, taking into account (79), the Young inequality with $\frac{1}{r-1}, \frac{1}{2-r} > 1$, and (94), we get

$$||(\mathscr{R}u^m)_t; L^2(Q_{0,T})|| \leq \varkappa_1 ||u_{tt}^m; L^2(Q_{0,T})|| + C_{38}(\varkappa_1), \quad \varkappa_1 > 0, \quad (95)$$

where $C_{38}(\varkappa_1) > 0$ is independent of m, ε.

Step 4. Assume that the positive numbers $\{\widehat{\sigma_m}\}_{m\in\mathbb{N}}$ are determined from Remark 1. Hence, in particular, (28) hold. Using (73_3) and the condition $v^\mu|_{t=0} = 0$, we have

$$\int\limits_{Q_{0,T}} \varepsilon u_t^m v_t^\mu\, dxdt = \int\limits_\Omega \varepsilon u_t^m v^\mu\, dx\Big|_{t=0}^{t=T} - \int\limits_{Q_{0,T}} \varepsilon u_{tt}^m v^\mu\, dxdt = -\int\limits_{Q_{0,T}} \varepsilon u_{tt}^m v^\mu\, dxdt.$$

Then we multiply (75) by $\widehat{\sigma_\mu}\,\varphi_\mu^m$, use (28), and sum in $\mu=\overline{1,m}$. Taking into account (74) and (71), we obtain

$$\int\limits_{Q_{0,T}}\Big[-\varepsilon u_{tt}^m(D_{tt}-u_{tt}^m)+(\mathscr{R}u^m)_t(D_{tt}-u_{tt}^m)+a(\nabla u^m,\nabla(-z_{tt}^m))+\mathscr{N}u^m(D_{tt}-u_{tt}^m)\Big]\,dxdt=\int\limits_{Q_{0,T}} f(D_{tt}-u_{tt}^m)\,dxdt. \quad (96)$$

The Young inequality and condition **(Φ)** imply that

$$|u_{tt}^m D_{tt}|\le\frac{1}{2}|u_{tt}^m|^2+\frac{1}{2}|D_{tt}|^2,\quad |fD_{tt}|\le\frac{1}{2}|f|^2+\frac{1}{2}|D_{tt}|^2,\quad |fu_{tt}^m|\le\varkappa_2|u_{tt}^m|^2+\frac{1}{4\varkappa_2}|f|^2,\quad \varkappa_2>0,$$
$$|(\mathscr{R}u^m)_t(D_{tt}-u_{tt}^m)|\le\frac{\varkappa_3}{2}|u_{tt}^m|^2+C_{39}(\varkappa_3)\Big(|(\mathscr{R}u^m)_t|^2+|D_{tt}|^2\Big),\quad \varkappa_3>0, \quad (97)$$
$$|\mathscr{N}u^m(D_{tt}-u_{tt}^m)|\le\varkappa_4|u_{tt}^m|^2+C_{40}(\varkappa_4)(|\mathscr{N}u^m|^2+|D_{tt}|^2),\quad \varkappa_4>0.$$

By (97) and (95) with $\varkappa_1>0$ small enough, it follows that

$$\Big|\int\limits_{Q_{0,T}}(\mathscr{R}u^m)_t(D_{tt}-u_{tt}^m)\,dxdt\Big|\le\varkappa_3\int\limits_{Q_{0,T}}|u_{tt}^m|^2\,dxdt+C_{41}(\varkappa_3)\Big(1+\int\limits_{Q_{0,T}}|D_{tt}|^2\,dxdt\Big).$$

Since $\partial\Omega\in C^4$, Remark 1 yields that $\{v^m\}_{m\in\mathbb{N}}\subset H^4(Q_{0,T})$. Integrating by parts and using ($72_{2,3}$), we get

$$\int\limits_{Q_{0,T}}a(\nabla u^m,-\nabla z_{tt}^m)\,dxdt=-\int\limits_{Q_{0,T}}a(\nabla D+\nabla z^m,\nabla z_{tt}^m)\,dxdt=I_1-\int\limits_{Q_{0,T}}a(\nabla z^m,\nabla z_{tt}^m)\,dxdt$$
$$=I_1-\int\limits_{\Omega}a(\nabla z^m,\nabla z_t^m)\,dx\Big|_{t=0}^{t=T}+\int\limits_{Q_{0,T}}a|\nabla z_t^m|^2\,dxdt=I_1+\int\limits_{Q_{0,T}}a|\nabla u_t^m-\nabla D_t|^2\,dxdt$$
$$\ge I_1+\frac{a}{2}\int\limits_{Q_{0,T}}|\nabla u_t^m|^2\,dxdt-C_{42}\int\limits_{Q_{0,T}}|\nabla D_t|^2\,dxdt, \quad (98)$$

where

$$I_1:=-\int\limits_{Q_{0,T}}a(\nabla D,\nabla z_{tt}^m)\,dxdt.$$

Integrating by parts in t, using (72_3), and using (73_2), we obtain

$$I_1=-\int\limits_{\Omega}a(\nabla D,\nabla z_t^m)\,dx\Big|_{t=0}^{t=T}+\int\limits_{Q_{0,T}}a(\nabla D_t,\nabla z_t^m)\,dxdt=I_2+\int\limits_{Q_{0,T}}a(\nabla D_t,\nabla z_t^m)\,dxdt.$$

where

$$I_2:=\int\limits_{\Omega}a(\nabla D(0),\nabla z_t^m(0))\,dx.$$

Integrating by parts in x and using condition **(U)**, we get

$$I_2=\int\limits_{\partial\Omega}a(\nabla D(0),\nu)z_t^m(0)\,dS-\int\limits_{\Omega}a\big(\Delta D(0)\big)\,z_t^m(0)\,dx=-\int\limits_{\Omega}az_t^m(0)\,\Delta D(0)\,dx.$$

Moreover, $|a(\nabla D_t,\nabla z_t^m)| = |a(\nabla D_t,\nabla u_t^m-\nabla D_t)| \le \varkappa_5|\nabla u_t^m|^2 + C_{43}(\varkappa_5)|\nabla D_t|^2$. Whence,

$$|I_1| \le \int\limits_{\Omega}\Big[\varkappa_6|z_t^m(0)|^2 + C_{44}(\varkappa_6)|\Delta D(0)|^2\Big]\,dx + \int\limits_{Q_{0,T}}\Big[\varkappa_5|\nabla u_t^m|^2 + C_{45}(\varkappa_5)|\nabla D_t|^2\Big]\,dxdt,\quad \varkappa_5,\varkappa_6>0.$$

By the estimate of type (30), (72_3), and (71), it follows that

$$\int\limits_{\Omega}|z_t^m(0)|^2\,dx \le C_{46}\Big(\int\limits_{\Omega}|z_t^m(T)|^2\,dx + \int\limits_{Q_{0,T}}|z_{tt}^m|^2\,dxdt\Big) = C_{46}\int\limits_{Q_{0,T}}|u_{tt}^m-D_{tt}|^2\,dxdt \le C_{47}\int\limits_{Q_{0,T}}\Big[|u_{tt}^m|^2+|D_{tt}|^2\Big]\,dxdt,$$

where $C_{47}>0$ is independent of m.

Finally, from (98) we get

$$\int\limits_{Q_{0,T}} a(\nabla u^m,-\nabla z_{tt}^m)\,dxdt \ge \int\limits_{Q_{0,T}}\Big[\Big(\frac{a}{2}-\varkappa_5\Big)|\nabla u_t^m|^2 - C_{47}\varkappa_6|u_{tt}^m|^2\Big]\,dxdt$$
$$-C_{48}(\varkappa_5,\varkappa_6)\Big(\int\limits_{\Omega}|\Delta D(0)|^2\,dx + \int\limits_{Q_{0,T}}\Big[|\nabla D_t|^2+|D_{tt}|^2\Big]\,dxdt\Big).$$

Since $\int_\Omega|\Delta D(0)|^2\,dx + \int_{Q_{0,T}}[|f|^2+|D_{tt}|^2+|\nabla D_t|^2]\,dxdt \le C_{49}$, by (93), (95), and (96), it follows that

$$\int\limits_{Q_{0,T}}\Big[\Big(\varepsilon-\frac{\varepsilon}{2}-\varkappa_2-\varkappa_3-\varkappa_4-C_{47}\varkappa_6\Big)|u_{tt}^m|^2 + \Big(\frac{a}{2}-\varkappa_5\Big)|\nabla u_t^m|^2\Big]\,dxdt \le C_{50}(\varkappa_2,\dots,\varkappa_6).$$

Hence, choosing $\varkappa_2,\dots,\varkappa_6>0$ small enough, we have

$$\int\limits_{Q_{0,T}}\Big[|u_{tt}^m|^2+|\nabla u_t^m|^2\Big]\,dxdt \le C_{51}. \tag{99}$$

Thus, (95) yields that

$$||(\mathscr{R}u^m)_t;L^2(Q_{0,T})|| \le C_{52}, \tag{100}$$

Here $C_{50},\dots,C_{52}>0$ are independent of m.

Step 5. Assume that the positive numbers $\{\widehat{\lambda_m}\}_{m\in\mathbb{N}}$ are determined from Remark 1. Hence, in particular, (28) hold. Clearly (here and below ν denotes the outward unit vector field on $\partial\Omega$),

$$\int\limits_{Q_{0,T}} a(\nabla u^m,\nabla v^\mu)\,dxdt = \int\limits_{\Sigma_{0,T}} a(\nabla u^m,\nu)v^\mu\,dSdt - \int\limits_{Q_{0,T}} a\Delta u^m v^\mu\,dxdt = -\int\limits_{Q_{0,T}} a\Delta u^m v^\mu\,dxdt.$$

Multiply (75) by $\widehat{\lambda_\mu}\,\varphi_\mu^m$, use (28) and (74), and sum in $\mu=\overline{1,m}$. Then we get

$$\int\limits_{Q_{0,T}}\Big[\varepsilon u_t^m(-\Delta z_t^m) + (\mathscr{R}u^m)_t(\Delta D-\Delta u^m) - a\Delta u^m(\Delta D-\Delta u^m)$$
$$+\mathscr{N}u^m(\Delta D-\Delta u^m)\Big]\,dxdt = \int\limits_{Q_{0,T}} f(\Delta D-\Delta u^m)\,dxdt. \tag{101}$$

Clearly (we recall that $\{z^m\}_{m\in\mathbb{N}} \subset H^4(Q_{0,T})$),

$$-\int\limits_{Q_{0,T}} \varepsilon u_t^m \Delta z_t^m \,dxdt = -\int\limits_{\Sigma_{0,T}} \varepsilon u_t^m (\nabla z_t^m, \nu)\, dSdt + \int\limits_{Q_{0,T}} \varepsilon(\nabla u_t^m, \nabla z_t^m)\, dxdt$$

$$= \int\limits_{Q_{0,T}} \varepsilon(\nabla u_t^m, \nabla u_t^m - \nabla D_t)\, dxdt \ge \frac{\varepsilon}{2}\int\limits_{Q_{0,T}} \left[|\nabla u_t^m|^2 - |\nabla D_t|^2\right] dxdt.$$

The Young inequalities yields that

$$|f\Delta D| \le \frac{1}{2}|f|^2 + \frac{1}{2}|\Delta D|^2, \qquad |a\Delta u^m \Delta D| \le \varkappa_1 |\Delta u^m|^2 + \frac{a^2}{4\varkappa_1}|\Delta D|^2, \quad \varkappa_1 > 0,$$

$$|f\Delta u^m| \le \varkappa_2|\Delta u^m|^2 + \frac{1}{4\varkappa_2}|f|^2, \quad \varkappa_2 > 0,$$

$$|(\mathscr{R}u^m)_t(\Delta D - \Delta u^m)\Big| \le \varkappa_3|\Delta u^m|^2 + C_{53}(\varkappa_3)(|(\mathscr{R}u^m)_t|^2 + |\Delta D|^2\Big), \quad \varkappa_3 > 0,$$

$$|\mathscr{N}u^m(\Delta D - \Delta u^m)| \le \varkappa_4|\Delta u^m|^2 + C_{54}(\varkappa_4)(|\mathscr{N}u^m|^2 + |\Delta D|^2\Big), \quad \varkappa_4 > 0.$$

Since $\int_{Q_{0,T}}[|f|^2 + |\Delta D|^2 + |\nabla D_t|^2]\, dxdt \le C_{55}$, by (93), (95), and (101), it follows that

$$\int\limits_{Q_{0,T}} \left[\frac{\varepsilon}{2}|\nabla u_t^m|^2 + \Big(a - \varkappa_1 - \varkappa_2 - \varkappa_3 - \varkappa_4\Big)|\Delta u^m|^2\right] dxdt \le C_{56}(\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4),$$

where $C_{56} > 0$ is independent of m. Therefore, choosing $\varkappa_1,\ldots,\varkappa_4 > 0$ small enough, we obtain

$$\int\limits_{Q_{0,T}} |\Delta u^m|^2\, dxdt \le C_{57}, \tag{102}$$

where $C_{57} > 0$ is independent of m.

By (100) and (87), it follows that

$$(\mathscr{R}u^{m_j})_t \underset{j\to\infty}{\longrightarrow} (\mathscr{R}u)_t \quad \text{weakly in} \quad L^2(Q_{0,T}). \tag{103}$$

By (99) and (102), we get

$$u^{m_j} \underset{j\to\infty}{\longrightarrow} u \quad \text{weakly in} \quad H^2(Q_{0,T})$$

and so $z^{m_j} \underset{j\to\infty}{\longrightarrow} z$ weakly in $H^2(Q_{0,T})$. Since every function z^m satisfies (72), the convergence above implies that $z \in U_0(Q_{0,T})$. Thus, $u - D = z \in U_0(Q_{0,T})$ and Theorem 6 is proved. □

4 Proof of main Theorem

The Case $r = 2$ is trivial. Assume that $r \in (1,2)$. The solution will be constructed via the elliptic regularization method.

Step 1. For every $\varepsilon > 0$ let us denote by u^ε a weak solution to problem (62)-(63) (see Theorem 6). Then (66) holds, $u^\varepsilon \in H^2(Q_{0,T}) \cap H^1(0,T;H_0^1(\Omega))$, $\mathscr{R}u^\varepsilon \in H^1(0,T;L^2(\Omega))$, and $Gu^\varepsilon, \phi(Eu^\varepsilon) \in L^2(Q_{0,T})$. By (79) and (80), we get

$$\varepsilon\int\limits_{Q_{0,T}} |u_t^\varepsilon|^2\, dxdt + \int\limits_{Q_{0,T}} \left[|\nabla u^\varepsilon|^2 + |u^\varepsilon|^2 + |u^\varepsilon|^r + |u^\varepsilon|^{q(x)}\right] dxdt \le C_{58}, \tag{104}$$

$$||u^\varepsilon;L^{q(x)}(Q_{0,T})||\le C_{59}. \tag{105}$$

Whence, similarly to (81), we obtain

$$||Gu^\varepsilon;L^{q'(x)}(Q_{0,T})||\le C_{60},\qquad ||\mathscr{R}u^\varepsilon;L^{r'}(Q_{0,T})||\le C_{61}. \tag{106}$$

In addition, by (16), **(Φ)**, and (104), it follows that

$$||\phi(Eu^\varepsilon);L^2(Q_{0,T})||\le C_{62}. \tag{107}$$

Here $C_{58},\dots,C_{62}>0$ are independent of ε.

Estimates (104)-(107) implies that there exists a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}\subset\mathbb{R}_+$ such that $\varepsilon_j \underset{j\to\infty}{\longrightarrow} 0$ and

$$u^{\varepsilon_j}\underset{j\to\infty}{\longrightarrow} u \quad \text{weakly in} \quad L^2(0,T;H_0^1(\Omega))\cap L^r(Q_{0,T})\cap L^{q(x)}(Q_{0,T}), \tag{108}$$

$$Gu^{\varepsilon_j}\underset{j\to\infty}{\longrightarrow}\chi_1 \quad \text{weakly in} \quad L^{q'(x)}(Q_{0,T}), \tag{109}$$

$$\mathscr{R}u^{\varepsilon_j}\underset{j\to\infty}{\longrightarrow}\chi_2 \quad \text{weakly in} \quad L^{r'}(Q_{0,T}), \tag{110}$$

$$\phi(Eu^{\varepsilon_j})\underset{j\to\infty}{\longrightarrow}\chi_3 \quad \text{weakly in} \quad L^2(Q_{0,T}), \tag{111}$$

$$\sqrt{\varepsilon_j}\,u_t^{\varepsilon_j}\underset{j\to\infty}{\longrightarrow}\chi_4 \quad \text{weakly in} \quad L^2(Q_{0,T}). \tag{112}$$

Step 2. Notice that the constants in (99), (100), and (102) depend on ε and so we can not use these estimates here. We shall prove the additional estimates.

Since $u^\varepsilon\in H^2(Q_{0,T})$ and $2\ge 2(\frac{r}{2}+1-1)$ holds, Case 2.ii of Theorem 5 with $p=2$ and $\frac{r}{2}+1$ instead of r yields that

$$|u^\varepsilon|^{\frac{r}{2}-1}u^\varepsilon\in H^1(0,T;L^2(\Omega)),\qquad \left(|u^\varepsilon|^{\frac{r}{2}-1}u^\varepsilon\right)_t=\frac{r}{2}\,|u^\varepsilon|^{\frac{r}{2}-1}u_t^\varepsilon. \tag{113}$$

Since $u^\varepsilon\in H^2(Q_{0,T})$ and $\frac{3}{2}<r\le 2$ holds, similarly to (74), we have

$$\mathscr{R}u^\varepsilon\in H^1(0,T;L^2(\Omega)),\qquad (\mathscr{R}u^\varepsilon)_t=|u^\varepsilon|^{r-2}u_t^\varepsilon. \tag{114}$$

Moreover, $u^\varepsilon\in H^1(Q_{0,T})$, $2\ge r$, and so similarly to (70), we obtain

$$|u^\varepsilon|^r\in W^{1,\frac{2}{r}}(0,T;L^{\frac{2}{r}}(\Omega)),\qquad \left(|u^\varepsilon|^r\right)_t=r\,|u^\varepsilon|^{r-2}u^\varepsilon\,u_t^\varepsilon. \tag{115}$$

Continuing in the same way (we recall that $u^\varepsilon,u_t^\varepsilon,u_{x_1}^\varepsilon,\dots,u_{x_n}^\varepsilon\in H^1(Q_{0,T})$), using Case 1.ii of Theorem 2 with $p(x)\equiv 2$ and $r(x)\equiv 2$, we see that

$$|u^\varepsilon|^2\in W^{1,1}(0,T;L^1(\Omega)),\qquad \left(|u^\varepsilon|^2\right)_t=2\,u^\varepsilon\,u_t^\varepsilon, \tag{116}$$

$$|u_t^\varepsilon|^2\in W^{1,1}(0,T;L^1(\Omega)),\qquad \left(|u_t^\varepsilon|^2\right)_t=2\,u_t^\varepsilon\,u_{tt}^\varepsilon, \tag{117}$$

$$|\nabla u^\varepsilon|^2\in W^{1,1}(0,T;L^1(\Omega)),\qquad \left(|\nabla u^\varepsilon|^2\right)_t=2\,(\nabla u^\varepsilon,\nabla u_t^\varepsilon). \tag{118}$$

Finally, since $u^\varepsilon \in H^1(Q_{0,T})$ and $q^0 \le 2$ holds, Case 1.ii of Theorem 2 with $r(x) = q(x)$ and $p(x) \equiv 2$ yields that

$$|u^\varepsilon|^{q(x)}, (|u^\varepsilon|^{q(x)})_t \in L^{\frac{2}{q(x)}}(Q_{0,T}), \qquad (|u^\varepsilon|^{q(x)})_t = q(x)|u^\varepsilon|^{q(x)-2}u^\varepsilon u_t^\varepsilon. \tag{119}$$

Integrating by parts (see (114)) and using (63_3), we obtain

$$\int\limits_{Q_{0,T}} \left[\varepsilon u_t^\varepsilon v_t - \mathscr{R}u^\varepsilon v_t\right] dxdt = \int\limits_{\Omega} \left[\varepsilon u_t^\varepsilon v - \mathscr{R}u^\varepsilon v\right] dx\Big|_{t=0}^{t=T} - \int\limits_{Q_{0,T}} \left[\varepsilon u_{tt}^\varepsilon v - (\mathscr{R}u^\varepsilon)_t v\right] dxdt$$
$$= -\int\limits_{\Omega} \mathscr{R}u^\varepsilon(T)v(T)\, dx + \int\limits_{Q_{0,T}} \left[-\varepsilon u_{tt}^\varepsilon v + (\mathscr{R}u^\varepsilon)_t v\right] dxdt, \quad v \in U_0(Q_{0,T}).$$

Then from (66) it follows that

$$\int\limits_{Q_{0,T}} \left[-\varepsilon u_{tt}^\varepsilon v + (\mathscr{R}u^\varepsilon)_t v + a(\nabla u^\varepsilon, \nabla v) + Gu^\varepsilon\, v + \phi(Eu^\varepsilon)v\right] dxdt = \int\limits_{Q_{0,T}} fv\, dxdt, \quad v \in U_0(Q_{0,T}). \tag{120}$$

Assume that $w \in H_0^1(\Omega)$, $\varphi \in C^1([0,T])$, and $\varphi(0) = 0$. Substituting $v(x,t) = w(x)\varphi(t)$ in (120), we get

$$\int\limits_0^T \Big(\int\limits_{\Omega_t} \left[-\varepsilon u_{tt}^\varepsilon w + (\mathscr{R}u^\varepsilon)_t w + a(\nabla u^\varepsilon, \nabla w) + Gu^\varepsilon\, w + \phi(Eu^\varepsilon)w - fw\right] dx\Big)\varphi(t)\, dt = 0. \tag{121}$$

Take a function $\psi \in C_0^\infty((0,T))$. Substituting $\varphi = \psi$ in (121), we have the following equality in $H^{-1}(\Omega) := [H_0^1(\Omega)]^*$:

$$-\varepsilon u_{tt}^\varepsilon(t) + \big(\mathscr{R}u^\varepsilon(t)\big)_t + Au^\varepsilon(t) + Gu^\varepsilon(t) + \phi(Eu^\varepsilon(t)) = f(t), \quad t \in (0,T), \tag{122}$$

where the operator $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ is defined by the rule

$$\langle Av, w\rangle_{H_0^1(\Omega)} = \int\limits_{\Omega} a(\nabla v(x), \nabla w(x))\, dx, \quad v, w \in H_0^1(\Omega). \tag{123}$$

Since $u_t^\varepsilon \in L^2(0,T; H_0^1(\Omega))$, by (122) and (114), it follows that

$$\int\limits_{Q_{0,T}} \left[-\varepsilon u_{tt}^\varepsilon u_t^\varepsilon + |u^\varepsilon|^{r-2}|u_t^\varepsilon|^2 + a(\nabla u^\varepsilon, \nabla u_t^\varepsilon) + Gu^\varepsilon u_t^\varepsilon + \phi(Eu^\varepsilon)u_t^\varepsilon\right] dxdt = \int\limits_{Q_{0,T}} fu_t^\varepsilon\, dxdt. \tag{124}$$

Integrating by parts (see (117)) and using (63_3), we obtain

$$\int\limits_{Q_{0,T}} -\varepsilon u_{tt}^\varepsilon u_t^\varepsilon\, dxdt = -\frac{\varepsilon}{2}\int\limits_{\Omega} |u_t^\varepsilon|^2\, dx\Big|_{t=0}^{t=T} = -0 + \frac{\varepsilon}{2}\int\limits_{\Omega} |u_t^\varepsilon(0)|^2\, dx \ge 0.$$

From the Young inequality (note that $\frac{1}{2} + \frac{1}{r'} + \frac{r'-2}{2r'} = 1$), it follows that

$$|fu_t^\varepsilon| = |u^\varepsilon|^{\frac{r}{2}-1}|u_t^\varepsilon||f||u^\varepsilon|^{1-\frac{r}{2}} \le \varkappa_1|u^\varepsilon|^{r-2}|u_t^\varepsilon|^2 + C_{63}(\varkappa_1)(|f|^{r'} + |u^\varepsilon|^{(1-\frac{r}{2})\frac{2r'}{r'-2}})$$
$$= \varkappa_1|u^\varepsilon|^{r-2}|u_t^\varepsilon|^2 + C_{63}(\varkappa_1)(|f|^{r'} + |u^\varepsilon|^{r}), \quad \varkappa_1 > 0,$$

Assume that the constant $\gamma \in (\frac{4}{3}, 2]$ is defined by (11). Then the Young inequality with $\frac{4-r}{2}, \frac{4-r}{2-r} > 1$ implies that

$$|u_t^\varepsilon|^\gamma = |u^\varepsilon|^{\frac{2(r-2)}{4-r}} |u_t^\varepsilon|^{\frac{4}{4-r}} |u^\varepsilon|^{\frac{2(2-r)}{4-r}} \leq \varkappa_2 |u^\varepsilon|^{r-2}|u_t^\varepsilon|^2 + C_{64}(\varkappa_2)|u^\varepsilon|^2, \quad \varkappa_2 > 0. \tag{125}$$

Whence, since ϕ is bounded, the Young inequality and (104) yield that

$$\Big| \int\limits_{Q_{0,T}} \phi(Eu^\varepsilon) u_t^\varepsilon \, dxdt \Big| \leq C_{65} \int\limits_{Q_{0,T}} |u_t^\varepsilon| \, dxdt \leq \int\limits_{Q_{0,T}} |u_t^\varepsilon|^\gamma \, dxdt + C_{66} \leq \varkappa_2 \int\limits_{Q_{0,T}} |u^\varepsilon|^{r-2}|u_t^\varepsilon|^2 \, dxdt + C_{67}(\varkappa_2).$$

Taking into account the transformations above, by (124), (118), and (119), we obtain

$$\int\limits_{Q_{0,T}} \Big[(1-\varkappa_1-\varkappa_2)|u^\varepsilon|^{r-2}|u_t^\varepsilon|^2 + \frac{a}{2}(|\nabla u^\varepsilon|^2)_t + \frac{g(x,t)}{q(x)}(|u^\varepsilon|^{q(x)})_t \Big] \, dxdt \leq C_{68}(\varkappa_1, \varkappa_2). \tag{126}$$

Choosing $\varkappa_1, \varkappa_2 > 0$ small enough and integrating by parts, we get

$$\int\limits_{\Omega_T} \Big[\frac{a}{2}|\nabla u^\varepsilon|^2 + \frac{g_0}{q^0}|u^\varepsilon|^{q(x)} \Big] \, dx + \frac{1}{2} \int\limits_{Q_{0,T}} |u^\varepsilon|^{r-2}|u_t^\varepsilon|^2 \, dxdt \leq C_{69}\Big(1 + \int\limits_{\Omega} \Big[|\nabla u_0|^2 + |u_0|^{q(x)} \Big] \, dx + \int\limits_{Q_{0,T}} \frac{|g_t(x,t)|}{q(x)} |u^\varepsilon|^{q(x)} \, dxdt \Big).$$

Thus,

$$\int\limits_{Q_{0,T}} |u^\varepsilon|^{r-2}|u_t^\varepsilon|^2 \, dxdt \leq C_{70}, \tag{127}$$

and so (113) yields that

$$\int\limits_{Q_{0,T}} \Big| \big(|u^\varepsilon|^{\frac{r}{2}-1}u^\varepsilon \big)_t \Big|^2 \, dxdt = \frac{|r|^2}{4} \int\limits_{Q_{0,T}} |u^\varepsilon|^{r-2}|u_t^\varepsilon|^2 \, dxdt \leq C_{71}. \tag{128}$$

Moreover, from (104) it follows that

$$\int\limits_{Q_{0,T}} \Big| |u^\varepsilon|^{\frac{r}{2}-1}u^\varepsilon \Big|^2 \, dxdt = \int\limits_{Q_{0,T}} |u^\varepsilon|^r \, dxdt \leq C_{72}. \tag{129}$$

In addition, (125), (127), and (104) imply that

$$\int\limits_{Q_{0,T}} |u_t^\varepsilon|^\gamma \, dxdt \leq C_{73}. \tag{130}$$

Here $C_{63}, \ldots, C_{73} > 0$ are independent of ε.

Estimates (104) and (127)-(130) yield that

$$|u^{\varepsilon_j}|^{\frac{r}{2}-1}u^{\varepsilon_j} \underset{j\to\infty}{\longrightarrow} \chi_5 \quad \text{weakly in} \quad H^1(0,T;L^2(\Omega)), \tag{131}$$

$$u^{\varepsilon_j} \underset{j\to\infty}{\longrightarrow} u \quad \text{weakly in} \quad W^{1,\gamma}(Q_{0,T}). \tag{132}$$

Hence, using the Rellich-Kondrachov theorem (see Lemma 1.28 [27, p. 47]) and Lemma 1.18 [27, p. 39], we obtain

$$u^{\varepsilon_j} \underset{j\to\infty}{\longrightarrow} u \quad \text{strongly in} \quad L^\gamma(Q_{0,T}) \quad \text{and almost everywhere in} \quad Q_{0,T}, \tag{133}$$

Therefore, (see Proposition 4, (109), (110), and (131)) $\chi_1 = Gu$, $\chi_2 = \mathscr{R}u$, and $\chi_5 = |u|^{\frac{r}{2}-1}u$. By (132) and the Aubin theorem (see Proposition 7), it follows that

$$u^{\varepsilon_j} \underset{j\to\infty}{\longrightarrow} u \quad \text{in} \quad C([0,T];L^{\gamma}(\Omega)). \tag{134}$$

It is easy to prove that the operator E (see (3)) is continuous from $L^{\gamma}(Q_{0,T})$ into $L^{\gamma}(Q_{0,T})$. Then, (Φ) and (134) imply that $\phi(Eu^{\varepsilon_j}) \underset{j\to\infty}{\longrightarrow} \phi(Eu)$ strongly in $L^{\gamma}(Q_{0,T})$. Thus, $\chi_3 = \phi(Eu)$.

Step 3. We substitute $\varepsilon = \varepsilon_j$ in (66) and tend $j \to \infty$ if $v \in H_0^1(Q_{0,T})$. Taking into account (108)-(112), we obtain (15). By (132) and condition (63), it follows that the function u satisfies (2). Thus, u is a weak solution to problem (1)-(2). By (131) and (132), we get $|u|^{\frac{r}{2}-1}u \in H^1(0,T;L^2(\Omega))$ and $u \in W^{1,\gamma}(Q_{0,T})$. This completes the proof of Theorem 1. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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