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# A new characterization of curves on dual unit sphere

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**Abstract:** The position vectors of unit speed spherical curves can be expressed with the help of their Frenet frame vectors. In this paper, we classify such curves and get certain consequences for $T$-constant and $N$-constant types of curves in dual space $\mathbb{D}^3$.

**Keywords:** Dual unit sphere, curve, dual number, Frenet frame vectors.

## 1 Introduction

W. K. Clifford (1849–1879) presented dual numbers [6]. Afterwards, E. Study used dual numbers and vectors for his studies on geometry and mechanics, and he defined the mapping, E. Study's mapping, which gives the relation between the set of oriented straight lines in $\mathbb{E}^3$ and the dual points on the surface of a dual unit sphere $\widetilde{S}^2$ in the dual space $\mathbb{D}^3$. Also, in [19], the geometric sense of dual numbers are studied in detail. Recently, dual numbers have been the subject of many studies such as [1,11,15,17,20].

The notion of *T-constant* and *N-constant* types of curves is given by B. Y. Chen in [5]. If the norm of the tangential component (normal component) is constant, the curve is called as *T-constant* ( *N-constant* ) in $\mathbb{E}^n$ [5]. Also, if this norm is equal to zero, then the curve is a *T-constant* (*N-constant*) curve of first kind, otherwise second kind [10]. Recently the authors study with these types of curves in [2,3,4,10,12].

Screw lines or helices, called as *W-curves* by F. Klein and S. Lie [13], are curves with constant curvature and constant torsion, and they are mentioned in [7,8].

In this paper, we deal with unit speed spherical curves on dual unit sphere $\widetilde{S}^2$ with respect to their Frenet frame $\{T, N_1, N_2\}$. Since $\{T, N_1, N_2\}$ is an orthonormal basis in $\mathbb{D}^3$, we write the position vector of the curve as

$$\overrightarrow{\Gamma}(s) = \widehat{m_0}(s)\,\overrightarrow{T}(s) + \widehat{m_1}(s)\,\overrightarrow{N_1}(s) + \widehat{m_2}(s)\,\overrightarrow{N_2}(s)$$

for some differentiable dual functions $\widehat{m_i}(s)$, $0 \le i \le 2$ on dual unit sphere $\widetilde{S}^2$. We characterize such curves with regard to their curvature functions $\widehat{m_i}(s)$ and give certain consequences about $T$-constant and $N$-constant curves on dual unit sphere $\widetilde{S}^2$.

## 2 Basic notations

In this part, we give some fundamental properties of dual numbers and some notions about space curves.

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**Definition 21** *The combination $P = p + \varepsilon p^*$ of the real numbers $p$ and $p^*$ is said to be a dual number where $\varepsilon^2 = 0$ and $\varepsilon = (0,1)$ is a dual unit.*

The set

$$\mathbb{D} = \{P = p + \varepsilon p^* : p, p^* \in \mathbb{R}\}$$

is called dual number system and constitutes a commutative ring over the real number field with the following addition and multiplication operations

$$(p + \varepsilon p^*) + (q + \varepsilon q^*) = p + q + \varepsilon(p^* + q^*)$$

and

$$(p + \varepsilon p^*)(q + \varepsilon q^*) = pq + \varepsilon(pq^* + p^*q),$$

respectively.

For the equality of $P$ and $Q$, we have $P = Q \Longleftrightarrow p = q \wedge p^* = q^*$ [6].

**Definition 22** *Let $\mathbb{D}^3$ be the set of all triples of dual numbers, i.e.,*

$$\mathbb{D}^3 = \left\{ \overrightarrow{P} = \overrightarrow{p} + \varepsilon \overrightarrow{p^*} : \overrightarrow{p}, \overrightarrow{p^*} \in \mathbb{R}^3 \right\}.$$

*The set $\mathbb{D}^3$ whose elements are dual vectors is a module over the ring $\mathbb{D}$ and called as $\mathbb{D}$-module or dual space .*

*A dual vector $\overrightarrow{P} = \overrightarrow{p} + \varepsilon \overrightarrow{p^*}$ has another statement*

$$\overrightarrow{P} = (p_1 + \varepsilon p_1^*, p_2 + \varepsilon p_2^*, p_3 + \varepsilon p_3^*),$$

*where vectors $\overrightarrow{p} = (p_1, p_2, p_3)$, $\overrightarrow{p^*} = (p_1^*, p_2^*, p_3^*)$.*

**Definition 23** *Let $\overrightarrow{P}$ and $\overrightarrow{Q}$ be two dual vectors. Then the scalar product and the cross product of these vectors are defined as*

$$\left\langle \overrightarrow{P}, \overrightarrow{Q} \right\rangle = \langle \overrightarrow{p}, \overrightarrow{q} \rangle + \varepsilon \left( \left\langle \overrightarrow{p}, \overrightarrow{q^*} \right\rangle + \left\langle \overrightarrow{p^*}, \overrightarrow{q} \right\rangle \right)$$

and

$$\overrightarrow{P} \times \overrightarrow{Q} = \overrightarrow{p} \times \overrightarrow{q} + \varepsilon(\overrightarrow{p} \times \overrightarrow{q^*} + \overrightarrow{p^*} \times \overrightarrow{q}),$$

respectively.

**Definition 24** *The norm of a dual vector is defined as*

$$\left\| \overrightarrow{P} \right\| = \| \overrightarrow{p} \| + \varepsilon \frac{\left\langle \overrightarrow{p}, \overrightarrow{p^*} \right\rangle}{\| \overrightarrow{p} \|}, \| \overrightarrow{p} \| \neq 0.$$

A dual vector $\overrightarrow{P}$ with $\left\| \overrightarrow{P} \right\| = 1$ is called a dual unit vector.

**Proposition 25** *[9] $\overrightarrow{P}$ is a dual unit vector if and only if $\| \overrightarrow{p} \| = 1$ and $\left\langle \overrightarrow{p}, \overrightarrow{p^*} \right\rangle = 0$.*

**Definition 26** *The set*

$$\left\{ \overrightarrow{P} = \overrightarrow{p} + \varepsilon \overrightarrow{p^*} \in \mathbb{D}^3 : \left\| \overrightarrow{P} \right\| = 1 \right\}$$

*is called dual unit sphere in $\mathbb{D}^3$.*

**Theorem 27** *[18] (E. Study) A unit vector* $\overrightarrow{P} = \overrightarrow{p} + \varepsilon\overrightarrow{p^*}$ *on dual unit sphere corresponds to one and only oriented line in* $\mathbb{R}^3$*, where the real part* $\overrightarrow{p}$ *shows the direction of the line and the dual part* $\overrightarrow{p^*}$ *shows the vectorial moment of the unit vector* $\overrightarrow{a}$ *with respect to the origin.*

**Definition 28** *Let* $\gamma_i(t)$ *and* $\gamma_i^*(t)$*,* $1 \le i \le 3$ *are real valued differentiable functions. Then*

$$\begin{aligned}&\overrightarrow{\Gamma} : I \subseteq \mathbb{R} \to \mathbb{D}^3,\\ &t \to \overrightarrow{\Gamma}(t) = (\gamma_1(t) + \varepsilon\gamma_1^*(t), \gamma_2(t) + \varepsilon\gamma_2^*(t), \gamma_3(t) + \varepsilon\gamma_3^*(t))\\ &\quad = \overrightarrow{\gamma}(t) + \varepsilon\overrightarrow{\gamma}^*(t)\end{aligned}$$

*is a differentiable dual space curve in* $\mathbb{D}^3$*. The real part* $\overrightarrow{\gamma}(t)$ *is called the indicatrix of* $\overrightarrow{\Gamma}(t)$*. The dual arclength of the curve* $\overrightarrow{\Gamma}(t)$ *from* $t_1$ *to* $t$ *is defined as*

$$\widehat{s} = \int_{t_1}^{t} \left\| \overrightarrow{\Gamma}'(t) \right\| dt = \int_{t_1}^{t} \left\| \overrightarrow{\gamma}'(t) \right\| dt + \varepsilon \int_{t_1}^{t} \left\langle \overrightarrow{t}, \left(\overrightarrow{\gamma}^*(t)\right)' \right\rangle dt = s + \varepsilon s^*,$$

*where* $\overrightarrow{t}$ *is the unit tangent vector of* $\overrightarrow{\gamma}(t)$*. Henceforth, instead of* $t$*,* $s$ *will be used as the parameter of* $\overrightarrow{\gamma}(t)$ *[14].*

**Theorem 29** *Let* $\overrightarrow{\Gamma} : I \subseteq \mathbb{R} \to \mathbb{D}^3$ *be a unit speed spherical curve on dual unit sphere on* $\widetilde{S}^2$ *and* $\left\{\overrightarrow{T}, \overrightarrow{N}_1, \overrightarrow{N}_2\right\}$ *be the Frenet frame of the curve* $\overrightarrow{\Gamma}$*. Then, the Frenet equations are given as follows:*

$$\begin{aligned}\overrightarrow{T}' &= \widehat{\kappa_1}\overrightarrow{N_1},\\ \overrightarrow{N_1}' &= -\widehat{\kappa_1}\overrightarrow{T} + \widehat{\kappa_2}\overrightarrow{N_2},\\ \overrightarrow{N_2}' &= -\widehat{\kappa_2}\overrightarrow{N_1},\end{aligned} \tag{1}$$

*where* $\widehat{\kappa_1} = \kappa_1 + \varepsilon\kappa_1^*$ *is nowhere pure dual curvature and* $\widehat{\kappa_2} = \kappa_2 + \varepsilon\kappa_2^*$ *is nowhere pure dual torsion of the curve* $\overrightarrow{\Gamma}$ *[14].*

**Definition 210** *Let* $\overrightarrow{\gamma} : I \subset \mathbb{R} \to \mathbb{E}^n$ *be a unit speed curve in* $\mathbb{E}^n$*. If the norm of the tangential component of* $\overrightarrow{\gamma}$*, i.e.* $\|\overrightarrow{\gamma}^T\|$*, is constant, then* $\overrightarrow{\gamma}$ *is a* $T$*-constant curve [5]. Further, if this norm is equal to zero, i.e.* $\|\overrightarrow{\gamma}^T\| = 0$*, then the curve is a* $T$*-constant curve of first kind, otherwise second kind [10].*

**Definition 211** *Let* $\overrightarrow{\gamma} : I \subset \mathbb{R} \to \mathbb{E}^n$ *be a unit speed curve in* $\mathbb{E}^n$*. If the norm of the normal component of* $\overrightarrow{\gamma}$*, i.e.* $\|\overrightarrow{\gamma}^N\|$*, is constant, then* $\overrightarrow{\gamma}$ *is a* $N$*-constant curve [5]. Further, if this norm is equal to zero, i.e.* $\|\overrightarrow{\gamma}^N\| = 0$*, then the curve is a* $N$*-constant curve of first kind, otherwise second kind [10].*

## 3 Curves on dual unit sphere

In this part, we classify the unit speed spherical curves with regards to their curvatures on dual unit sphere $\widetilde{S}^2$. Let $\overrightarrow{\Gamma}(s) = \overrightarrow{\gamma}(s) + \varepsilon\overrightarrow{\gamma}^*(s)$ be a unit speed spherical curve on dual unit sphere.

Since $\left\{\overrightarrow{T}, \overrightarrow{N}_1, \overrightarrow{N}_2\right\}$ is an orthonormal basis in the dual space $\mathbb{D}^3$, the position vector of the curve can be written as

$$\overrightarrow{\Gamma}(s) = \widehat{m_0}(s)\overrightarrow{T}(s) + \widehat{m_1}(s)\overrightarrow{N_1}(s) + \widehat{m_2}(s)\overrightarrow{N_2}(s), \tag{2}$$

where $\widehat{m_i}(s)$, $0 \leq i \leq 2$, are differentiable dual functions of the dual arclength $s$. Differentiating (2) with attention to (1), one can get

$$\begin{aligned}\overrightarrow{T}(s) &= \left(\widehat{m_0}'(s) - \widehat{\kappa_1}(s)\widehat{m_1}(s)\right)\overrightarrow{T}(s) \\ &+ \left(\widehat{m_1}'(s) + \widehat{\kappa_1}(s)\widehat{m_0}(s) - \widehat{\kappa_2}(s)\widehat{m_2}(s)\right)\overrightarrow{N_1}(s) \\ &+ \left(\widehat{m_2}'(s) + \widehat{\kappa_2}(s)\widehat{m_1}(s)\right)\overrightarrow{N_2}(s),\end{aligned}$$

which yields

$$\begin{aligned}\widehat{m_0}'(s) - \widehat{\kappa_1}(s)\widehat{m_1}(s) &= 1, \\ \widehat{m_1}'(s) + \widehat{\kappa_1}(s)\widehat{m_0}(s) - \widehat{\kappa_2}(s)\widehat{m_2}(s) &= 0, \\ \widehat{m_2}'(s) + \widehat{\kappa_2}(s)\widehat{m_1}(s) &= 0.\end{aligned} \tag{3}$$

From the fact that $\left\langle \overrightarrow{\Gamma}, \overrightarrow{\Gamma} \right\rangle = 1$, the equation

$$\widehat{m_0}^2(s) + \widehat{m_1}^2(s) + \widehat{m_2}^2(s) = 1$$

holds. Differentiating this equation with account of (3), we get

$$\widehat{m_0} = 0. \tag{4}$$

By (3) and in view of (2), the curve $\overrightarrow{\Gamma}$ is given in the form:

$$\overrightarrow{\Gamma}(s) = \widehat{m_1}(s)\overrightarrow{N_1}(s) + \widehat{m_2}(s)\overrightarrow{N_2}(s) \tag{5}$$

[1]. Thus, the equation (3) becomes

$$\begin{aligned}-\widehat{\kappa_1}(s)\widehat{m_1}(s) &= 1, \\ \widehat{m_1}'(s) - \widehat{\kappa_2}(s)\widehat{m_2}(s) &= 0, \\ \widehat{m_2}'(s) + \widehat{\kappa_2}(s)\widehat{m_1}(s) &= 0.\end{aligned} \tag{6}$$

**Theorem 31** *Let $\overrightarrow{\Gamma}(s) = \overrightarrow{\gamma}(s) + \varepsilon\overrightarrow{\gamma}^*(s)$ be a unit speed spherical curve on dual unit sphere $\widetilde{S}^2$. Then $\overrightarrow{\Gamma}$ is a $T$-constant curve of first kind.*

*Proof.* Let $\overrightarrow{\Gamma}(s) = \overrightarrow{\gamma}(s) + \varepsilon\overrightarrow{\gamma}^*(s)$ be a unit speed spherical curve on dual unit sphere $\widetilde{S}^2$. From the equation (4), $\widehat{m_0} = 0$. Thus, $\overrightarrow{\Gamma}$ is a *$T$-constant* curve of first kind.

**Corollary 32** *There is no $T$-constant curve of second kind on the dual unit sphere $\widetilde{S}^2$.*

**Theorem 33** *Let $\overrightarrow{\Gamma}(s) = \overrightarrow{\gamma}(s) + \varepsilon\overrightarrow{\gamma}^*(s)$ be a unit speed spherical curve on dual unit sphere $\widetilde{S}^2$. Then $\overrightarrow{\Gamma}$ is a $N$-constant curve of second kind.*

*Proof.* Let $\overrightarrow{\Gamma}(s) = \overrightarrow{\gamma}(s) + \varepsilon\overrightarrow{\gamma}^*(s)$ be a unit speed spherical curve on dual unit sphere $\widetilde{S}^2$. From the fact that $\left\langle \overrightarrow{\Gamma}, \overrightarrow{\Gamma} \right\rangle = 1$, we get

$$\widehat{m_1}^2 + \widehat{m_2}^2 = 1,$$

which means $\overrightarrow{\Gamma}$ is a *$N$-constant* curve of second kind.

**Corollary 34** *There is no N-constant curve of first kind on the dual unit sphere* $\widetilde{S}^2$.

**Theorem 35** *Let* $\overrightarrow{\Gamma}(s) = \overrightarrow{\gamma}(s) + \varepsilon \overrightarrow{\gamma}^*(s)$ *be a unit speed spherical curve on dual unit sphere* $\widetilde{S}^2$. *Then*

$$\left\{ \frac{\widehat{\kappa_1}'(s)}{\widehat{\kappa_1}^2(s)\,\widehat{\kappa_2}(s)} \right\}' - \frac{\widehat{\kappa_2}(s)}{\widehat{\kappa_1}(s)} = 0$$

*holds.*

*Proof.* Let $\overrightarrow{\Gamma}(s) = \overrightarrow{\gamma}(s) + \varepsilon \overrightarrow{\gamma}^*(s)$ be a unit speed spherical curve on dual unit sphere $\widetilde{S}^2$. From the equations (6), $\widehat{m}_1 = -\frac{1}{\widehat{\kappa_1}}$ and $\widehat{m}_2 = \frac{\widehat{\kappa_1}'}{\widehat{\kappa_1}^2\widehat{\kappa_2}}$. Writing these equations in (6), we get the result.

**Theorem 36** *There is no unit speed W-curve on dual unit sphere* $\widetilde{S}^2$.

*Proof.* Let $\overrightarrow{\Gamma}(s) = \overrightarrow{\gamma}(s) + \varepsilon \overrightarrow{\gamma}^*(s)$ be a unit speed spherical $W$-curve on dual unit sphere $\widetilde{S}^2$. From the equations (6), $\widehat{m}_1 = -\frac{1}{\widehat{\kappa_1}}$. Since $\overrightarrow{\Gamma}$ is a spherical curve, $\widehat{\kappa_2}$ cannot be zero. Hence, from the second equation of (6), $\widehat{m}_2 = 0$, a contradiction. Thus, there is no unit speed $W$-curve on dual unit sphere.

## Competing interests

The authors declare that they have no competing interests.

## Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

## References

[1] R. A. Abdel Baky, *An Explicit Characterization of Dual Spherical Curve*, Commun. Fac. Sci. Univ. Ank. Series A1, 51(2), 1-9, (2002).

[2] S. Büyükkütük, I. Kişi, V. N. Mishra and G. Öztürk, *Some Characterizations of Curves in Galilean 3-Space* $\mathbb{G}_3$, Facta Universitatis, Series: Mathematics and Informatics, 31(2), 503-512,(2016).

[3] S. Buyukkutuk, I. Kisi and G. Ozturk, *A Characterization of Curves According to Parallel Transport Frame in Euclidean n-Space* $\mathbb{E}^n$, New Trends in Mathematical Sciences, 5(2), 61-68, (2017).

[4] S. Büyükkütük and G. Öztürk, *Constant Ratio Curves According to Bishop Frame in Euclidean 3-Space* $\mathbb{E}^3$ , Gen. Math. Notes, 28(1), 81-91, (2015).

[5] B. Y. Chen, *Geometry of Warped Products as Riemannian Submanifolds and Related Problems*, Soochow J. Math., 28, 125-156, (2002).

[6] W. K. Clifford, *Preliminary Sketch of bi-Quaternions*, Proceedings of London Mathematical Society, 65, 361-395, (1873).

[7] A. Gray, *Modern Differential Geometry of Curves and Surface*, CRS Press, Inc., (1993).

[8] H. Gluck, *Higher Curvatures of Curves in Euclidean Space*, Amer. Math. Monthly, 73, 699-704, (1966).

[9] H. W. Guggenheimer, *Differential Geometry*, McGraw-Hill Comp., (1963).

[10] S. Gürpınar, K. Arslan and G. Öztürk, *A Characterization of Constant-Ratio Curves in Euclidean 3-space* $\mathbb{E}^3$ , Acta Universitatis Apulensis, 44, 39-51, (2015).

[11] M. Kazaz and H. H. Uğurlu, *A Special Motion on Dual Hyperbolic Unit Sphere* $\mathbb{H}_0^2$, Gen. Math. Notes, 33(2), 24-39, (2016).

[12] İ. Kişi and G. Öztürk, *Constant Ratio Curves According to Bishop Frame in Minkowski 3-space* $\mathbb{E}_1^3$, Facta Universitatis, Series: Mathematics and Informatics, 30(4), 527-538, (2015).

[13] F. Klein and S. Lie, *Uber Diejenigen Ebenenen Kurven Welche Durch Ein Geschlossenes System Von Einfach Unendlich Vielen Vartauschbaren Linearen Transformationen in Sich Ubergehen*, Math. Ann. 4, 50-84, (1871).

[14] Ö. Köse, Ş. Nizamoğlu and M. Sezer, *An Explicit Characterization of Dual Spherical Curves*, Doğa Mat. 12(3), 105-113, (1988).

[15] S. Özkaldi, K. İlarslan and Y. Yaylı, *On Mannheim Partner Curve in Dual Space*, An. St. Univ. Ovidius Constanta, 17(2), 131-142, (2009).

[16] G. Öztürk, İ. Kişi and S. Büyükkütük, *Constant Ratio Quaternionic Curves in Euclidean Spaces*, Advances in Applied Clifford Algebras, 27, 1659-1673, (2017).

[17] G. Öztürk, A. Küçük and K. Arslan, *Some Characteristic Properties of AW(k)-type Curves on Dual Unit Sphere*, Extracta mathematicae, 29,(1 - 2), 167-175, (2014).

[18] E. Study, *Geometrie der Dynamen*, Leibzig, (1903).

[19] I. M. Yaglom, *A Simple Non-Euclidean Geometry and Its Physical Basis*, Springer-Verlag, New-York, (1979).

[20] A. Yücesan, N. Ayyıldız and A. C. Çöken, *On Rectifying Dual Space Curves*, Revista Matematica Complutense, 20, 497-506, (2007).