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A numerical study of a fluid flow by transforming a non-uniform wavy channel

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Abstract: An incompressible viscous flow in a non-uniform wavy channel with slowly varying cross section with absorbing walls is numerically studied. The governing equations are solved using finite difference method by transforming the irregular boundary of the region of the problem to rectangular region. The effect of different parameters on the transverse velocity and the mean pressure drop is analyzed. The results obtained using this method is in a good argument with results obtained by other numerical methods without transforming the wavy channel to rectangular channel. The results are presented by using tables and graphs.

Keywords: Fluid flow, non-uniform channel, finite difference method.

1 Introduction

Kidneys are important organs with many functions in the body, including producing hormones, absorbing minerals, and filtering blood and producing urine. The actual filtering occurs in tiny units inside the kidneys called nephrons. Every kidney has a million nephrons. In the nephron, tiny blood vessels called capillaries intertwine with tiny urine carrying tubes called tubules. A complicated chemical exchange takes place as waste materials and water leave the blood and enter in to unitary system.

Researchers have been realized that to understand the mechanism of kidney it is sufficient to study the function of nephron. Many researchers are devoted on the study of the function of nephron using mathematical models.

Ludwig, 1861 developed a theory of urine formation which consisted of filtration through the walls of glomerular capillaries and reabsorption, which takes place in the renal tubules. According to Babsky et al., 1970, this theory has been confirmed by many experiments. The hydrodynamical problem in the renal tubule has been studied by several researchers such as Wesson, 1954, Kelman, 1962, Macey, 1963, Macey, 1965 considering different models for reabsorption in the tubules. Wesson, 1954 discussed renal model theoretically assuming a constant rate of reabsorption. Macey, 1963 was the first who studied the flow of an incompressible viscous fluid through a renal tubule using mathematical model. He obtained exact solution using a circular tube with linear rate of reabsorption. Kelman, 1962 proposed the bulk flow in the proximal tubule decays exponentially with the axial distance. Later, Macey, 1965 used the condition of Kelman, 1962 and solved the transport equations to find velocity components and pressure drop. Radhakrishnamacharya et al., 1981 considered a non-uniform geometry to model renal tubule. They made an attempt to understand the flow through the renal tubule by studding the hydrodynamical aspect of an incompressible viscous fluid in a circular tube of varying cross section with reabsorption at the wall. With similar approach Chandra and Prasad, 1992 analyzed fluid flow in rigid tube of slowly varying cross section on different geometries. Chaturani and Ranganatha,

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1991 studied fluid flow through a diverging /converging tube with variable wall permeability. They obtained approximate analytical solution for the case that flux at the wall depends on wall permeability and transboundary pressure drop. Muthu and Tesfahun, 2011 have studied a mathematical model of fluid flow in a non- uniform rigid wavy channel of varying cross section and presented the effects of slope parameter, reabsorption coefficient on the transverse velocity and mean pressure drop. The governing equations are solved numerically by using the finite difference technique related to the method of Takabatake Ayukawe, 1982. Sinha and Getachew, 2009, Getachew and Sinha, 2011, Getachew and Sinha, 2012, numerically analyzed the combined effect of thermal and surface roughness on the performance of a slider bearing using finite difference method by transforming the irregular domain of the bearing to rectangular domain.

As per the knowledge of the authors there is no a numerical study of a fluid flow in a non-uniform channel by transforming the irregular region to regular region of the problem. In this paper, the governing equations of an incompressible viscous fluid through a wavy non-uniform permeable channel are solved numerically by transforming the wavy non –uniform channel to a rectangular channel. The transformation made the numerical computation simple, and the finite difference method to give a better approximation.

Nomenclature A amplitude
d half width of the channel at the inlet
k slope parameter
R_e Reynolds number Q_o Flux across the cross section at $x = 0$
P pressure
u fluid velocity in the direction of x-axis
v fluid velocity in the direction of y -axis
δ wave number
ε amplitude ratio
ψ stream function
ψ vorticity function
ν kinmatic viscosity
ρ density of the lubricant

2 Governing equations

The schematic diagram of a channel with slowly varying cross section is shown in Figure 1.

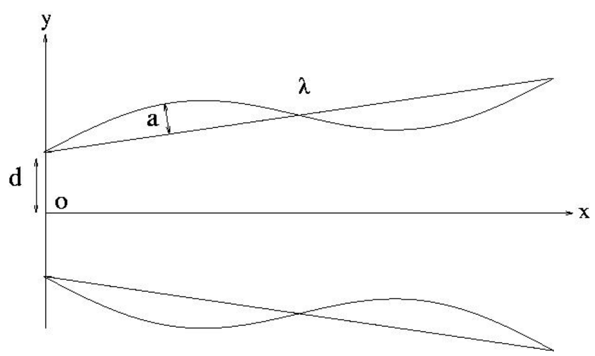

Fig. 1: Geometry of the problem

The boundary of the channel walls are assumed to be symmetric about x axis described by the function

$$\eta(x) = d + k_1 x + a \sin\left(\frac{2\pi x}{\lambda}\right) \tag{1}$$

where d is the half width of the channel at the inlet, x = 0. k_1 is a constant whose magnitude depends on the length of the channel exit and inlet dimensions, a is the amplitude and is the wave length. To model the slowly varying slope is assumed very small positive number $(k << 1)$.

An incompressible viscous fluid flow through a channel with slowly varying cross section is considered. The motion of the fluid is assumed to be laminar, steady and symmetric. The channel is assumed to be long enough so that the initial and the end effects are neglected.

In view of the above assumptions and the usual lubrication approximations, the Navier Stoke's equation is reduced to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{3}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{4}$$

where u and v are the velocity of the fluid along the x axis and the y axis respectively, p is the pressure, ρdensity of the fluid and $\nu = \frac{\mu}{\rho}$ is kinematic viscosity.

According to [11] the following boundary conditions are considered.

(i) The tangential velocity at the wall is zero.

$$u + \frac{\partial \eta}{\partial x} v = 0 \quad \text{at} \quad y = \eta(x) \tag{5}$$

(ii) The regularity condition requires

$$v = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = 0 \tag{6}$$

(iii) The reabsorption has been accounted for by considering the bulk flow as a decreasing function of x. That is, the flux across a cross-section is given as

$$Q(x) = \int_0^{\eta(x)} u(x,y)dy = Q_o F(\alpha x) \tag{7}$$

where when $\alpha = 0$ and decreases with x, $\alpha \geq 0$ is the reabsorption coefficient and is a constant, and $Q_o$is the flux across the cross section $x = 0$.

Introducing stream function ψ and the vorticity ω by $u = \frac{\partial \psi}{\partial y}$

$$u = \frac{\partial \psi}{\partial y}, v = \frac{\partial \psi}{\partial x} \quad \text{and} \quad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{8}$$

and the following non dimensional quantities

$$x' = \frac{x}{\lambda}, y' = \frac{y}{d}, \eta' = \frac{\eta}{d}, \psi' = \frac{\psi}{Q_o}, p' = \frac{d^2}{\mu Q_o} p, \omega' = \frac{d^2}{Q_o}\omega$$

the above governing equations in dimensionless form are reduced (after dropping the primes) to the following.

$$\delta^2 \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega \tag{9}$$

$$\delta^2 \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} = \delta R_e \left(\frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} \right) \tag{10}$$

and the boundary conditions are reduced to the following form.

$$\frac{\partial \psi}{\partial y} = \delta(k_1 + A\cos 2\pi x)\frac{\partial \psi}{\partial x} \quad \text{at} \quad y = \eta(x) = 1 + kx + \varepsilon \sin 2\pi x \tag{11}$$

$$\psi = 0 \quad \text{and} \quad \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \text{at} \quad y = 0 \tag{12}$$

and

$$\psi = F(\alpha x) \quad \text{at} \quad y = \eta(x) = 1 + kx + \varepsilon \sin 2\pi x \tag{13}$$

$$u = \frac{Q_o}{d}\frac{\partial \psi}{\partial y}, v = -\frac{Q_o}{\lambda}\frac{\partial \psi}{\partial x} \tag{14}$$

where

$$\delta = \frac{d}{\lambda}, A = \frac{2\pi a}{\lambda}, \varepsilon = \frac{a}{d}, k = \frac{k_1 \lambda}{d}.$$

The parameter R_e is the Reynolds number and δ is the wave number (the ratio of inlet width to the wavelength). ε is the ratio of the amplitude to the inlet width and k is slope parameter. In this paper, exponentially decaying bulk flow is considered similar to [11]. That is, in equation 13) F is taken to be

$$F(\alpha x) = e^{-\alpha x}. \tag{15}$$

Using equation (3) and equation (8) the non-dimensional pressure $p(x,y)$ is given by

$$p(x,y) = \delta \frac{\partial u}{\partial x} + \frac{1}{\delta}\int \frac{\partial^2 u}{\partial y^2 dx} - R_e \int u \frac{\partial u}{\partial x} dx + \int v \frac{\partial u}{\partial y} dx \tag{16}$$

and the mean pressure is obtained by $\bar{p}(x) = \int p(x,y)dy$.

Further, the mean pressure drop between $x = 0$ and $x = x_o$ is obtained by

$$\Delta p(x_o) = \bar{p}(0) - \bar{p}(x_o)$$

3 Formulation of the problem

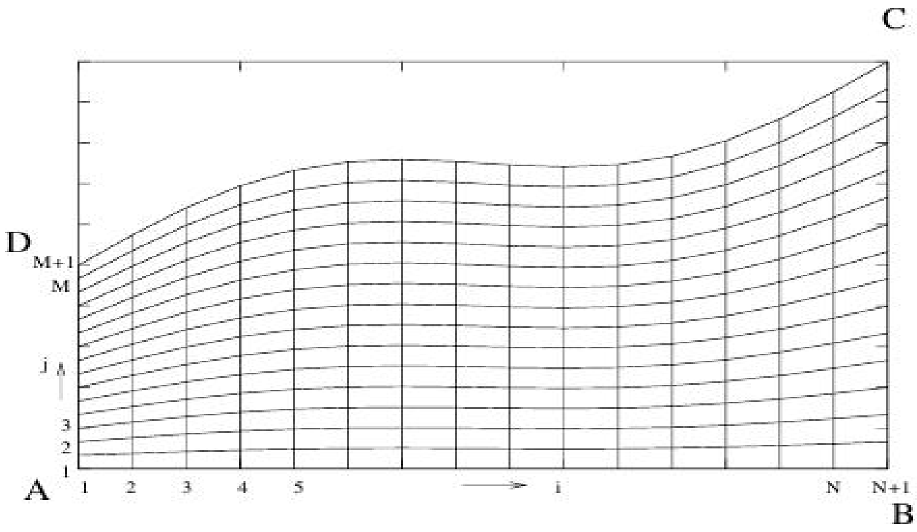

Fig. 2: Region of the problem.

Muthu and Tesfahun, 2011 solved the governing equations (9) and (10) together with boundary conditions (11)-(13) in the region of Fig. 2 which is nonrectangular.

The following boundary conditions are used for the numerical computation similar to [11],

$$\psi = 0, \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \text{on} \quad AB \tag{17}$$

$$\psi = f(y), \frac{\partial \psi}{\partial x} = 0 \quad \text{on} \quad AD \tag{18}$$

$$\frac{\partial \psi}{\partial y} = \delta\left(k_1 + A\cos 2\pi x\right)\frac{\partial \psi}{\partial x} \quad \text{on} \quad CD \tag{19}$$

$$\psi = g(y), \frac{\partial \psi}{\partial x} = 0 \quad \text{on} \quad BC, \tag{20}$$

Where $f(y)$ and $g(y)$ are prescribed functions such that u is parabolic at AD and BC. These functions are also assumed to satisfy the boundary conditions so that the solution is free from discontinuities.

However, it is well known that finite difference approximation method is easy and effective if the region of the problem is rectangular. In this paper, the region is transformed to rectangular region so that the numerical computation becomes simple and effective.

The following transformation is used to transform the region (Figure 2) to a rectangular region (Figure 3).

$$y = y^*\eta(x),\ 0 \le y^* \le 1$$
$$x = x*$$

The new coordinate system consists of two spatial variables x^* and y^*. Under these transformations, the lower boundary of the region is mapped onto y*=0 and the upper boundary of the region $y = \eta(x)$is transformed onto $y* = 1$

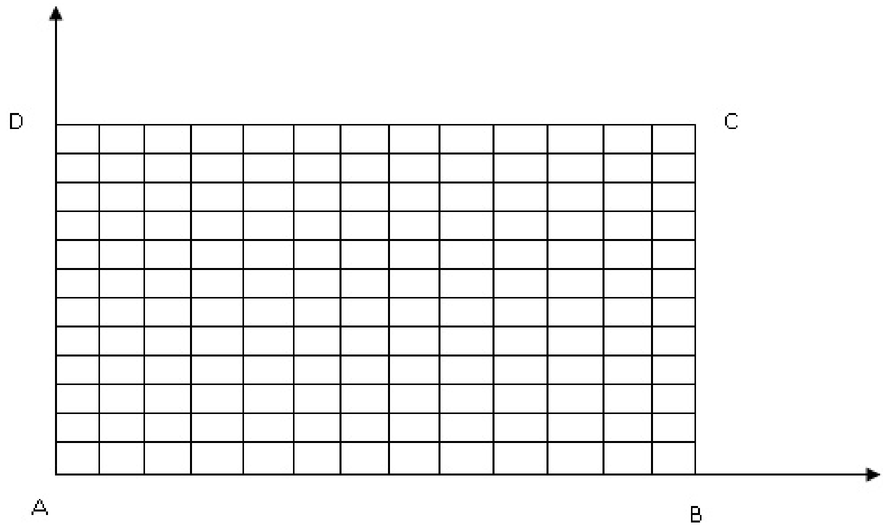

Fig. 3: The region of the problem after the transformation.

Using these transformations equations (9-10) have the following form:

$$\delta^2 \left(\frac{\partial^2 \psi}{\partial x*^2} + \left(\frac{y*\eta'(x)}{\eta(x)} \right)^2 \frac{\partial^2 \psi}{\partial y*^2} + \left(\frac{2y*(\eta'(x))^2}{(\eta(x))^2} - \frac{y*\eta(x)\eta''(x)}{(\eta(x))^2} \right) \frac{\partial \psi}{\partial y*} - \frac{2y*\eta'(x)}{\eta(x)} \frac{\partial^2 \psi}{\partial y*\partial x} \right) + \frac{1}{\eta(x)} \frac{\partial^2 \psi}{\partial y*^2} = -\omega \tag{21}$$

$$\begin{gathered} \delta^2 \left(\frac{\partial^2 \omega}{\partial x^2} + \left(\frac{y*\eta'(x)}{\eta(x)} \right)^2 \frac{\partial^2 \omega}{\partial y*^2} + \left(\frac{2y*(\eta'(x))^2}{(\eta(x))^2} - \frac{y*\eta(x)\eta''(x)}{(\eta(x))^2} \right) \frac{\partial \omega}{\partial y*} - \frac{2y*\eta'(x)}{\eta(x)} \frac{\partial^2 \omega}{\partial y*\partial x} \right) + \\ \frac{1}{\eta(x)} \frac{\partial^2 \omega}{\partial y*^2} = \delta \mathrm{R_e} \left(\frac{1}{\eta(x)} \frac{\partial \psi}{\partial y*} \left(\frac{\partial \omega}{\partial x} - \frac{y*\eta'(x)}{\eta(x)} \frac{\partial \omega}{\partial y*} \right) - \left(\frac{\partial \psi}{\partial x} - \frac{y*\eta'(x)}{\eta(x)} \frac{\partial \psi}{\partial y*} \right) \frac{1}{\eta(x)} \frac{\partial \omega}{\partial y*} \right) \end{gathered} \tag{22}$$

The corresponding boundary conditions with respect to the new coordinate system have the following form

$$\psi = 0, \ \frac{\partial^2 \psi}{\partial y*^2} = 0 \quad \text{on} \quad y* = 0 \tag{23}$$

$$\frac{\partial \psi}{\partial x} - \frac{y*\eta'(x)}{\eta(x)} \frac{\partial \psi}{\partial y*} = 0 \quad \text{on} \quad x = 0 \tag{24}$$

$$\psi = Q_o e^{-\alpha x}, \ \frac{1}{\eta(x)} \frac{\partial \psi}{\partial y*} = \delta (k_1 + A\cos 2\pi x) \left(\frac{\partial \psi}{\partial x} - \frac{y*\eta'(x)}{\eta(x)} \frac{\partial \psi}{\partial y*} \right) \quad \text{on} \quad y* = 1 \tag{25}$$

$$\psi = g(y*\eta(x)), \quad \text{on} \quad BC \tag{26}$$

with respect to the new coordinate system the u and $v-$velocities are given by

$$u = \frac{Q_o}{d} \frac{1}{\eta(x)} \frac{\partial \psi}{\partial y*}, \quad v = -\frac{Q_0}{\lambda} \frac{1}{\eta(x)} \left(\frac{\partial \psi}{\partial x} - \frac{y^*\eta'(x)}{\eta(x)} \frac{\partial \psi}{\partial y^*} \right) \tag{27}$$

and the pressure is given as

$$p(x,y) = \delta\left(\frac{\partial u}{\partial x} - \frac{y*\eta'(x)}{\eta(x)}\frac{\partial u}{\partial y*}\right) + \frac{1}{\delta}\int \frac{1}{(\eta(x))^2}\frac{\partial^2 u}{\partial y*^2}dx - R_e\int u\left(\frac{\partial u}{\partial x} - \frac{y*\eta'(x)}{\eta(x)}\frac{\partial u}{\partial y*}\right)dx + \int v\frac{1}{\eta(x)}\frac{\partial u}{\partial y*}dx \quad (28)$$

4 Treatment of the solution

The system of equations was discretized and solved simultaneously using finite difference representations. All the derivatives are represented by central differences and a direct iterative approach is used to obtain the distributions of the variables.

The results have been obtained to an accuracy of $Tol = 10^{-6}$, where $Max\left(\frac{\varsigma_j^{new}-\varsigma_j^{old}}{\varsigma_j^{new}}\right) \leq Tol, \varsigma_j$, are the variables. The iteration is carried out for $Tol = 10^{-5}$, $Tol = 10^{-6}$, $Tol = 10^{-7}$.

Algorithm

(Step 1) Initialization
- (a) Input data:
- (b) Set boundary conditions
- (c) Set fictitious values for to the remaining grid points

(Step 2) Evaluate ψ^{new}using ψ^{old}, ω^{old}
(Step 3) Evaluate ω^{new}usingψ^{new}, ω^{old}
(Step 4) Test for convergence.
(Step 5) Repeat steps 2-3 till convergence is obtained on all field variables.
(Step 6) Evaluate u and$v$using
(Step 7) Evaluate $p$using u,v and equation (28)
(Step 8) Evaluate the pressure drop

5 Results and discussion

In the present study the non-uniform channel is transformed in to uniform channel which added complexity to the governing equation. The resulting $\psi - \omega$ form of the governing equation is solved by using finite difference method. Therefore, for computation purpose we consider the following parameters constant in all the results.

As in the case of [11], the wave number $\delta$is taken as 0.1 in view that it is a small perturbation parameter. The Reynolds number Re is taken as 1 to indicate that the flow considers a low Reynolds number as the flow is laminar.

Velocity Profile (u & v). It can be observed from figure 4 that the reabsorption coefficient has a significant effect on the transversal velocity (v). That is, a rise in the reabsorption coefficient alpha increases the transverse velocity. This agrees with the natural phenomenon that due to rise of alpha the pressure drops which in turn results an increase in the transverse velocity. Figure 5 shows the behavior of the transverse velocity for different values of x along the channel. It can be seen that the velocity slows down as the fluid pass all the way from the entrance to the exit of the channel.

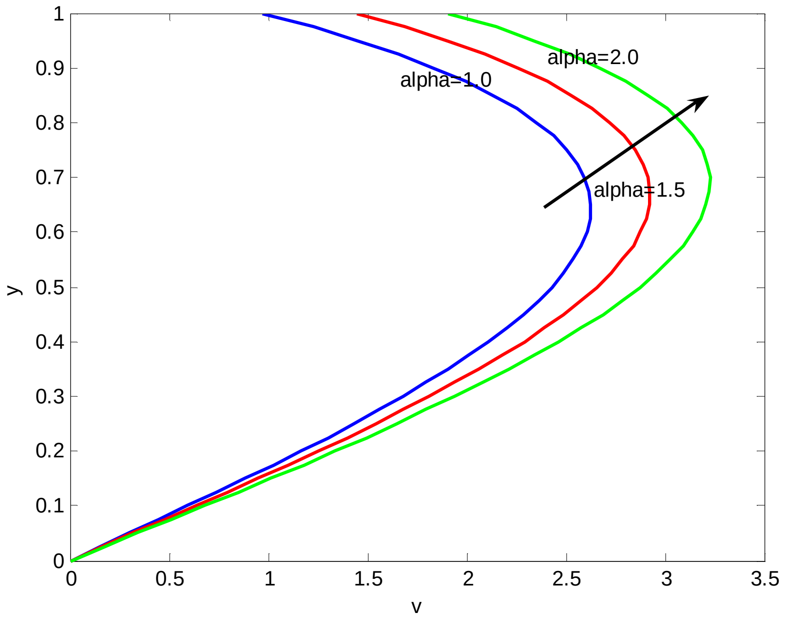

Fig. 4: Transverse velocity (v) with y for alpha variation at the entrance of the channel.

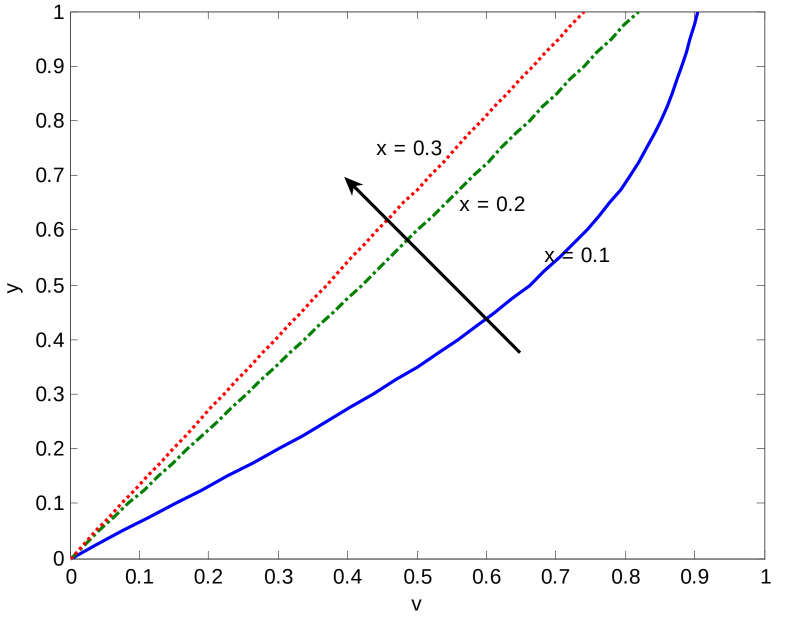

Fig. 5: Transverse velocity (v) with y along various points of x through the channel.

The effect of the reabsorption coefficient alpha on the longitudinal velocity (u) is shown in figure 6. The velocity is more for low values of alpha than higher values of alpha. The reabsorption coefficient alpha has a reverse effect on the transverse velocity (v) and longitudinal velocity (u). That is an increment in alpha increases v whereas decreases u. On the other hand, Figure 7 shows the behavior of the longitudinal velocity for different values of x along the channel. It can be seen that the velocity significantly decreases as the fluid pass all the way from the entrance to the exit of the channel.

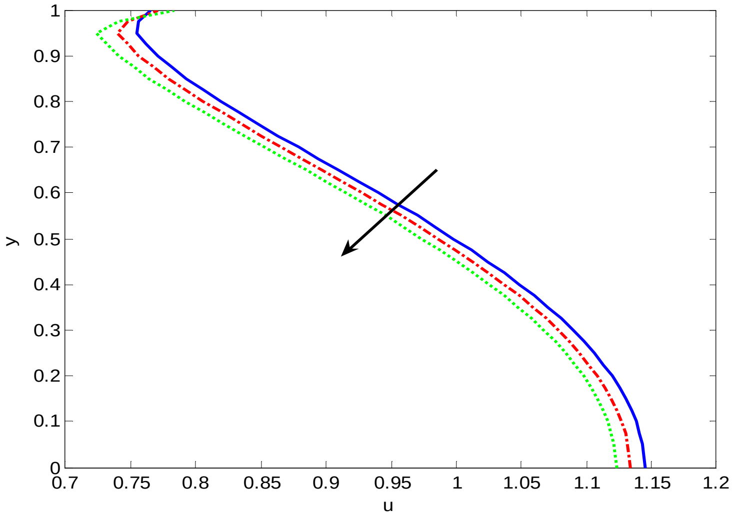

Fig. 6: Longitudinal velocity (u) with y for alpha variation at the entrance of the channel.

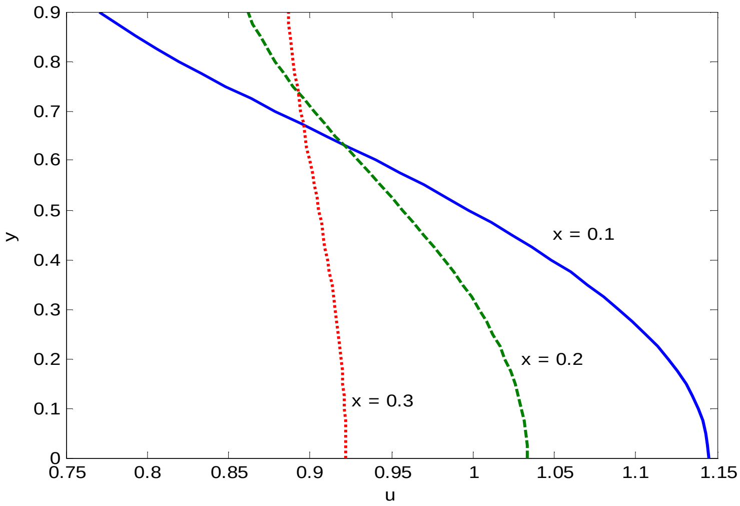

Fig. 7: Longitudinal velocity (u) with y along various points of x through the channel.

Mean Pressure Drop (mpd). The value of the mean pressure drop over the length of the channel is calculated for different values of alpha. It can be noted from figure 8 that an increase in the reabsorption coefficient raises the value of the mean pressure drop. Particularly at the exit the drop increases significantly. This is because the mean pressure decreases all the way from entrance to exit which resulted in a rise of mean pressure drop. On the other hand it is worthwhile to mention that this result is opposite to what has been reported on the paper by Muthu and Tesfahun, 2011 which corrects an incorrect result reported.

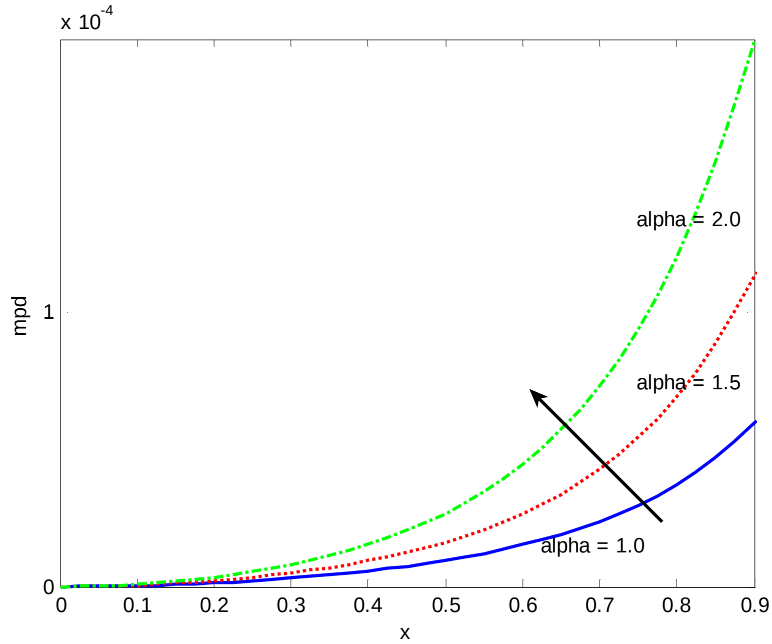

Fig. 8: Mean pressure drop *mpd* with x for alpha variation.

6 Conculusion

In the present analysis an incompressible viscous flow in a non-uniform wavy channel with slowly varying cross section with absorbing walls is numerically studied. The governing equations are solved using finite difference method by transforming the irregular boundary of the region of the problem to rectangular region. The main contribution of this study is to make the numerical computation simple by transforming the wavy non –uniform channel to a rectangular channel. It is observed that the reabsorption coefficient has a significant effect on the transversal velocity (v). That is, a rise in the reabsorption coefficient alpha increases the transverse velocity. This agrees with the natural phenomenon that due to rise of alpha the pressure drops which in turn results an increase in the transverse velocity. The reabsorption coefficient alpha has a reverse effect on the transverse velocity (v) and longitudinal velocity (u). That is an increment in alpha increases v whereas decreases u. Moreover an increase in the reabsorption coefficient raises the value of the mean pressure drop. Particularly at the exit the drop increases significantly. This is because the mean pressure decreases all the way from entrance to exit which resulted in a rise of mean pressure drop.

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