New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2017.205

Multiplicative Newton's Methods with Cubic Convergence

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Received: 8 December 2016, Accepted: 2 April 2017
Published online: 17 October 2017.

Abstract: In this paper, we develop some modifications of the multiplicative Newton method which are third-order convergence. We use the multiplicative Newton Theorem and Newton Cotes quadrature formulas to present these new modifications of the multiplicative Newton method. Using the multiplicative Taylor expansion, we give also the convergence analysis of these new methods. Furthermore, we compare the multiplicative Newton methods with the classical Newton methods in details.

Keywords: Multiplicative derivative, multiplicative Newton method, cubic convergence, iterative method, nonlinear equation.

1 Introduction

Solving the nonlinear equation and system of equations is highly important in science and engineering. Besides, finding approximately a real root of a function is a substantial situation in numerical analysis. In this work, since we work in multiplicative analysis we use equation $f(x)+1=1$ to search a simple root equation $f(x)=0$. As known, the Newton method is well known and widely used method for approximating a real root of a function. The classical Newton method is given by

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}, n=0,1,\ldots.$$

The Newton method has quadratic convergence at the simple root. In the literature, there are some variants of Newton method obtained for faster convergence. For having the third-order convergence, some of these methods are used in the second-order derivative of the function. For instance, Halley method [1], Euler method [2], Householder method [3], super-Newton method [4] and super-Halley method [5] are used in the second-order derivative of the function to obtain cubic convergence. However, calculating the second-order derivative of complicated functions is difficult and loss of time. Therefore, without needing to calculate the second-order derivative of function, some variants of Newton method with cubic convergence are derived. These iterative methods are obtained using different approaches to definite integral in Newton theorem. For instance, arithmetic mean Newton method [6], harmonic mean Newton method [7] and geometric mean Newton method [8] by the help of trapezoidal rule, midpoint Newton method [7] and iterative method in [9] via midpoint rule, iterative methods in [10] and [11] using Simpson rule and different quadrature formulae are obtained. Besides, the method which based on usage of the Newton method for the inverse function has cubically convergence and is developed in [12]. For multiple roots cases, some of these mentioned methods are studied in [13, 14]. In addition to these, for multivariate cases some variants of Newton method are investigated in [15, 16]. Here, we give some basic definitions and properties of the multiplicative derivative theory which can be found in [17, 18].

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Definition 1. *Let $f:\mathbb{R}\to\mathbb{R}^{+}$ be a positive function. The multiplicative derivative of the function f is given by:*

$$\frac{d^{*}f}{dt}(t)=f^{*}(t)=\lim_{h\to 0}\left(\frac{f(t+h)}{f(t)}\right)^{\frac{1}{h}}.$$

Theorem 1. *(Multiplicative Taylor theorem) Let A be an open interval and let $f:A\to\mathbb{R}$ be $n+1$ times multiplicative differentiable on A. Then for any $x,x+h\in A$, there exists a number $\theta\in(0,1)$ such that*

$$f(x+h)=\left(\prod_{k=0}^{n}f^{*(k)}(x)^{\frac{h^k}{k!}}\right)f^{*(n+1)}(x+\theta h)^{\frac{h^{n+1}}{(n+1)!}}$$

The multiplicative Newton method is used to find a real root of equation $f(x)=1$. In literature [17], the multiplicative iterative formula is given as

$$x_{n+1}=x_n-\frac{ln f(x_n)}{ln f^{*}(x_n)},\quad n=0,1,\cdots.$$

Besides, the convergence of multiplicative Newton method is proved and perturbed root-finding methods via multiplicative calculi are presented in [18]. As noted in the above formula, it is seen that in the interval of convergence of the function must be defined positive.

Theorem 2. *(see [17]) Assume that $f\in C^2[a,b]$ and f is positive function for all $x\in(a,b)$. Assume that there exists a number $\alpha\in[a,b]$ such that $f(\alpha)=1$. If $f'(\alpha)?0$, then there exists a $\delta>0$, such that the following iteration*

$$x_{n+1}=x_n-\frac{ln f(x_n)}{ln f^{*}(x_n)}$$

will converge to α for any initial approximation $x_0\in[\alpha+\delta,\alpha-\delta]$.

In Section 2, we present some new approaches of multiplicative Newton method which are the three-order convergence. In Section 3, we give the convergence analysis of them. In Section 4, we present numerical examples which compare introduced method to classical multiplicative Newton method, Newton method and a modification Newton method corresponding to the introduced method.

Definition 2. *(see [20]) If the sequence $\{x_n\}$ tends to a limit α in such a way that*

$$\lim_{n\to\infty}\frac{x_{n+1}-\alpha}{(x_n-\alpha)^p}=C$$

for some $C?0$ and $p?1$, p is called as the order of convergence of the sequence and C is known as the asymptotic error constant.

Definition 3. *(see [6]) Let α be a root of the function $f(x)$ and suppose that x_{n+1}, x_n and x_{n-1} are three consecutive iterations closer to the root α. Then, the computational order of convergence (COC) ρ can be approximated using the formula*

$$\rho\approx\frac{ln|(x_{n+1}-\alpha)/(x_n-\alpha)|}{ln|(x_n-\alpha)/(x_{n-1}-\alpha)|}.$$

2 Numerical schemes

Let α be a simple root of nonlinear equation $f(x)=1$ (or $g(x)=f(x)-1=0$). From the multiplicative analysis [19], it is clear that the multiplicative Newton theorem is

$$f(x)=f(x_n)\int_{x_n}^{x} f^*(t)^{dt}=f(x_n)\exp\left(\int_{x_n}^{x}(\ln f(t))'dt\right). \tag{1}$$

In Equation (1), if the zeroth degree of Newton Cotes quadrature for definite integral is used, it can be written

$$\int_{x_n}^{x} f^*(t)^{dt}=\exp\left(\int_{x_n}^{x}(\ln f(t))'dt\right)\approx\exp\left((x-x_n)(\ln f(x_n))'\right)=(f^*(x_n))^{x-x_n}.$$

Since $f(x)=1$, the Explicit Multiplicative Newton (MN) Method is obtained as

$$x_{n+1}=x_n-\frac{\ln f(x_n)}{\ln f^*(x_n)}. \tag{2}$$

In equation (1), According to the first degree of Newton Cotes quadrature for definite integral, it follows as

$$\int_{x_n}^{x} f^*(t)^{dt}=\exp\left(\int_{x_n}^{x}(\ln f(t))'dt\right)\approx\exp\left(\frac{1}{2}(x-x_n)\left((\ln f(x))'+(\ln f(x_n))'\right)\right)=[f^*(x)f^*(x_n]^{\frac{1}{2}(x-x_n)}$$

that gives Arithmetic Multiplicative Newton (AMN) Method as

$$x_{n+1}=x_n-\frac{2\ln f(x_n)}{\ln f^*(x_n)+\ln f^*(z_{n+1})} \tag{3}$$

where

$$z_{n+1}=x_n-\frac{\ln f(x_n)}{\ln f^*(x_n)}.$$

Now, as in [7], if we consider this approximation as the arithmetic mean of $\ln f^*(x_n)$ and $\ln f^*(z_{n+1})$ and take the harmonic mean instead, we get the harmonic mean Multiplicative Newton (HMN) Method as

$$x_{n+1}=x_n-\frac{\ln f(x_n)\left(\ln f^*(x_n)+\ln f^*(z_{n+1})\right)}{2\ln f^*(x_n)\ln f^*(z_{n+1})} \tag{4}$$

where

$$z_{n+1}=x_n-\frac{\ln f(x_n)}{\ln f^*(x_n)}.$$

If the multiplicative midpoint approximation is used, it follows from

$$1\approx f(x_n)\left(f^*\left(\frac{x_n+\alpha}{2}\right)\right)^{\alpha-x_n}$$

that gives midpoint multiplicative Newton (MpMN) Method as

$$x_{n+1}=x_n-\frac{\ln f(x_n)}{\ln f^*(\frac{x_n+z_{n+1}}{2})} \tag{5}$$

Finally, we can write the multiplicative Newton theorem in (1) as

$$f(x) = f(y_n) \int_{y_n}^{x} f^*(t)^{dt} \tag{6}$$

such that $y_n = x_n + \frac{ln f(x_n)}{ln f^*(x_n)}$. If we use the multiplicative midpoint rule for the multiplicative integral in the right-hand side of Equation (6)

$$\int_{y_n}^{x} f^*(t)^{dt} \approx f^*(\frac{x+y_n}{2})^{x-y_n}$$

and since $f(\alpha) = 1$, we obtain a modification of the multiplicative Newton (mMN) method as

$$x_{n+1} = y_n - \frac{ln f(y_n)}{ln f^*(\frac{y_n+z_{n+1}}{2})}$$

where $z_{n+1} = x_n - \frac{ln f(x_n)}{ln f^*(x_n)}$. Hence, the iterative formula of modification the multiplicative Newton method is written as

$$x_{n+1} = x_n - \frac{ln f\left(x_n + \frac{ln f(x_n)}{ln f^*(x_n)}\right) - ln f(x_n)}{ln f^*(x_n)}. \tag{7}$$

3 Convergence analysis

Let α be a simple root of equation $f(x) = 1$ and $x_n = \alpha + e_n$. Here, we only prove that the multiplicative Newton method given in (3) cubically converges by following the process steps in [6]. Similarly, for the modified multiplicative Newton methods defined in (4) and (5), it can be proved by using the process steps in [7]. Furthermore, the convergence as cubic of the modification multiplicative Newton method presented in (7) can be given as in [9].
Using the multiplicative Taylor expansions, it can be written

$$f(x_n) = f(\alpha + e_n) = f(\alpha)\left(f^*(\alpha)\right)^{e_n}\left(f^{(2*)}(\alpha)\right)^{\frac{e_n^2}{2!}}\left(f^{(3*)}(\alpha)\right)^{\frac{e_n^3}{3!}} O^*(e_n^4).$$

Now, taking the natural logarithm in both sides, we have

$$\begin{aligned} ln f(x_n) &= ln f(\alpha) + ln f^*(\alpha)e_n + ln f^{(2*)}(\alpha)\frac{e_n^2}{2!} + ln f^{(3*)}(\alpha)\frac{e_n^3}{3!} + O(e_n^4) \\ &= ln f^*(\alpha)\left(e_n + \frac{1}{2!}\frac{ln f^{(2*)}(\alpha)}{ln f^*(\alpha)}e_n^2 + \frac{1}{3!}\frac{ln f^{(3*)}(\alpha)}{ln f^*(\alpha)}e_n^3 + O(e_n^4)\right) \\ &= ln f^*(\alpha)\left(e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4)\right) \end{aligned} \tag{8}$$

where $C_j = \frac{1}{j!}\frac{ln f^{(j*)}(\alpha)}{ln f^*(\alpha)}$. On the other hand, we get

$$\begin{aligned} ln f^*(x_n) &= ln f^*(\alpha) + ln f^{(2*)}(\alpha)e_n + ln f^{(3*)}(\alpha)\frac{e_n^2}{2!} + O(e_n^3) \\ &= ln f^*(\alpha)\left(1 + \frac{ln f^{(2*)}(\alpha)}{ln f^*(\alpha)}e_n + \frac{1}{2!}\frac{ln f^{(3*)}(\alpha)}{ln f^*(\alpha)}e_n^2 + O(e_n^3)\right) \\ &= ln f^*(\alpha)\left(1 + 2C_2 e_n + 3C_3 e_n^2 + O(e_n^3)\right). \end{aligned} \tag{9}$$

If Equation (8) are divided by Equation (9), the following equation

$$\frac{ln f(x_n)}{ln f^*(x_n)} = \frac{\left(e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4)\right)}{\left(1 + 2C_2 e_n + 3C_3 e_n^2 + O(e_n^3)\right)} = e_n - C_2 e_n^2 + (2C_2^2 - 2C_3)e_n^3) + O(e_n^4)$$

is obtained. So, we have

$$z_{n+1} = \alpha + C_2 e_n^2 + 2(C_3 - C_2^2)e_n^3 + O(e_n^4).$$

Again expanding $f^*(z_{n+1})$ about α and using the multiplicative Taylor expansion we obtain

$$\begin{aligned}
f^*(z_{n+1}) &= f^*(\alpha)\left(f^{(2*)}(\alpha)\right)^{(z_{n+1}-\alpha)}\left(f^{(3*)}(\alpha)\right)^{\frac{((z_{n+1}-\alpha)^2}{2!}}\cdots \\
ln f^*(z_{n+1}) &= ln f^*(\alpha) + ln f^{(2*)}(\alpha)(z_{n+1}-\alpha) + ln f^{(3*)}(\alpha)\frac{(z_{n+1}-\alpha)^2}{2!} + \cdots \\
&= ln f^*(\alpha) + \left(C_2 e_n^2 + 2(C_3 - C_2^2)e_n^3 + O(e_n^4)\right) ln f^{(2*)}(\alpha) + O(e_n^4) \\
&= ln f^*(\alpha)\left(1 + 2C_2 e_n^2 + 4\left(C_2 C_3 - C_2^3\right)e_n^3 + O(e_n^4)\right).
\end{aligned} \tag{10}$$

Adding (9) and (10), we have

$$ln f^*(x_n) + ln f^*(z_{n+1}) = 2ln f^*(\alpha)\left(1 + C_2 e_n + \left(C_2^2 + \frac{3}{2}C_3\right)e_n^2 + O(e_n^3)\right). \tag{11}$$

From (8) and (11), we get

$$\frac{2ln f(x_n)}{ln f^*(x_n) + ln f^*(z_{n+1})} = \frac{\left(e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4)\right)}{\left(1 + C_2 e_n + \left(C_2^2 + \frac{3}{2}C_3\right)e_n^2 + O(e_n^3)\right)} = e_n - \left(C_2^2 + \frac{1}{2}C_3\right)e_n^3 + O(e_n^4),$$

as required. This implies that the method defined by the formula (3) converges as cubic.
It is clear that the number of function evaluation in per iteration for methods in (3), (4), (5) and (7) is three. According to the definition of efficiency index [21] which is $\sqrt[p]{NFE}$ (where p is order of convergence of method and NFE is number of function evaluation), the efficiency index of methods defined in this work is $\sqrt[3]{3} \cong 1.442$. Hence, the efficiency index of these new methods is better than the efficiency index of Newton method and multiplicative Newton method which are $\sqrt{2} \cong 1.414$ and are same as the ones of methods defined in [6], [7] and [9].

4 Numerical examples

In this section, we use Newton method , arithmetic Newton (AN) method defined in [6], harmonic Newton (HN) method and midpoint Newton (MpN) method presented in [7], a modification Newton (mN) method given in [9], multiplicative Newton (MN) method and the new methods obtained in this work to solve some non-linear equation $f(x) = 0$. Summing the number of evaluations of function f with the number of evaluations of its derivative, the number of function evaluation (NFE) in per iteration is found. The results obtained via these methods are showed in Table **??**. The used stopping criterion is $|x_{n+1} - \alpha| + |f(x_{n+1})| < 10^{-14}$. We accept an approximate root to 15 decimal places rather than the exact root in test functions.
The results show that modification multiplicative methods converging cubically can compete with Newton method, MN, AN, HN, MpN methods. Besides, mMN method is better than other methods in some case where mN only converges, such as example (b). Further, in certain problems such as example (c), (d), (e), (f) and (g), modified multiplicative

Newton methods give better results compared with modified Newton methods. Also if we look closely to example (c), the multiplicative Newton method and its modification that is developed in this work reach to the root in one step. This is not actually an amazing result. This case originates from multiplicative calculus.

Test functions:

(a) $f(x) = lnx - sinx$ and multiplicative version: $f^*(x) = lnx - sinx + 1$
(b) $f(x) = arctanx$ and multiplicative version: $f^*(x) = arctanx + 1$
(c) $f(x) = e^{1-x} - 1$ and multiplicative version: $f^*(x) = e^{1-x}$
(d) $f(x) = e^x - sin^2x$ and multiplicative version: $f^*(x) = e^x - sin^2x + 1$
(e) $f(x) = (x+2)e^x - 1$ and multiplicative version: $f^*(x) = (x+2)e^x$
(f) $f(x) = (x-1)^6 - 1$ and multiplicative version: $f^*(x) = (x-1)^6$
(g) $f(x) = e^{x^3+7x-30} - 1$ and multiplicative version: $f^*(x) = e^{x^3+7x-30}$

(a), $x_0 = 3$	n	NFE	COC	x_n
N	5	10	1.99	2.219107148913746
AN	3	9	2.86	2.219107148913746
HN	3	9	3.07	2.219107148913746
MpN	3	9	2.75	2.219107148913746
mN	4	12	2.96	2.219107148913746
MN	5	10	1.99	2.219107148913746
AMN	3	9	3.17	2.219107148913746
HMN	3	9	3.16	2.219107148913746
MpMN	3	9	3.13	2.219107148913746
mMN	4	12	3.01	2.219107148913746

(b), $x_0 = 3$	n	NFE	COC	x_n
N	-	-	-	divergence
AN	-	-	-	divergence
HN	-	-	-	divergence
MpN	-	-	-	divergence
mN	5	15	2.99	0.000000000000000
MN	-	-	-	divergence
AMN	-	-	-	divergence
HMN	-	-	-	divergence
MpMN	-	-	-	divergence
mMN	4	12	3.00	0.000000000000000

(c), $x_0 = 10$	n	NFE	COC	x_n
N	-	-	-	divergence
AN	NC	-	-	-
HN	-	-	-	divergence
MpN	NC	-	-	-
mN	12	36	3.00	1.000000000000000
MN	1	2	ND	1
AMN	1	3	ND	1
HMN	1	3	ND	1
MpMN	1	3	ND	1

mMN	1	3	ND	1

(d), $x_0 = 3$	n	NFE	COC	x_n
N	9	18	2.00	-0.755265684142554
AN	7	21	3.00	-0.755265684142554
HN	5	15	3.01	-0.755265684142554
MpN	5	15	3.06	-0.755265684142554
mN	6	18	2.98	-0.755265684142554
MN	7	14	1.99	-0.755265684142554
AMN	4	12	2.98	-0.755265684142554
HMN	4	12	2.95	-0.755265684142554
MpMN	5	15	3.00	-0.755265684142554
mMN	4	12	2.99	-0.755265684142551

(e), $x_0 = 3.5$	n	NFE	COC	x_n
N	10	20	1.91	-0.442854401002389
AN	7	21	2.98	-0.442854401002389
HN	6	18	3.01	-0.442854401002389
MpN	6	18	2.91	-0.442854401002389
mN	6	18	2.94	-0.442854401002389
MN	5	10	2.00	-0.442854401002389
AMN	4	12	2.92	-0.442854401002389
HMN	4	12	2.96	-0.442854401002389
MpMN	3	9	2.73	-0.442854401002389
mMN	4	12	3.10	-0.442854401002389

(f), $x_0 = 1.5$	n	NFE	COC	x_n
N	15	30	1.99	2.000000000000000
AN	467	1401	3.07	2.000000000000000
HN	7	21	3.01	2.000000000000000
MpN	58	174	3.11	2.000000000000000
mN	-	-	-	-
MN	5	10	2.00	2.000000000000000
AMN	4	12	3.00	2.000000000000000
HMN	3	9	3.16	2.000000000000000
MpMN	3	9	3.16	2.000000000000000
mMN	4	12	2.97	2.000000000000000

(g), $x_0 = 3.5$	n	NFE	COC	x_n
N	43	86	1.99	2.374101653848877
AN	29	87	2.93	2.374101653848877
HN	23	69	3.02	2.374101653848877
MpN	26	78	2.96	2.374101653848877
mN	25	75	2.98	2.374101653848877
MN	5	10	1.99	2.374101653848877
AMN	4	12	2.99	2.374101653848877

HMN	3	9	3.01	2.374101653848877
MpMN	4	12	2.96	2.374101653848877
mMN	4	12	2.99	2.374101653848877

Table 1: Comparison of methods

where,

N: Newton method
n: Number of iterations
AN: Arithmetic Newton method
NFE: Number of functional evaluations
HN: Harmonic Newton method
COC: Computational order of convergence
MpN: Midpoint Newton method
ND: Not defined
mN: Modification Newton method in [9]
NC: Not convergence
MN: Multiplicative Newton method
x_n- the approximation root in n. iteration
MpMN: Midpoint multiplicative Newton method
AMN: Arithmetic multiplicative Newton method
mMN: Method which presented in this work
HMN: Harmonic multiplicative Newton method

5 Conclusion

In this study, we present some modifications of the multiplicative Newton method using multiplicative Newton theorem and Newton Cotes quadrature formulas. Also, we prove the cubical converge of modified multiplicative Newton methods obtained in this text. Like methods in [6], [7] and [9], computing second or higher derivatives of function does not require in modification multiplicative Newton methods, either. The efficiency indexes of the presented methods are better than the classical Newton and the multiplicative Newton methods. This case implies that modified multiplicative Newton methods are preferable to classical Newton and the multiplicative Newton methods. Especially example (c) shows that the natural of the fundamental calculus plays an essential role in solving to equation $f(x)=0$ ($f(x)+1=1$ for multiplicative calculus). The corresponding example for the classical Newton method is the function $f(x)=1-x$. So indeed, the root of this function is obtained in exactly one step by the classical Newton method and its modifications. Moreover, methods in this work give results better than the methods presented in [6], [7] and [9] in certain problems such as examples (c), (d), (e), (f) and (g).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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