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On approximation properties for non-linear integral operators

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Abstract: We investigate the problem of pointwise convergence of the family of non-linear integral operators:

$$L_\lambda(f,x)=\int_a^b\sum_{m=1}^N f^m(t)K_{\lambda,m}(x,t)dt, x\in(a,b),$$

where $N\geq 1$ is a finite natural number, λ is a non-negative real parameter, $K_{\lambda,m}(x,t)$ is a non-negative kernel and f is the function in $L_1(a,b)$. We consider two cases such that (a,b) denotes finite interval of $\mathbb{R}$ and (a,b) denotes the whole real axis.

Keywords: Pointwise convergence, non-linear integral operators, lebesgue point.

1 Introduction

In [6] the concept of singularity was studied by including the case of nonlinear integral operators such that

$$T_wf(s)=\int_G K_w(t-s,f(t))dt, s\in G,$$

and the assumption of linearity of the operators was replaced by an assumption of Lipschitz condition for K_w with respect to the second variable. Later on, Swiderski and Wachnicki [9] investigated the pointwise convergence of the operators T_wf in $L_p(-\pi,\pi)$ and $L_p(\mathbb{R})$ at a point of continuity and a Lebesgue point of f.

In [3], Karsli studied both the pointwise convergence and the rate of pointwise convergence of above operators at a $\mu-generalized$ Lebesgue point of $f\in L_1\langle a,b\rangle$ as $(x,\lambda)\to(x_0,\lambda_0)$. In [4], the rate of convergence for the same operators is studied at a point x, where the being approximated function f has a discontinuity of the first kind, as $\lambda\to\lambda_0$. For general analysis on non-linear integral operators in different spaces and settings the book [1] is recommended. Also, for some recent works, we refer the reader to see [2,5] and [7]. Recently, Esen Almali investigated the problem of pointwise convergence at lebesgue points of functions for the family of singular integrals involving infinitive sum in [8].

The aim of this article is to obtain pointwise convergence results for a family of non-linear operators of the form:

$$L_\lambda(f,x)=\sum_{m=1}^N\int_a^b f^m(t)K_{\lambda,m}(x,t)dt,\ x\in(a,b), \tag{1}$$

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where $K_{\lambda,m}(x,t)$ is a family of kernels depending on λ.We study convergence of the family (1) at every Lebesgue point of the function f in the spaces of $L_1(a,b)$ and $L_1(-\infty,\infty)$. Here, the number $N \geq 1$ is finite arbitrary natural number.

Now, we give the following definition:

Definition 1. *(Class A) Let $m = 1,2,...,N$. We take a family $(K_\lambda)_{\lambda\in\Lambda}$ of functions $K_{\lambda,m}(x,t) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$. We will say that the function $K_\lambda(x,t)$ belongs the class A, if the following conditions are satisfied:*

(a) For every m, $K_{\lambda,m}(x,t)$ is a non-negative function defined for all t on (a,b) and $\lambda \in \Lambda$.
(b) For every m, as function of t, $K_{\lambda,m}(x,t)$ is non-decreasing on $[a,x]$ and non-increasing on $[x,b]$ for any fixed x.
(c) For every m and for any fixed x, $\lim\limits_{\lambda\to\infty}\int_a^b K_{\lambda,m}(x,t)dt = C_m$, where C_m are finite non-negative real numbers.
(d) For every m and $\lim\limits_{\lambda\to\infty} K_{\lambda,m}(x,y) = 0$ whenever $y \neq x$.

2 Main results

We are going to prove the family of non-linear integral operators (1) with the positive kernel convergence to the functions $f \in L_1(a,b)$

Theorem 1. *Suppose that $f \in L_1(a,b)$ and f is bounded on (a,b). If $K_{\lambda,m}$ belongs to Class A then, for the operator $L_\lambda(f,x)$ which is defined in (1) the relation*

$$\lim_{\lambda\to\infty} L_\lambda(f,x) = \sum_{m=1}^{N} C_m f^m(x)$$

holds at every Lebesgue point $x \in (a,b)$ of f.

Proof. For integral (1), we can write

$$L_\lambda(f,x) - \sum_{m=1}^{N} C_m f^m(x) = \sum_{m=1}^{N}\int_a^b f^m(t) - f^m(x)K_{\lambda,m}(x,t)dt + \sum_{m=1}^{N} f^m(x)\left[\int_a^b K_{\lambda,m}(x,t)dt - C_m\right],$$

and in view of (a), we may write

$$\begin{aligned}\left|L_\lambda(f,x) - \sum_{m=1}^{N} C_m f^m(x)\right| &\leq \sum_{m=1}^{N}\int_a^b |f^m(t) - f^m(x)|K_{\lambda,m}(x,t)dt + \sum_{m=1}^{N} |f^m(x)|\left|\int_a^b K_{\lambda,m}(x,t)dt - C_m\right| \\ &= I_1(x,\lambda) + I_2(x,\lambda).\end{aligned}$$

It is sufficient to show that terms on right hand side of the last inequality tend to zero as $\lambda\to\infty$.By property (c) ,it is clear that $I_2(x,\lambda)$ tends to zero as $\lambda\to\infty$.

Now, we consider $I_1(x,\lambda)$. For any fixed $\delta > 0$, we can write $I_1(x,\lambda)$ as follows:

$$\begin{aligned}I_1(x,\lambda) &= \sum_{m=1}^{N}\left[\int_a^{x-\delta} + \int_{x-\delta}^{x} + \int_x^{x+\delta} + \int_{x+\delta}^{b}\right] |f^m(t) - f^m(x)|K_{\lambda,m}(x,t)dt \qquad (1) \\ &= I_{11}(x,\lambda,m) + I_{12}(x,\lambda,m) + I_{13}(x,\lambda,m) + I_{14}(x,\lambda,m).\end{aligned}$$

Firstly, we shall calculate $I_{11}(x,\lambda,m)$, that is

$$I_{11}(x,\lambda,m)=\sum_{m=1}^{N}\int_{a}^{x-\delta}|f^m(t)-f^m(x)|K_{\lambda,m}(x,t)dt.$$

By the condition (b), we have

$$I_{11}(x,\lambda,m)\leq\sum_{m=1}^{N}K_{\lambda,m}(x,x-\delta)\left\{\int_{a}^{x-\delta}|f^m(t)|dt+x\int_{a}^{x-\delta}|f^m(x)|dt\right\},$$

and

$$\leq\sum_{m=1}^{N}K_{\lambda,m}(x,x-\delta)\left\{\|f^m\|_{L_1(a,b)}+|f^m(x)|(b-a)\right\} \tag{3}$$

In the same way, we can estimate $I_{14}(x,\lambda,m)$. From property (b)

$$\begin{aligned}I_{14}(x,\lambda,m)&\leq\sum_{m=1}^{N}K_{\lambda,m}(x,x+\delta)\left\{\int_{x+\delta}^{b}|f^m(t)|dt+\int_{x+\delta}^{b}|f^m(x)|dt\right\}\\&\leq\sum_{m=1}^{N}K_{\lambda,m}(x,x+\delta)\left\{\|f^m\|_{L_1(a,b)}+|f^m(x)|(b-a)\right\}.\end{aligned} \tag{2}$$

On the other hand, Since x is a Lebesgue point of f, for every $\varepsilon>0$, there exists a $\delta>0$ such that

$$\int_{x}^{x+h}|f(t)-f(x)|dt<\varepsilon h \tag{5}$$

and

$$\int_{x-h}^{x}|f(t)-f(x)|dt<\varepsilon h \tag{6}$$

for all $0<h\leq\delta$. Now let us define a new function as follows

$$F(t)=\int_{x}^{t}|f(u)-f(x)|du.$$

Then from (5), for $t-x\leq\delta$ we have

$$F(t)\leq\varepsilon(t-x).$$

Also, since f is bounded, there exists $M>0$ such that

$$|f^m(t)-f^m(x)|\leq M|f(t)-f(x)|.$$

is satisfied. Therefore, we can estimate $I_{13}(x,\lambda,m)$ as follows

$$I_{13}(x,\lambda,m)\leq M\sum_{m=1}^{N}\int_{x}^{x+\delta}|f(t)-f(x)|K_{\lambda,m}(x,t)dt\leq M\sum_{m=1}^{N}\int_{x}^{x+\delta}K_{\lambda,m}(x,t)dF(t).$$

We apply integration by parts, then we obtain the following result

$$|I_{13}(x,\lambda,m)| \leq M \sum_{m=1}^{N} \left\{ F(x+\delta,x)K_{\lambda,m}(x+\delta,x) + \int_{x}^{x+\delta} F(t)d\left(-K_{\lambda,m}(x,t)\right) \right\}.$$

Since $K_{\lambda,m}$ is decreasing on $[x,b]$, it is clear that $-K_{\lambda,m}$ is increasing. Therefore, we can write

$$|I_{13}(x,\lambda,m)| \leq M \sum_{m=1}^{N} \left\{ \varepsilon\delta K_{\lambda,m}(x+\delta,x) + \varepsilon \int_{x}^{x+\delta} (t-x)d\left(-K_{\lambda,m}(x,t)\right) \right\}.$$

Using integration by parts again, we have the following inequality

$$|I_{13}(x,\lambda,m)| \leq \varepsilon M \sum_{m=1}^{N} \int_{x}^{x+\delta} K_{\lambda,m}(x,t)dt \leq \varepsilon M \sum_{m=1}^{N} \int_{a}^{b} K_{\lambda,m}(x,t)dt.$$

Now, we can use similar method for evaluation $I_{12}(x,\lambda,m)$. Let

$$G(t) = \int_{t}^{x} |f(y) - f(x)|\, dy.$$

Then, the statement

$$dG(t) = -|f(t) - f(x)|\, dt.$$

is satisfied. For $x - t \leq \delta$, by using (6), it can be written as follows:

$$G(t) \leq \varepsilon\, |x-t|.$$

Hence, we get

$$I_{12}(x,\lambda,m) \leq M \sum_{m=1}^{N} \int_{x-\delta}^{x} |f(t) - f(x)| K_{\lambda,m}(x,t)dt.$$

Then, we shall write

$$|I_{12}(x,\lambda,m)| \leq M \sum_{m=1}^{N} \left[-\int_{x-\delta}^{x} K_{\lambda,m}(x,t)dG(t) \right].$$

By using integration by parts, we have

$$|I_{12}(x,\lambda,m)| \leq M \sum_{m=1}^{N} \left\{ G(x-\delta K_{\lambda,m}(x-\delta,x) + \int_{x-\delta}^{x} G(t)d_t(K_{\lambda,m}(x,t)) \right\}.$$

From (6), we obtain

$$|I_{12}(x,\lambda,m)| \leq M \sum_{m=1}^{N} \left\{ \varepsilon\delta K_{\lambda,m}(x,x-\delta) + \varepsilon \int_{x-\delta}^{x} (x-t)d_t(K_{\lambda,m}(x,t)) \right\}.$$

By using integration by parts again, we see that

$$|I_{12}(x,\lambda,m)| \leq \varepsilon M \sum_{m=1}^{N} \int_{a}^{b} K_{\lambda,m}(x,t)dt. \tag{8}$$

Combining (7) and (8), we get

$$|I_{12}(x,\lambda,m)| + |I_{13}(x,\lambda,m)| \leq 2\varepsilon M \sum_{m=1}^{N} \int_{a}^{b} K_{\lambda,m}(x,t)dt. \tag{9}$$

Hence from (3), (4) and (9), the terms on right hand side of these inequalities tend to 0 as $\lambda \to \infty$. That is

$$\lim_{\lambda \to \infty} L_{\lambda}(f,x) = \sum_{m=1}^{N} C_m f^m(x).$$

Thus, the proof is completed.

In this theorem, specifically we take $a = -\infty$ and $b = \infty$. In this case, we can give the following theorem:

Theorem 2. *Let $f \in L_1(-\infty,\infty)$ and f is bounded on $\mathbb{R}$. If $K_{\lambda,m}$ belongs to Class A and satisfy also the following properties for every $m = 1,...,N$:*

$$\lim_{\lambda \to \infty} \int_{-\infty}^{x-\delta} K_{\lambda,m}(t,x)dt = 0, \tag{10}$$

and

$$\lim_{\lambda \to \infty} \int_{x+\delta}^{\infty} K_{\lambda,m}(t,x)dt = 0. \tag{11}$$

Then,

$$\lim_{\lambda \to \infty} L_{\lambda}(f,x) = \sum_{m=1}^{N} C_m f^m(x)$$

holds at every Lebesgue point $x \in \mathbb{R}$ of f.

Proof. Easily, we can write

$$\begin{aligned}\left| L_{\lambda}(f,x) - \sum_{m=1}^{N} C_m f^m(x) \right| &\leq \sum_{m=1}^{N} \int_{-\infty}^{\infty} |f^m(t) - f^m(x)| K_{\lambda,m}(x,t)dt + \sum_{m=1}^{N} |f^m(x)| \left| \int_{-\infty}^{\infty} K_{\lambda,m}(x,t)dt - C_m \right| \\ &= A_1(x,\lambda) + A_2(x,\lambda).\end{aligned}$$

It is clear that $A_2(x,\lambda) \to 0$ as $\lambda \to \infty$.

For a fixed $\delta > 0$, we divide the integral $A_1(x,\lambda)$ such that

$$\begin{aligned}A_1(x,\lambda) &= \sum_{m=1}^{N} \left[\int_{-\infty}^{x-\delta} + \int_{x-\delta}^{x} + \int_{x}^{x+\delta} + \int_{x+\delta}^{\infty} \right] |f^m(t) - f^m(x)| K_{\lambda,m}(x,t)dt \\ &= A_{11}(x,\lambda,m) + A_{12}(x,\lambda,m) + A_{13}(x,\lambda,m) + A_{14}(x,\lambda,m).\end{aligned}$$

The integrals $A_{12}(x,\lambda,m)$ and $A_{13}(x,\lambda,m)$ are calculated as in the previous proof. For the remaing integrals, we have to to show that $A_{11}(x,\lambda,m)$ and $A_{14}(x,\lambda,m)$ tend to zero as $\lambda\to\infty$.

Firstly, we consider $A_{11}(x,\lambda,m)$. Since f is bounded and by the property (b), the following expression holds:

$$\begin{aligned} A_{11}(x,\lambda,m) &\leq M\sum_{m=1}^{N}\int_{-\infty}^{x-\delta}|f(t)-f(x)|K_{\lambda,m}(x,t)dt \\ &\leq M\sum_{m=1}^{N}K_{\lambda,m}(x,x-\delta)\left\{\int_{-\infty}^{x-\delta}|f(t)|\right\}+M|f(x)|\sum_{m=1}^{N}\int_{-\infty}^{x-\delta}K_{\lambda,m}(x,t)dt \\ &\leq \|f\|_{L_1(-\infty,\infty)}M\sum_{m=1}^{N}K_{\lambda,m}(x,x-\delta)+M|f(x)|\sum_{m=1}^{N}\int_{-\infty}^{x-\delta}K_{\lambda,m}(x,t)dt. \end{aligned}$$

In addition, we obtain the following inequality:

$$\begin{aligned} A_{14}(x,\lambda,m) &\leq M\sum_{m=1}^{N}\int_{x+\delta}^{\infty}|f(t)-f(x)|K_{\lambda,m}(x,t)dt \\ &\leq \|f\|_{L_1(-\infty,\infty)}M\sum_{m=1}^{N}K_{\lambda,m}(x,x+\delta)+M|f(x)|\sum_{m=1}^{N}\int_{x+\delta}^{\infty}K_{\lambda,m}(x,t)dt. \end{aligned}$$

According to the conditions (d), (10) and (11), we find that $A_{11}(x,\lambda,m)+A_{14}(x,\lambda,m)\to 0$ as $\lambda\to\infty$. This completes the proof.

3 Conclusions

In this paper, we obtained the pointwise convergence for the specifically chosen family of non-linear integral operators. For this aim, we defined a class of kernel functions called *Class A*. For each $m=1,2,...,N$, the functions from this class satisfies the properties similar to classical approximate identities. From another point of view, the operators defined by (1) are of type summation-integral type operators, since they include powers of f. Under the hypotheses of Theorem 1 and Theorem 2, we saw that the convergence is obtained at every Lebesgue of $f\in L_1(a,b)$ and $f\in L_1(-\infty,\infty)$, respectively.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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