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Refinements of fractional integral inequalities obtained for p-convex functions

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Abstract: This paper is about obtaining the fractional integral inequalities obtained for p-convex functions with the help of functionals. Firstly, definitions and theorems necessary for our study are given. In the findings of the work, the left sides of the Hermite-Hadamard and Hermite-Hadamard-Fejér inequalities obtained using Riemann-Liouville fractional integrals for p-convex functions were obtained through functionals.

Keywords: Functionals, Hermite-Hadamard inequality, Hermite-Hadamard-FejÃ©r inequality, Integral inequalities, Convex function, p-convex functions, Riemann-Liouville fractional integral

1 Introduction

Definition 1. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality*

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$ If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$ This definition is well known in the literature.

It is well known that theory of convex sets and convex functions play an important role in mathematics and the other pure and applied sciences.
If $f : I \to \mathbb{R}$ is a convex function on the interval I, then for any $a, b \in I$ with $a < b$, we have the following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

This double inequalities is known in the literature as Hermite–Hadamard integral inequalities for convex functions. Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave. For some results which generalize, improve and extend the inequalities (1) we refer the reader to the recent papers [1,2,3,4,5,6,10,17,18,19,20,21,22] and references therein.

Definition 2. *Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval. A functionf $: I \to \mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq t f(y) + (1-t) f(x)$$

for all $x, y \in I$ andt $\in [0,1]$ If this inequality is reversed, then f is said to be harmonically concave. [6]

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Definition 3. *Let $I \subset (0,\infty)$ be a real interval and $p \in \mathbb{R} \backslash \{0\}$. A function $f : I \to \mathbb{R}$ is said to be a p-convex function, if*

$$f\left(\left[tx^p + (1-t)y^p\right]^{\frac{1}{p}}\right) \le tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0,1]$. If this inequality is reversed, then f is said to be p -concave. [9]

According to this definition, It can be easily seen that for $p = 1$ and $p = -1$, p-convexity reduces to ordinary convexity and harmonically convexity of functions defined on $I \subset (0,\infty)$, respectively.
Hermite-Hadamard inequality for the p-convex function is following,

Theorem 1. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a p -convex function, $p \in \mathbb{R} \backslash \{0\}$ and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then we have [9]*

$$f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx \le \frac{f(a) + f(b)}{2}.$$

These inequalities are sharp [13, 14]. If these inequalities are reversed, then f is said to be p-concave. We refer the reader to the recent papers related to p-convexity [12, 15, 16, 17, 18, 19, 20, 21] and references therein.

Theorem 2. *Let f is convex on $[a,b]$.Then H is convex, increasing on $[0,1]$,and for all $t \in [0,1]$, we have [2]*

$$f\left(\frac{a+b}{2}\right) \le H(0) \le H(t) \le H(1) \le \frac{1}{b-a} \int_a^b f(x)\,dx \tag{2}$$

where

$$H(t) = \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx.$$

An analogous result for convex functions which refines the second inequality of (1) is obtained by G. S. Yang and M. C. Hong in [22] as follows.
G. S. Yang and K. L. Tseng in [21] established some generalizations of (2) based on the following results.

Theorem 3. *[21] Let $f : [a,b] \to \mathbb{R}$ be a convex function, $0 < \alpha < 1$, $0 < \beta < 1$*

$$A = \alpha a + (1-\alpha)b, \;\; u_0 = (b-a)min\left\{\frac{\alpha}{1-\beta}, \frac{1-\alpha}{\beta}\right\}$$

and let h be defined by

$$h(t) = (1-\beta) f(A - \beta t) + \beta f(A + (1-\beta)t),$$

for $t \in [0,u_0]$. Then h is convex, increasing on $[0,u_0]$ and for all $t \in [0,u_0]$

$$f(\alpha a + (1-\alpha)b) \le h(t) \le \alpha f(a) + (1-\alpha) f(b).$$

It is remarkable that M. Z. Sarıkaya et al. [19] proved the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 4. *[21] Let $f : [a,b] \to \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$ then the following inequalities for fractional integrals hold*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \le \frac{fa) + f(b)}{2} \tag{3}$$

with $\alpha > 0$.

We remark that the symbols $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \ge 0$ with $a \ge 0$ which are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \qquad x > a,$$

and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \qquad x<b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha)=\int_0^\infty e^{-t}t^{\alpha-1}dt$.

Theorem 5. *[7] Let $a,b\in I$ with $a<b$ $f:I\subseteq(0,\infty)\to\mathbb{R}$ be a function and $f\in L[a,b]$ then the following inequalities for fractional integrals holds*

$$f\left(\frac{2ab}{a+b}\right)\le\frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J_{\frac{1}{a}-}^{\alpha}(fog)\left(\frac{1}{b}\right)+J_{\frac{1}{b}+}^{\alpha}(fog)\left(\frac{1}{a}\right)\right]\le\frac{fa)+f(b)}{2}.$$

where $\alpha>0$ and $g(x)=\frac{1}{x}$, $x\in\left[\frac{1}{a},\frac{1}{b}\right]$

Ruiyin Xiang [20] proved the following Lemma and Theorem for interesting inequalities of Hermite-Hadamard type inequalities for convex functions via Riemann-Liouville fractional integrals.

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function and h be defined by*

$$h(t)=\frac{1}{2}\left[f\left(\left(\frac{a+b}{2}\right)-\frac{t}{2}\right)+f\left(\left(\frac{a+b}{2}\right)+\frac{t}{2}\right)\right].$$

Then $h(t)$ is convex, increasing on $[0,b-a]$ and for al $lt\in[0,b-a]$,

$$f\left(\frac{a+b}{2}\right)\le h(t)\le\frac{f(a)+f(b)}{2}.$$

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $a<b$ and $f\in L[a,b]$. If f is a convex function on $[a,b]$,then WH is convex and monotonically increasing on $[0,1]$ and*

$$f\left(\frac{a+b}{2}\right)=WH(0)\le WH(t)\le WH(1)=\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]$$

with $\alpha>0$, where

$$WH(t)=\frac{\alpha}{2(b-a)^{\alpha}}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]dx.$$

Hermit-Hadamard-Fejér type inequalities obtained for convex and harmonically convex functions via fractional integrals are given as follow

Theorem 7. *[8] Let $f:[a,b]\to\mathbb{R}$ be a convex function with $a<b$ and $f\in L[a,b]$. If $w:[a,b]\to\mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$, then the following inequalities for fractional integrals holds*

$$\begin{aligned}f\left(\frac{a+b}{2}\right)\left[J_{a+}^{\alpha}w(b)+J_{b-}^{\alpha}w(a)\right]&\le\left[J_{a+}^{\alpha}(fw)(b)+J_{b-}^{\alpha}(fw)(a)\right]\\&\le\frac{fa)+f(b)}{2}\left[J_{a+}^{\alpha}w(b)+J_{b-}^{\alpha}w(a)\right],\end{aligned}$$

with $\alpha>0$.

Theorem 8. *[11] Let $f:[a,b]\to\mathbb{R}$ be a harmonically convex function with $a<b$ and $f\in L[a,b]$. If $w:[a,b]\to\mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{2ab}{a+b}$, then the following inequalities for fractional integrals holds*

$$\begin{aligned}f\left(\frac{2ab}{a+b}\right)\left[J_{\frac{1}{b}+}^{\alpha}(wog)\left(\frac{1}{a}\right)+J_{\frac{1}{a}-}^{\alpha}(wog)\left(\frac{1}{b}\right)\right]&\le\left[J_{\frac{1}{b}+}^{\alpha}(fwog)\left(\frac{1}{a}\right)+J_{\frac{1}{a}-}^{\alpha}(fwog)\left(\frac{1}{b}\right)\right]\\&\le\frac{fa)+f(b)}{2}\left[J_{\frac{1}{b}+}^{\alpha}(wog)\left(\frac{1}{a}\right)+J_{\frac{1}{a}-}^{\alpha}(wog)\left(\frac{1}{b}\right)\right],\end{aligned}$$

with $\alpha>0$ and $g(x)=\frac{1}{x}$, $x\varepsilon\left[\frac{1}{b},\frac{1}{a}\right]$

In this paper, we establish some new Hermite-Hadamard type inequalities for convex functions via Riemann-Liouville fractional integrals which refine the inequalities of (3).

2 The left hand sides of the Hermite-Hadamard and Hermite-Hadamard-Fejér inequalities via functionals

Throughout this section, $\|w\|_\infty = \sup_{t\in[a,b]} |w(t)|$ for the continuous function $w : [a,b] \to \mathbb{R}$.

Definition 4. *[13] Let $p \in \mathbb{R}\setminus\{0\}$. A function $w : [a,b] \subset (0,\infty) \to \mathbb{R}$ is said to be p -symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}$ if*

$$w(x) = w\left([a^p + b^p - x^p]^{\frac{1}{p}}\right)$$

holds for all $x \in [a,b]$.

Lemma 2. *[13] Let $p \in \mathbb{R}\setminus\{0\}$, $\alpha > 0$ and $w : [a,b] \subset (0,\infty) \to \mathbb{R}$ is integrable, be p -symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}$, then*

(i) If $p > 0$

$$J^{\alpha}_{a^p+}(wog)(b^p) = J^{\alpha}_{b^p-}(wog)(a^p) = \frac{1}{2}\left[J^{\alpha}_{a^p+}(wog)(b^p) + J^{\alpha}_{b^p-}(wog)(a^p)\right],$$

with $g(x) = x^{\frac{1}{p}}$, $x\varepsilon\,[a^p, b^p]$.

(ii) If $p < 0$,

$$J^{\alpha}_{b^p+}(wog)(a^p) = J^{\alpha}_{a^p-}(wog)(b^p) = \frac{1}{2}\left[J^{\alpha}_{b^p+}(wog)(a^p) + J^{\alpha}_{a^p-}(wog)(b^p)\right],$$

with $g(x) = x^{\frac{1}{p}}$, $x\varepsilon\,[b,^p a^p]$.

Lemma 3. *[9] Let $p > 0$. If $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ is p-convex function, then function $g : [a^p, b^p] \to \mathbb{R}$, $g(t) = f\left(t^{\frac{1}{p}}\right)$ is convex for $p < 0$.*

Lemma 4. *Let $p > 0$, $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ be a p -convex function and $h : [0, b^p - a^p] \to \mathbb{R}$, $(p < 0)$ be defined by*

$$h(t) = \frac{1}{2}\left[f\left[\left(\frac{a^p+b^p}{2} - \frac{t}{2}\right)^{\frac{1}{p}}\right] + f\left[\left(\left(\frac{a^p+b^p}{2}\right) + \frac{t}{2}\right)^{\frac{1}{p}}\right]\right],$$

Then $h(t)$ is convex, increasing on $[0, b^p - a^p]$ and for all $t \in [0, b^p - a^p]$

$$f\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right) \le h(t) \le \frac{f(a)+f(b)}{2}.$$

Proof. If the function $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ is p-convex, then according to the Lemma 3 the function $g(t) = f\left(t^{\frac{1}{p}}\right)$ is convex on $[a^p, b^p]$ for $p < 0$. Then the function h defined by

$$\begin{aligned} h(t) &= \frac{1}{2}\left[g\left(\left(\frac{a^p+b^p}{2}\right) - \frac{t}{2}\right) + g\left(\left(\frac{a^p+b^p}{2} + \frac{t}{2}\right)\right)\right] \\ &= \frac{1}{2}\left[f\left[\left(\frac{a^p+b^p}{2} - \frac{t}{2}\right)^{\frac{1}{p}}\right] + f\left[\left(\left(\frac{a^p+b^p}{2}\right) + \frac{t}{2}\right)^{\frac{1}{p}}\right]\right] \end{aligned}$$

is convex and increasing on $[0, b^p - a^p]$ *and for all* $t \in [0, b^p - a^p]$

$$f\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right) \le h(t) \le \frac{f(a)+f(b)}{2}.$$

Theorem 9. *Let $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ be a function and $f\in L[a,b]$. If the function f is p -convex on $[a,b]$, then $W_p:[0,1]\to\mathbb{R}$ defined by*

$$W_p(t)=\begin{cases}\frac{\alpha}{2(b^p-a^p)^\alpha}\int_{a^p}^{b^p} f\left[\left(\left(tx+(1-t)\left(\frac{a^p+b^p}{2}\right)\right)\right)^{\frac{1}{p}}\right]\times\left[(b^p-x)^{\alpha-1}+(x-a^p)^{\alpha-1}\right]dx, & p>0\\ \frac{\alpha}{2(a^p-b^p)^\alpha}\int_{a^p}^{b^p} f\left[\left(\left(tx+(1-t)\left(\frac{a^p+b^p}{2}\right)\right)\right)^{\frac{1}{p}}\right]\times\left[(x-b^p)^{\alpha-1}+(a^p-x)^{\alpha-1}\right]dx, & p<0\end{cases}$$

is convex and monotonically increasing on $[0,1]$ and

$$f\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)=W_p(0)\le W_p(t)\le W_p(1)=\begin{cases}\frac{\Gamma(\alpha+1)}{2(b^p-a^p)^\alpha}\left[J^\alpha_{a^p+}(fog)(b^p)+J^\alpha_{b^p-}(fog)(a^p)\right], & p>0,\\ \frac{\Gamma(\alpha+1)}{2(a^p-b^p)^\alpha}\left[J^\alpha_{b^p+}(fog)(a^p)+J^\alpha_{a^p-}(fog)(b^p)\right], & p<0,\end{cases}$$

for all $t\in[0,1]$ .Where $g(x)=x^{\frac{1}{p}}$, $x\in[a^p,b^p]$ and $\alpha>0$.

Proof. Firstly, let $t_1,t_2,\beta\in[0,1]$. For $p>0$, we need to show that

$$W_p((1-\beta)t_1+\beta t_2)\le(1-\beta)W_p(t_1)+\beta W_P(t_2).$$

Using the definition of W_p, we can write the following

$$\begin{aligned}W_p((1-\beta)t_1+\beta t_2)&=\frac{\alpha}{2(b^p-a^p)^\alpha}\int_{a^p}^{b^p} f\{[((1-\beta)t_1+\beta t_2)x\\&+[(1-\beta)(1-t_1)\ +\ \beta(1-t_2)]\left(\frac{a^p+b^p}{2}\right)]^{\frac{1}{p}}\Big\}\\&\times\left((b^p-x)^{\alpha-1}+(x-a^p)^{\alpha-1}\right)dx\\&=\frac{\alpha}{2(b^p-a^p)^\alpha}\int_{a^p}^{b^p} f\left\{\left[(1-\beta)\left[t_1x+(1-t_1)\left(\frac{a^p+b^p}{2}\right)\right]\right.\right.\\&\left.\left.+\ \beta\left[t_2x+(1-t_2)\left(\frac{a^p+b^p}{2}\right)\right]\right]^{\frac{1}{p}}\right\}\\&\times\left((b^p-x)^{\alpha-1}+(x-a^p)^{\alpha-1}\right)dx.\end{aligned}$$

Since the function f is p-convex, we get

$$\begin{aligned}W_p((1-\beta)t_1+\beta t_2)&\le\frac{\alpha}{2(b^p-a^p)^\alpha}\int_{a^p}^{b^p}\left[(b^p-x^p)^{\alpha-1}+(x^p-a^p)^{\alpha-1}\right]\\&\times\left\{(1-\beta)f\left(\left(t_1x+(1-t_1)\left(\frac{a^p+b^p}{2}\right)\right)^{\frac{1}{p}}\right)\right.\\&\left.+\beta f\left(\left(t_2x+(1-t_2)\left(\frac{a^p+b^p}{2}\right)\right)^{\frac{1}{p}}\right)\right\}dx\\&=(1-\beta)W_p(t_1)+\beta W_p(t_2)\end{aligned}$$

from which we get W_p is convex on $[0,1]$. By elementary calculus, we have

$$\begin{aligned}
W_p(t) &= \frac{\alpha}{2(b^p-a^p)^\alpha}\int_{a^p}^{b^p} f\left[\left(tx+(1-t)\left(\frac{a^p+b^p}{2}\right)\right)^{\frac{1}{p}}\right] \\
&\times\left[(b^p-x)^{\alpha-1}+(x-a^p)^{\alpha-1}\right]dx \\
&= \frac{\alpha}{2(b^p-a^p)^\alpha}\int_{a^p}^{\frac{a^p+b^p}{2}} f\left[\left(tx+(1-t)\left(\frac{a^p+b^p}{2}\right)\right)^{\frac{1}{p}}\right] \\
&\times\left[(b^p-x)^{\alpha-1}+(x-a^p)^{\alpha-1}\right]dx \\
&+ \frac{\alpha}{2(b^p-a^p)^\alpha}\int_{\frac{a^p+b^p}{2}}^{b^p} f\left[\left(tx+(1-t)\left(\frac{a^p+b^p}{2}\right)\right)^{\frac{1}{p}}\right] \\
&\times\left[(b^p-x)^{\alpha-1}+(x-a^p)^{\alpha-1}\right]dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{\alpha}{2(b^p-a^p)^\alpha}\int_0^{b^p-a^p}\left[f\left(\frac{a^p+b^p}{2}-\frac{tu}{2}\right)^{\frac{1}{p}}\right] \\
&\times\left[\left(\frac{b^p-a^p}{2}+\frac{u}{2}\right)^{\alpha-1}+\left(\frac{b^p-a^p}{2}-\frac{u}{2}\right)^{\alpha-1}\right]\frac{du}{2} \\
&+ \frac{\alpha p}{2(b^p-a^p)^\alpha}\int_0^{b^p-a^p}\left[f\left(\frac{a^p+b^p}{2}+\frac{tu}{2}\right)^{\frac{1}{p}}\right] \\
&\times\left[\left(\frac{b^p-a^p}{2}-\frac{u}{2}\right)^{\alpha-1}+\left(\frac{b^p-a^p}{2}+\frac{u}{2}\right)^{\alpha-1}\right]\frac{du}{2}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\alpha}{2(b^p-a^p)^\alpha}\int_0^{b^p-a^p}\frac{1}{2}\left[f\left(\left[\frac{a^p+b^p}{2}-\frac{tu}{2}\right]^{\frac{1}{p}}\right)+f\left(\left[\frac{a^p+b^p}{2}+\frac{tu}{2}\right]^{\frac{1}{p}}\right)\right] \\
&\times\left[\left(\frac{b^p-a^p}{2}-\frac{u}{2}\right)^{\alpha-1}+\left(\frac{b^p-a^p}{2}+\frac{u}{2}\right)^{\alpha-1}\right]du \\
&= \frac{\alpha}{2(b^p-a^p)^\alpha}\int_0^{b^p-a^p}\frac{1}{2}\left[f\left(\left[\frac{a^p+b^p}{2}-\frac{tx}{2}\right]^{\frac{1}{p}}\right)+f\left(\left[\frac{a^p+b^p}{2}+\frac{tx}{2}\right]^{\frac{1}{p}}\right)\right] \\
&\times\left[\left(\frac{b^p-a^p}{2}-\frac{x}{2}\right)^{\alpha-1}+\left(\frac{b^p-a^p}{2}+\frac{x}{2}\right)^{\alpha-1}\right]dx
\end{aligned}$$

It follows from Lemma (4) that function

$$h(x)=\frac{1}{2}\left[f\left(\left[\frac{a^p+b^p}{2}-\frac{tx}{2}\right]^{\frac{1}{p}}\right)+f\left(\left[\frac{a^p+b^p}{2}+\frac{tx}{2}\right]^{\frac{1}{p}}\right)\right]$$

is increasing on $[0,b^p-a^p]$. Since $\left[\left(\frac{b^p-a^p}{2}-\frac{x}{2}\right)^{\alpha-1}+\left(\frac{b^p-a^p}{2}+\frac{x}{2}\right)^{\alpha-1}\right]$ is nonne-gative, hence $W_p(t)$ is monotonically increasing on $[0,1]$. Finally, from

$$W_p(0)=f\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)$$

and

$$\begin{aligned}W_p(1) &= \frac{\alpha}{2(b^p-a^p)^\alpha}\int_{a^p}^{b^p} f\left(x^{\frac{1}{p}}\right)\left[(b^p-x)^{\alpha-1}+(x-a^p)^{\alpha-1}\right]\,dx\\ &= \frac{\alpha}{2(b^p-a^p)^\alpha}\left[\int_{a^p}^{b^p}(b^p-x)^{\alpha-1}(fog)(x)\,dx+\int_{a^p}^{b^p}(x-a^p)^{\alpha-1}(fog)(x)\,dx\right]\\ &= \frac{\Gamma(\alpha+1)}{2(b^p-a^p)^\alpha}\left[J^\alpha_{a^p+}(fog)(b^p)+J^\alpha_{b^p-}(fog)(a^p)\right],\end{aligned}$$

This completes proof of theorem. Similar proof can be made for $p<0$.

In Theorem (9), If we take $p=1$, then we get the following result for convex functions.

Corollary 1. *Let $f:[a,b]\subseteq(0,\infty)\to\mathbb{R},\ f\in L[a,b]$. If f is convex, then*

$$\begin{aligned}W_1(t) = {} & \frac{\alpha}{2(b-a)^\alpha}\int_a^b f\left[\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)\right]\\ & \times\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]dx.\end{aligned}$$

Moreover

$$\begin{aligned}f\left(\left(\frac{a+b}{2}\right)\right) &= W_1(0)\le W_1(t)\le W_1(1)\\ &= \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)\right].\end{aligned}$$

Remark. The inequality obtained in Corollary (1) gives us the left hand side of the inequality obtained in Theorem (4).

In Theorem (9), If we take $p=-1$, then we get the following result for harmonically convex functions.

Corollary 2. *Let $f:[a,b]\subseteq(0,\infty)\to\mathbb{R},\ f\in L[a,b]$. If the function f is convex, then*

$$\begin{aligned}W_{-1}(t) = {} & \frac{\alpha}{2(\frac{1}{a}-\frac{1}{b})^\alpha}\int_{\frac{1}{a}}^{\frac{1}{b}}\left[f\left(tx+(1-t)\left(\frac{a+b}{2ab}\right)\right)^{-1}\right]\\ & \times\left[\left(x-\frac{1}{b}\right)^{\alpha-1}+\left(\frac{1}{a}-x\right)^{\alpha-1}\right]dx.\end{aligned}$$

Moreover

$$\begin{aligned}f\left(\frac{2ab}{a+b}\right) &= W_{-1}(0)\le W_{-1}(t)\le W_{-1}(1)\\ &= \frac{\Gamma(\alpha+1)}{2(\frac{1}{a}-\frac{1}{b})^\alpha}\left[J^\alpha_{\frac{1}{b}+}(fog)\left(\frac{1}{a}\right)+J^\alpha_{\frac{1}{a}-}(fog)\left(\frac{1}{b}\right)\right]\end{aligned}$$

where $g(x)=\frac{1}{x}$.

Remark. The inequality obtained in Corollary (2) gives us the left hand side of the inequality obtained in Theorem (5)

Theorem 10. *Let $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ be a function and $f\in L[a,b]$. If the function f is p -convex on $[a,b]$ and $w:[a,b]\to\mathbb{R}$ is integrable, nonnegative and p -symmetrically according to $\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}$. If the function f is convex, then*

$W_{p,w}:[0,1]\to\mathbb{R}$ *defined by*

$$W_{p,w}(t)=\begin{cases}\frac{\alpha}{2(b^p-a^p)^\alpha}\int_{a^p}^{b^p} f\left[\left(tx+(1-t)\left(\frac{a^p+b^p}{2}\right)\right)^{\frac{1}{p}}\right] \\ \qquad\times\left[(b^p-x)^{\alpha-1}+(x-a^p)^{\alpha-1}\right]w(x^{\frac{1}{p}})dx, \quad p>0 \\ \frac{\alpha}{2(a^p-b^p)^\alpha}\int_{a^p}^{b^p} f\left[\left(tx+(1-t)\left(\frac{a^p+b^p}{2}\right)\right)^{\frac{1}{p}}\right] \\ \qquad\times\left[(x-b^p)^{\alpha-1}+(a^p-x)^{\alpha-1}\right]w(x^{\frac{1}{p}})dx, \quad p<0\end{cases}$$

is convex and monotonically increasing on $[0,1]$ *and*

(i) Let $g(x)=x^{1/p}$, $\alpha>0$. If $p>0$,

$$\begin{aligned}&f\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)\left[J_{a^p+}^{\alpha}(wog)(b^p)+J_{b^p-}^{\alpha}(wog)(a^p)\right]\\&=W_{p,w}(0)\le W_{p,w}(t)\le W_{p,w}(1)\\&=\frac{\Gamma(\alpha+1)}{2(b^p-a^p)^\alpha}\left[J_{a^p+}^{\alpha}(fwog)(b^p)+J_{b^p-}^{\alpha}(fwog)(a^p)\right]\end{aligned}$$

for all $t\in[0,1]$.
(ii) Let $g(x)=x^{1/p}$, $\alpha>0$. If $p<0$,

$$\begin{aligned}&f\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)\left[J_{b^p+}^{\alpha}(wog)(a^p)+J_{a^p-}^{\alpha}(wog)(b^p)\right]\\&=W_{p,w}(0)\le W_{p,w}(t)\le W_{p,w}(1)\\&=\frac{\Gamma(\alpha+1)}{2(a^p-b^p)^\alpha}\left[J_{b^p+}^{\alpha}(fwog)(a^p)+J_{a^p-}^{\alpha}(fwog)(b^p)\right]\end{aligned}$$

for all $t\in[0,1]$.

Proof. **(i)** Firstly, suppose that $p>0$. Let $t_1,t_2,\beta\in[0,1]$. We need to show that

$$W_{p,w}(t)((1-\beta)t_1+\beta t_2)\le(1-\beta)W_{p,w}(t_1)+\beta W_{p,w}(t_2)$$

Using the definition of $W_{p,w}$ we can write the following

$$\begin{aligned}W_{p,w}(t)((1-\beta)t_1+\beta t_2)&=\frac{\alpha}{2(b^p-a^p)^\alpha}\int_{a^p}^{b^p} f\Bigg([((1-\beta)t_1+\beta t_2)x\\&+[(1-\beta)(1-t_1)\ +\ \beta(1-t_2)]\left(\frac{a^p+b^p}{2}\right)\Bigg]^{\frac{1}{p}}\Bigg)\\&\times\left((b^p-x)^{\alpha-1}+(x-a^p)^{\alpha-1}\right)w(x^{\frac{1}{p}})dx\\&=\frac{\alpha p}{2(b^p-a^p)^\alpha}\int_{a^p}^{b^p}\left[(b^p-x)^{\alpha-1}+(x-a^p)^{\alpha-1}\right]\\&\times f\left(\left[(1-\beta)\left[t_1x+(1-t_1)\left(\frac{a^p+b^p}{2}\right)\right]\right.\right.\\&\left.\left.+\beta\left[t_2x+(1-t_2)\right]\left(\frac{a^p+b^p}{2}\right)\right]^{\frac{1}{p}}\right)w(x^{\frac{1}{p}})dx\end{aligned}$$

Since the function $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ is p-convex, we get

$$\begin{aligned}
W_{p,w}(t)((1-\beta)t_1+\beta t_2) &\le \frac{\alpha}{2(b^p-a^p)^\alpha}\int_{a^p}^{b^p}\left[(b^p-x^p)^{\alpha-1}+(x^p-a^p)^{\alpha-1}\right]\\
&\times\left\{(1-\beta)f\left(\left(t_1x+(1-t_1)\left(\frac{a^p+b^p}{2}\right)\right)^{\frac{1}{p}}\right)\right.\\
&\left.+\beta f\left(\left(t_2x+(1-t_2)\left(\frac{a^p+b^p}{2}\right)\right)^{\frac{1}{p}}\right)\right\}w(x^{\frac{1}{p}})dx\\
&=(1-\beta)W_{p,w}(t_1)+\beta W_{P,w}(t_2)
\end{aligned}$$

from which we get $W_{p,w}$ is convex on $[0,1]$. By elementary calculus, we have

$$\begin{aligned}
W_{p,w}(t) &= \frac{\alpha}{2(b^p-a^p)^\alpha}\int_{a^p}^{b^p} f\left[\left(tx+(1-t)\left(\frac{a^p+b^p}{2}\right)\right)^{\frac{1}{p}}\right]\\
&\times\left[(b^p-x)^{\alpha-1}+(x-a^p)^{\alpha-1}\right]w(x^{\frac{1}{p}})dx\\
&= \frac{\alpha}{2(b^p-a^p)^\alpha}\int_{a^p}^{\frac{a^p+b^p}{2}} f\left[\left(tx+(1-t)\left(\frac{a^p+b^p}{2}\right)\right)^{\frac{1}{p}}\right]\\
&\times\left[(b^p-x)^{\alpha-1}+(x-a^p)^{\alpha-1}\right]w(x^{\frac{1}{p}})dx\\
&+ \frac{\alpha}{2(b^p-a^p)^\alpha}\int_{\frac{a^p+b^p}{2}}^{b^p} f\left[\left(tx+(1-t)\left(\frac{a^p+b^p}{2}\right)\right)^{\frac{1}{p}}\right]\\
&\times\left[(b^p-x)^{\alpha-1}+(x-a^p)^{\alpha-1}\right]w(x^{\frac{1}{p}})dx\\
&= \frac{\alpha}{2(b^p-a^p)^\alpha}\int_0^{b^p-a^p} f\left(\left[\frac{a^p+b^p}{2}-\frac{tu}{2}\right]^{\frac{1}{p}}\right)\\
&\times\left[\left(\frac{b^p-a^p+u}{2}\right)^{\alpha-1}+\left(\frac{b^p-a^p-u}{2}\right)^{\alpha-1}\right]w\left(\left(\frac{a^p+b^p-u}{2}\right)^{\frac{1}{p}}\right)\frac{du}{2}\\
&+ \frac{\alpha}{2(b^p-a^p)^\alpha}\int_0^{b^p-a^p} f\left(\left[\frac{a^p+b^p}{2}+\frac{tu}{2}\right]^{\frac{1}{p}}\right)\\
&\times\left[\left(\frac{b^p-a^p-u}{2}\right)^{\alpha-1}+\left(\frac{b^p-a^p+u}{2}\right)^{\alpha-1}\right]w\left(\left(\frac{a^p+b^p+u}{2}\right)^{\frac{1}{p}}\right)\frac{du}{2}
\end{aligned}$$

Since the function w is p-symmetrically according to the $\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}$

$$w\left(\left(\frac{a^p+b^p-u}{2}\right)^{\frac{1}{p}}\right)=w\left(\left(\frac{a^p+b^p+u}{2}\right)^{\frac{1}{p}}\right)$$

So

$$\begin{aligned}
W_{p,w}(t) &= \frac{\alpha}{2(b^p-a^p)^\alpha}\int_0^{b^p-a^p}\frac{1}{2}\left[f\left(\left[\frac{a^p+b^p}{2}-\frac{tx}{2}\right]^{\frac{1}{p}}\right)+f\left(\left[\frac{a^p+b^p}{2}+\frac{tx}{2}\right]^{\frac{1}{p}}\right)\right]\\
&\times\left[\left(\frac{b^p-a^p-x}{2}\right)^{\alpha-1}+\left(\frac{b^p-a^p+x}{2}\right)^{\alpha-1}\right]w\left(\left(\frac{a^p+b^p+x}{2}\right)^{\frac{1}{p}}\right)dx.
\end{aligned}$$

It follows from Lemma (4) that $h(x)=\frac{1}{2}\left[f\left(\left[\frac{a^p+b^p}{2}-\frac{tx}{2}\right]^{\frac{1}{p}}\right)+f\left(\left[\frac{a^p+b^p}{2}+\frac{tx}{2}\right]^{\frac{1}{p}}\right)\right]$ is increasing on $[0,b^p-a^p]$. Since $\left[\left(\frac{b^p-a^p}{2}-\frac{x}{2}\right)^{\alpha-1}+\left(\frac{b^p-a^p}{2}+\frac{x}{2}\right)^{\alpha-1}\right]$ is nonnegative, hence $W_{p,w}(t)$ is monotonically increasing on $[0,1]$. Finally, using by monotonically increasing of $W_{p,w}(t)$, we have

$$\begin{aligned} f\left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)\left[J_{a^p+}^{\alpha}(wog)(b^p)+J_{b^p-}^{\alpha}(wog)(a^p)\right] &= W_{p,w}(0)\le W_{p,w}(t)\le W_{p,w}(1)\\ &=\frac{\Gamma(\alpha+1)}{2(b^p-a^p)^{\alpha}}\left[J_{a^p+}^{\alpha}(fwog)(b^p)+J_{b^p-}^{\alpha}(fwog)(a^p)\right].\end{aligned}$$

(ii) Similar proof can be made for $p<0$.
We have completed the proof.

In Theorem (10), If we take $p=1$, then we get the following result for convex functions.

Corollary 3. *Let $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ be a function and $f\in L[a,b]$ and $w:[a,b]\to\mathbb{R}$ is integrable, nonnegative and symmetrically according to $\frac{a+b}{2}$. If the function f is convex, then $W_{1,w}:[0,1]\to\mathbb{R}$ defined by*

$$\begin{aligned} W_{1,w}(t)=\frac{\alpha}{2(b-a)^{\alpha}}\int_a^b f\left[\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)\right]\\ \times\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]w(x)dx\end{aligned}$$

is convex and monotonically increasing on $[0,1]$ and

$$\begin{aligned} f\left(\left(\frac{a+b}{2}\right)\right)\left[J_{a+}^{\alpha}w(b)+J_{b-}^{\alpha}w(a)\right] &= W_{1,w}(0)\le W_{1,w}(t)\le W_{1,w}(1)\\ &=\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}(fw)(b)+J_{b-}^{\alpha}(fw)(a)\right]\end{aligned}$$

where $g(x)=x$ and $\alpha>0$.

Remark. The inequality obtained in Corollary (3) gives us the left hand side of the inequality obtained in Theorem (4).

Corollary 4. *Let $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ be a function and $f\in L[a,b]$ and $w:[a,b]\to\mathbb{R}$ is integrable, nonnegative and symmetrically according to $\frac{a+b}{2}$. If the function f is convex, then $W_{-1,w}:[0,1]\to\mathbb{R}$ defined by*

$$\begin{aligned} W_{-1,w}(t)=\frac{\alpha}{2\left(\frac{1}{a}-\frac{1}{b}\right)^{\alpha}}\int_{\frac{1}{a}}^{\frac{1}{b}} f\left[\left(tx+(1-t)\left(\frac{a+b}{2ab}\right)\right)^{-1}\right]\\ \times\left[\left(x-\frac{1}{b}\right)^{\alpha-1}+\left(\frac{1}{a}-x\right)^{\alpha-1}\right]w(x^{-1})dx\end{aligned}$$

is convex and monotonically increasing on $[0,1]$ and

$$\begin{aligned} & f\left(\frac{2ab}{a+b}\right)\left[J_{\frac{1}{b}+}^{\alpha}(wog)\left(\frac{1}{a}\right)+J_{\frac{1}{a}-}^{\alpha}(wog)\left(\frac{1}{b}\right)\right]\\ &= W_{-1,w}(0)\le W_{-1,w}(t)\le W_{-1,w}(1)\\ &=\frac{\Gamma(\alpha+1)}{2\left(\frac{1}{a}-\frac{1}{b}\right)^{\alpha}}\left[J_{\frac{1}{b}+}^{\alpha}(fwog)\left(\frac{1}{a}\right)+J_{\frac{1}{a}-}^{\alpha}(fwog)\left(\frac{1}{b}\right)\right]\end{aligned}$$

where $g(x)=\frac{1}{x}$ and $\alpha>0$.

Remark. The inequality obtained in Corollary (4) gives us the left hand side of the inequality obtained in Theorem (8).

3 Conclusion

In this paper, we obtain some new Hermite-Hadamard and Hermite-Hadamard-Fej ér type inequalities for p-convex functions via Riemann-Liouville fractional integrals. We conclude that the results obtained in this work are the refinements of the earlier results. An interesting topic is whether we can use the methods in this paper to establish the left side hand of Hermite-Hadamard and Hermite-Hadamard-Fejér inequalities for p-convex functions via Hadamard fractional integrals.

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