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Non-homogeneous time fractional heat equation

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Abstract: In this article, the author considered certain non-homogeneous time fractional heat equation which is a generalization of the problem of a viscous ring damper for a freely processing satellite. Transform method is a powerful tool for solving partial fractional differential equations. The result reveals that the transform method is very convenient and effective.

Keywords: Caputo fractional derivative, Non-homogeneous time fractional heat equation, Laplace transform, Fourier transform.

1 Introduction

Fractional differential equations arise in the unification of diffusion and wave propagation phenomenon. The time fractional heat equation, which is a mathematical model of a wide range of important physical phenomena, is a partial differential equation obtained from the classical heat equation by replacing the first time derivative of a fractional derivative of order. In recent years, it has turned out that many phenomena in fluid mechanics, physics, biology, engineering and other areas of the sciences can be successfully modeled by the use of fractional derivatives. That is because of the fact that, a realistic modeling of a physical phenomenon having dependence not only at the time instant, but also the previous time history can be successfully achieved by using fractional calculus. In this work, we consider methods and results for the partial fractional diffusion equations which arise in applications. Several methods have been introduced to solve fractional differential equations, the popular Laplace transform method, [1,2,3]. Atanackovic and Stankovic [4,5] and Stankovic [13] used the Laplace transform in a certain space of distributions to solve a system of partial differential equations with fractional derivatives, and indicated that such a system may serve as a certain model for a visco-elastic rod. Wyss [15] and Schneider [12] considered the time fractional diffusion and wave equations and obtained the solution in terms of Fox functions. In recent years, the implementations of extended G/G- method for the solutions of nonlinear evolution equations, nonlinear Klein - Gordon equations, Boussinesq equations have been well-established by researchers [14].

1.1 Definitions And Notations

Definition 1. *The left Caputo fractional derivative of order* α $(0<\alpha<1)$ *of* $\phi(t)$ *is as follows [8]*

$$D_a^{c,\alpha}\phi(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t\frac{1}{(t-\xi)^{\alpha}}\phi'(\xi)d\xi. \tag{1}$$

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Definition 2. *The Laplace transform of function $f(t)$ is defined as follows*

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s) \tag{2}$$

If $\mathscr{L}\{f(t)\} = F(s)$, then $\mathscr{L}^{-1}\{F(s)\}$ is given by

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st} F(s)ds, \tag{3}$$

where $F(s)$ is analytic in the region $\mathrm{Re}(s) > c$. The above integral is known as Bromwich complex inversion formula

Lemma 1. *Let $\mathscr{L}\{f(t)\} = F(s)$ then, the following identities hold true.*

1. $\mathscr{L}^{-1}(e^{-k\sqrt{s}}) = \frac{k}{(2\sqrt{\pi})}\int_0^\infty e^{-t\xi-\frac{k^2}{4\xi}}d\xi,$
2. $e^{-\omega s^\beta} = \frac{1}{\pi}\int_0^\infty e^{-r^\beta(\omega cos\beta\pi)} sin(\omega r^\beta sin\beta\pi)(\int_0^\infty e^{-s\tau-r\tau}d\tau)dr,$
3. $\mathscr{L}^{-1}(F(s^\alpha)) = \frac{1}{\pi}\int_0^\infty f(u)\int_0^\infty e^{-tr-ur^\alpha cos\alpha\pi} sin(ur^\alpha sin\alpha\pi)drdu,$
4. $\mathscr{L}^{-1}(F(\sqrt{s}) = \frac{1}{2t\sqrt{\pi t}}\int_0^\infty ue^{-\frac{u^2}{4t}} f(u)du.$

Proof. [1,2]

Example 1. By using an appropriate integral representation for the modified Bessel's functions of the second kind of order ν, $K_\nu(s)$, show that

$$\mathscr{L}^{-1}\{\frac{K_\eta(a\sqrt{s-\mu})}{(s-\mu)^{\frac{\eta}{2}}}\frac{K_\nu(b\sqrt{s+\beta})}{(s+\beta)^{\frac{\nu}{2}}}\} = e^{(\mu-\beta)t}\int_0^t \frac{\tau^{\eta-1}e^{\mu\tau-\frac{a^2}{4\tau}}}{(2a)^{1+\eta}}\frac{(t-\tau)^{\nu-1}e^{-\beta(t-\tau)-\frac{b^2}{4(t-\tau)}}}{(2b)^{1+\nu}}d\tau. \tag{4}$$

Solution 1. It is well known that $K_\nu(a\sqrt{s})$ has the following integral representation [6]

$$K_\nu(a\sqrt{s}) = \frac{(a\sqrt{s})^\nu}{2^{\nu+1}}\int_0^\infty e^{-\xi-\frac{a^2 s}{4\xi}}\frac{d\xi}{\xi^{\nu+1}}. \tag{5}$$

At this stage, using complex inversion formula for the Laplace transforms and the above integral representation we get

$$\mathscr{L}^{-1}\{\frac{K_\eta(a\sqrt{s})}{s^{\frac{\eta}{2}}}\} = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\frac{1}{s^\eta}(\frac{(a\sqrt{s})^{0.5\eta}}{2^{\eta+1}}\int_0^\infty e^{-\xi-\frac{a^2 s}{4\xi}}\frac{d\xi}{\xi^{\eta+1}})ds. \tag{6}$$

Changing the order of integration and simplifying to obtain

$$\mathscr{L}^{-1}\{\frac{K_\eta(a\sqrt{s})}{s^{\frac{\eta}{2}}}\} = a^\eta\int_0^\infty \frac{e^{-\xi}}{\xi^{\eta+1}}(\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\frac{e^{ts-\frac{a^2 s}{4\xi}}}{2^{\eta+1}}ds)d\xi. \tag{7}$$

The value of the inner integral is $\delta(t-\frac{a^2}{4\xi})$, we have the following

$$\mathscr{L}^{-1}\{\frac{K_\eta(a\sqrt{s})}{s^{\frac{\eta}{2}}}\} = a^\eta\int_0^\infty \frac{e^{-\xi}}{\xi^{\eta+1}}\delta(t-\frac{a^2}{4\xi})d\xi, \tag{8}$$

making a change of variable $(t-\frac{a^2}{4\xi}) = u$ and using elementary properties of Dirac - delta function, we arrive at

$$\mathscr{L}^{-1}\{\frac{K_\eta(a\sqrt{s})}{s^{\frac{\eta}{2}}}\} = \frac{t^{\eta-1}e^{-\frac{a^2}{4t}}}{(2a)^{\eta+1}}. \tag{9}$$

Finally, using the shift and convolution theorem we obtain

$$\mathscr{L}^{-1}\{\frac{K_\eta(a\sqrt{s-\mu})}{(s-\mu)^{\frac{\eta}{2}}}\frac{K_\nu(b\sqrt{s+\beta})}{(s+\beta)^{\frac{\nu}{2}}}\}=e^{(\mu-\beta)t}\int_0^t\frac{\tau^{\eta-1}e^{\mu\tau-\frac{a^2}{4\tau}}}{(2a)^{1+\eta}}\frac{(t-\tau)^{\nu-1}e^{-\beta(t-\tau)-\frac{b^2}{4(t-\tau)}}}{(2b)^{1+\nu}}d\tau. \tag{10}$$

Definition 3. *The The Laplace transform of Caputo fractional derivatives of order non integer. For $n-1<\alpha\leq n$, we have the following identity [15]*

$$L\{{}_0^CD_t^\alpha f(t)\}=s^\alpha F(s)-\sum_{k=0}^{n-1}s^{\alpha-k-1}f^{(k)}(0). \tag{11}$$

Definition 4. *The The two-parameter function of the Mittag-Leffler type is defined by the series expansion*

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}, \tag{12}$$

when $\alpha,\beta,z\in C$. We have the following relationship

$$L\{t^{\beta-1}E_{\alpha,\beta}(\pm at^\alpha)\}=\frac{s^{\alpha-\beta}}{s^\alpha\mp a}(Re(s)>|a|^{\frac{1}{\alpha}}. \tag{13}$$

Definition 5. *The simplest Wright function is given by the series*

$$W(\alpha,\beta;z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(\alpha n+\beta)}, \tag{14}$$

when $\alpha,\beta,z\in C$. We have the following relationship

$$[L\{t^{\beta-1}E_{\alpha,\beta}(\pm at^\alpha)\}=\frac{s^{\alpha-\beta}}{s^\alpha\mp a}(Re(s)>|a|^{\frac{1}{\alpha}}). \tag{15}$$

Lemma 2. *The following identities hold true for $0<\nu<1$.*

$$\begin{aligned} g(t)=&\mathscr{L}^{-1}(\int_0^a\frac{e^{-(1+\frac{k}{s^\nu+k})\eta}}{s^\nu+k}d\eta)=..\\ =&e^{-kt}\int_0^\infty\int_0^a e^{-u}J_0(2\sqrt{ktu})(\int_0^\infty e^{-tr-\eta r^\nu cos\pi\nu}sin(\eta r^\nu sin\pi\nu)dr)du)d\eta. \end{aligned}$$

Proof. Let us assume that $F(s)=\int_0^a\frac{e^{-(1+\frac{k}{s+k})\eta}}{s+k}d\eta$
by complex inversion formula for the Laplace transforms, we have

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{ts}(\int_0^a\frac{e^{-(1+\frac{k}{s+k})\eta}}{s+k}d\eta)ds,$$

changing the order of integration which is permissible by Fubini's theorem, leads to

$$f(t)=\int_0^a e^{-\eta}(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{ts-\frac{k\eta}{s+k}}}{s+k}ds)d\eta,$$

in the inner integral by making change of variable $s+k=\xi$, we obtain

$$f(t)=e^{-kt}\int_0^a e^{-\eta}(\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{e^{t\xi-\frac{k\eta}{\xi}}}{\xi}d\xi)d\eta,$$

the value of the inner integral is $J_0(2\sqrt{kt\eta})$ [7,8] so that

$$f(t)=e^{-kt}\int_0^a e^{-\eta}J_0(2\sqrt{kt\eta})d\eta,$$

using part three of (1) and theorem of Titchmarsh [8] leads to

$$g(t)=e^{-kt}\int_0^\infty\int_0^a e^{-u}J_0(2\sqrt{ktu})(\int_0^\infty e^{-tr-\eta r^\nu cos\pi\nu}sin(\eta r^\nu sin\pi.\nu)dr)du)d\eta.$$

2 Main Results

In this section, the authors considered certain non-homogeneous time fractional heat equations which is a generalization of the problem of a viscous ring damper for a freely processing satellite studied by P.G.Bahuta [8]. In this study, only the Laplace transformation is considered as it is easily understood and being popular among engineers and scientists. The basic goal of this work has been to implement the Laplace transform method for studying the above mentioned problem. The goal has been achieved by formally deriving the exact analytical solution.

2.1 Non homogeneous time fractional heat equation

Problem 1. Let us solve the following partial fractional differential equation

$$\begin{gathered}D_t^{C,\alpha}u=\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})-\lambda u+\mu,\\ \lambda,\mu>0,0\le r\le a,u(r,0)=\beta,u(a,t)=\exp-\omega^2t,\\ lim_{r->0}|u(r,t)|\le M.\end{gathered}\tag{16}$$

Solution 2. Solution: In order to obtain the solution of the fractional heat equation, the Laplace transform is applied to PDE and boundary conditions to obtain

$$s^\alpha U(r,s)-\beta s^{\alpha-1}=\frac{1}{r}(U_r(r,s)+rU_{rr}(r,s))-\lambda Ur,s)+\frac{\mu}{s},\tag{17}$$

after simplifying, we get the following

$$U_{rr}(r,s)+\frac{1}{r}U_r(r,s)-(s^\alpha+\lambda)U(r,s)=-(\beta s^{\alpha-1}+\frac{\mu}{s}).\tag{18}$$

Hence, the homogeneous equation is

$$[U_{rr}(r,s)+\frac{1}{r}U_r(r,s)-(s^\alpha+\lambda)U(r,s)=0.\tag{19}$$

The equation (19) is modified Bessel differential equation of order zero. The general solution is

$$U_p(r,s) = AI_0(r\sqrt{s^\alpha+\lambda}) + BK_0(r\sqrt{s^\alpha+\lambda}). \tag{20}$$

The function K_0 for some r, is unbounded. However, $U_p(r,s)$ is a bounded function. Therefore $B=0$ and (20) read

$$U_p(r,s) = AI_0(r\sqrt{s^\alpha+\lambda}). \tag{21}$$

Now, in order to obtain the solution of nonhomogeneous equation (16), we suppose that $U_c(r,s)=\gamma$ is the solution to non-homogeneous equation.
Then we get the following

$$U_c(r,s) = \frac{s^{\alpha-1}\beta+\frac{\mu}{s}}{s^\alpha+\lambda} = \gamma. \tag{22}$$

From relations (21) and (22), we obtain

$$U(r,s) = U_p(r,s)+U_c(r,s) = AI_0(r\sqrt{s^\alpha+\lambda}) + \frac{s^{\alpha-1}\beta+\frac{\mu}{s}}{s^\alpha+\lambda}. \tag{23}$$

In order to obtain constant A, in relation (23), we use boundary condition to get

$$U(a,s) = \frac{1}{s+\omega^2} = AI_0(a\sqrt{s^\alpha+\lambda}) + \frac{s^{\alpha-1}\beta+\frac{\mu}{s}}{s^\alpha+\lambda}, \tag{24}$$

from the above relation, we get the value of constant A as below

$$A = \left(\frac{1}{s+\omega^2} - \frac{s^{\alpha-1}\beta+\frac{\mu}{s}}{s^\alpha+\lambda}\right)\frac{1}{I_0(a\sqrt{s^\alpha+\lambda})}. \tag{25}$$

By substitution of the value of A in relation (23), we obtain the general solution to non - homogeneous equation in the following form

$$U(r,s) = \left(\frac{1}{s+\omega^2} - \frac{s^{\alpha-1}\beta+\frac{\mu}{s}}{s^\alpha+\lambda}\right)\frac{I_0(r\sqrt{s^\alpha+\lambda})}{I_0(a\sqrt{s^\alpha+\lambda})} + \frac{s^{\alpha-1}\beta+\frac{\mu}{s}}{s^\alpha+\lambda}. \tag{26}$$

In case $\alpha=1$, we have

$$U(r,s) = \left(\frac{1}{s+\omega^2} - \frac{\beta+\frac{\mu}{s}}{s+\lambda}\right)\frac{I_0(r\sqrt{s+\lambda})}{I_0(a\sqrt{s+\lambda})} + \frac{\beta+\frac{\mu}{s}}{s+\lambda}. \tag{27}$$

At this stage, the Bromwich's integral is utilized to invert $U(r,s)$ as follows.

$$u(r,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\left(\frac{1}{s+\omega^2} - \frac{\beta+\frac{\mu}{s}}{s+\lambda}\right)\frac{I_0(r\sqrt{s+\lambda})}{I_0(a\sqrt{s+\lambda})} + \frac{\beta+\frac{\mu}{s}}{s+\lambda}\right)e^{ts}ds = .. \tag{28}$$
$$.... = \sum_k \left(Res\left[\left(\left(\frac{1}{s+\omega^2} - \frac{\beta+\frac{\mu}{s}}{s+\lambda}\right)\frac{I_0(r\sqrt{s+\lambda})}{I_0(a\sqrt{s+\lambda})} + \frac{\beta+\frac{\mu}{s}}{s+\lambda}\right)e^{ts}\right], s=s_k,\right).$$

The singularities of the integrand are $s=0, s=-\lambda, s=-\omega^2, s=-\frac{k_n^2}{a^2}, n=1,2,3,...$, then the residues at the singularities of (28) are as follows

1. At $s=0$, we have

$$b_{-1}^1 = lim_{s->0}\left(\left(\frac{1}{s+\omega^2} - \frac{\beta+\frac{\mu}{s}}{s+\lambda}\right)\frac{I_0(r\sqrt{s+\lambda})}{I_0(a\sqrt{s+\lambda})} + \frac{\beta+\frac{\mu}{s}}{s+\lambda}\right)se^{ts} = \frac{\mu}{\lambda}\left(1-\frac{I_0(r\sqrt{\lambda})}{I_0(a\sqrt{\lambda}}\right).$$

2. At $s=-\lambda$, we have

$$b_{-1}^{2}=lim_{s->-\lambda}((\frac{1}{s+\omega^2}-\frac{\beta+\frac{\mu}{s}}{s+\lambda})\frac{I_0(r\sqrt{s+\lambda})}{I_0(a\sqrt{s+\lambda})}+\frac{\beta+\frac{\mu}{s}}{s+\lambda})(s+\lambda)e^{ts}=0.$$

3. At $s=-\omega^2$, we have

$$b_{-1}^{3}=lim_{s->-\omega^2}((\frac{1}{s+\omega^2}-\frac{\beta+\frac{\mu}{s}}{s+\lambda})\frac{I_0(r\sqrt{s+\lambda})}{I_0(a\sqrt{s+\lambda})}+\frac{\beta+\frac{\mu}{s}}{s+\lambda})(s+\omega^2)e^{ts}=\frac{I_0(r\sqrt{\lambda-\omega^2})}{I_0(a\sqrt{\lambda-\omega^2})}e^{-\omega^2 t}.$$

4. If $k_1,k_2,k_3,.....$ are the roots of the function $J_0(\xi)$, then $J_0(k_n)=0$ for $n=1,2,3,....$ Using the fact that $I_0(\xi)=J_0(i\xi)$, one gets $\xi=-ik_n$, the roots of $I_0(a\sqrt{s+\lambda})$, are $s_n=-(\lambda+\frac{k_n^2}{a^2})$. Finally, the residues at $s_n=-(\lambda+\frac{k_n^2}{a^2})$, are At $s=-(\lambda+\frac{k_n^2}{a^2})$, we have

$$b_{-1}^{4}=lim_{s->-(\lambda+\frac{k_n^2}{a^2})}((\frac{1}{s+\omega^2}-\frac{\beta+\frac{\mu}{s}}{s+\lambda})\frac{I_0(r\sqrt{s+\lambda})}{I_0(a\sqrt{s+\lambda})}+\frac{\beta+\frac{\mu}{s}}{s+\lambda})(s+\lambda)e^{ts}=...$$

$$....=e^{-(\frac{k_n^2}{a^2}-\lambda)}\frac{J_0(\frac{r}{a}k_n)(\frac{k_n^2}{a^2})(\frac{k_n^2}{a^2}-\lambda)+\mu(\lambda+\frac{k_n^2}{a^2}-\omega^2)-\beta(\lambda+\frac{k_n^2}{a^2}-\omega^2)(\frac{k_n^2}{a^2})}{J_1(k_n)k_n(\lambda+\frac{k_n^2}{a^2})(\lambda+\frac{k_n^2}{a^2}-\omega^2)}.$$

Let us suppose that $\lambda=\mu=0$ and $\beta=1$. Hence, $u(r,t)$ is in the following form

$$u(r,t)=e^{-\omega^2 t}\frac{J_0(\omega r)}{J_1(\omega a)}+2a^2\sum_k\frac{e^{-(\frac{k_n^2}{a^2})}J_0((\frac{r}{a})k_n)}{k_n(a^2-\frac{k_n^2}{\omega^2})J_1(k_n)}. \tag{29}$$

In case $\alpha=0.5$, (semi - derivative) we have

$$U(r,s)=(\frac{1}{s+\omega^2}-\frac{\frac{\beta}{\sqrt{s}}+\frac{\mu}{s}}{\sqrt{s}+\lambda})\frac{I_0(r\sqrt{\sqrt{s}+\lambda})}{I_0(a\sqrt{\sqrt{s}+\lambda})}+\frac{\frac{\beta}{\sqrt{s}}+\frac{\mu}{s}}{\sqrt{s}+\lambda}, \tag{30}$$

the above relation can be re-writen as below

$$U(r,s)=\frac{\beta\sqrt{s}+\mu}{s(\sqrt{s}+\lambda)}+\frac{1}{s+\omega^2}\frac{I_0(r\sqrt{\sqrt{s}+\lambda})}{I_0(a\sqrt{\sqrt{s}+\lambda})}-\frac{\beta\sqrt{s}+\mu}{s(\sqrt{s}+\lambda)}\frac{I_0(r\sqrt{\sqrt{s}+\lambda})}{I_0(a\sqrt{\sqrt{s}+\lambda})}. \tag{31}$$

At this point, we find inversion of the above relation term wise, so that

$$u(r,t)=\mathscr{L}^{-1}(\frac{\beta}{\sqrt{s}(\sqrt{s}+\lambda)})+\mathscr{L}^{-1}(\frac{\mu}{s(\sqrt{s}+\lambda)})+\mathscr{L}^{-1}(\frac{1}{s+\omega^2})*\mathscr{L}^{-1}(\frac{I_0(r\sqrt{\sqrt{s}+\lambda})}{I_0(a\sqrt{\sqrt{s}+\lambda})})-.... \tag{32}$$

$$....-\mathscr{L}^{-1}(\frac{\beta\sqrt{s}+\mu}{s(\sqrt{s}+\lambda)}.\frac{I_0(r\sqrt{\sqrt{s}+\lambda})}{I_0(a\sqrt{\sqrt{s}+\lambda})}).$$

Let us introduce the following

$$G(r,\sqrt{s})=(\frac{\beta\sqrt{s}+\mu}{(\sqrt{s}+\lambda)}.\frac{I_0(r\sqrt{\sqrt{s}+\lambda})}{I_0(a\sqrt{\sqrt{s}+\lambda})}), \tag{33}$$

then

$$G(r,s)=(\frac{\beta s+\mu}{(s+\lambda)}.\frac{I_0(r\sqrt{s+\lambda})}{I_0(a\sqrt{s+\lambda})}), \tag{34}$$

and

$$g(r,t) = \mathscr{L}^{-1}[G(r,s)]. \tag{35}$$

At this point, in order to invert (33), we may use the tables of inverse Laplace -transforms or part four of (1) to get

$$v(r,t) = \mathscr{L}^{-1}[G(r,\sqrt{s})] = \int_0^{+\infty} \frac{\eta e^{-\frac{\eta^2}{4t}}}{2t\sqrt{\pi t}} g(r,\eta)d\eta,$$

in order to invert (34), we need to evaluate the residues at all the singularties. The singularties of (34) are as follows

1. A simple pole at $s = -\lambda$.
2. If $k_1, k_2, ...k_n, ..$ are the roots of the function $J_0(x)$, thus for $J_0(k_n) = 0$.and $I_0(x) = J_0(ix)$ therefore, ik_n are the roots of $I_0(x)$. Hence, the roots of $I_0(a\sqrt{s+\lambda})$ are $s_n = -\frac{a^2\lambda + k_n^2}{a^2}$.
At this point, we use the Bromwich's integral to invert (35) the residue at simple pole $s = -\lambda$ is (35) the residue at simple pole $s = -\lambda$ is

(i)

$$b^1_{-1} = lim_{s->-\lambda}(s+\lambda)(\frac{\beta s+\mu}{(s+\lambda)}.\frac{I_0(r\sqrt{s+\lambda})}{I_0(a\sqrt{s+\lambda})})e^{st} = 0,$$

the residue at poles $s = -\frac{\lambda a^2 + k_n^2}{a^2}$ is,
(ii)

$$b^2_{-1} = lim_{s->-\frac{\lambda a^2+k_n^2}{a^2}}(s+\frac{\lambda a^2+k_n^2}{a^2})(\frac{\beta s+\mu}{(s+\lambda)}.\frac{I_0(r\sqrt{s+\lambda})}{I_0(a\sqrt{s+\lambda})})e^{st},$$

the above limit can be written as follows

$$b^2_{-1} = lim_{s->-\frac{\lambda a^2+k_n^2}{a^2}} \frac{(\beta s+\mu)(I_0(r\sqrt{s+\lambda})}{(s+\lambda)} e^{st} \frac{1}{\frac{I_0(a\sqrt{s+\lambda})-I_0(ik_n)}{s-(-\frac{\lambda a^2+k_n^2}{a^2})}}).$$

By using the relations, $I_0(x) = J_0(ix)$ and $J_0' = J_1$, we obtain

$$(I_0(a\sqrt{s+\lambda}))' = (J_0(ia\sqrt{s+\lambda}))' = -\frac{ai}{2\sqrt{s+\lambda}}(J_1(ia\sqrt{s+\lambda}).$$

At this stage, let us take the limit as s tends to $s_n = -\frac{a^2\lambda+k_n}{a^2}$, we arrive at

$$lim_{s->s_n}(I_0(a\sqrt{s+\lambda}))' = \frac{a^2}{2k_n}J_1(k_n).$$

Finally, we may find the residue b^2_{-1} as below

$$b^2_{-1} = 2e^{-(\frac{a^2\lambda+k_n}{a^2})t}(\mu - \beta\lambda - \frac{\beta k_n^2}{a^2})\frac{J_0(\frac{rk_n}{a})}{J_1(k_n)},$$

by following the same procedure, we may find b^3_{-1} as below

$$\mathscr{L}^{-1}(\frac{I_0(r\sqrt{s+\lambda})}{I_0(a\sqrt{s+\lambda})}) = b^3_{-1} = lim_{s->-\frac{\lambda a^2+k_n^2}{a^2}}(s+\frac{\lambda a^2+k_n^2}{a^2})(\frac{I_0(r\sqrt{s+\lambda})}{I_0(a\sqrt{s+\lambda})})e^{st},$$

after simplifying we get

$$b_{-1}^3 = 2e^{-(\frac{a^2\lambda+k_n}{a^2})t}(\mu-\beta\lambda-\frac{\beta k_n^2}{a^2})\frac{J_0(\frac{rk_n}{a})}{(\frac{a^2}{2k_n})J_1(k_n)}.$$

After substitution of the value for each term in relation (32), we get the formal solution as follows

$$u(r,t) = \beta e^{\lambda^2 t}Erfc(\lambda\sqrt{t}) + \frac{\mu}{\sqrt{\pi}}\int_0^t \frac{e^{\lambda^2\eta}Erfc(\lambda\sqrt{\eta})}{\sqrt{t-\eta}}d\eta +$$

$$.....+\sum_{n=1}^{n=+\infty} e^{-\omega^2 t} * 2e^{-(\frac{a^2\lambda+k_n}{a^2})t}\frac{J_0(\frac{rk_n}{a})}{(\frac{a^2}{2k_n})J_1(k_n)} -$$

$$\sum_{n=1}^{n=+\infty} 2e^{-(\frac{a^2\lambda+k_n}{a^2})t}(\mu-\beta\lambda-\frac{\beta k_n^2}{a^2})\frac{J_0(\frac{rk_n}{a})}{J_1(k_n)}(H(t) * \frac{1}{t\sqrt{\pi t}}\int_0^\infty \xi e^{-\frac{\xi^2}{4t}-\frac{(k_n^2+a^2\lambda)\xi}{a^2}})d\xi.$$

In the above relation * is convolution for the Laplace transform and $H(.)$, stands for the Heaviside unit step function.

3 Conclusions

The main purpose of this work is to develop a method for finding an exact analytic solution of the time fractional heat equation. In this work, the author considered the time fractional heat equation (Time fractional in the Caputo sense). Many linear boundary value and initial value problems in applied mathematics, mathematical physics, and engineering science can be effectively solved by the use of the Fourier transform, the Laplace transform, the Fourier cosine/sine transform. The Fourier and Laplace type integral transform are wonderful alternative methods for solving different types of PDEs of fractional order. There are a lot of applications of PFDEs in the field of Visco elasticity as well.The paper is devoted to study applications of one dimensional Laplace transforms in details. One dimensional Laplace transforms provides a powerful method for analyzing linear systems. The transform method introduces a significant improvement in this field over existing techniques. We hope that it will also benefit many researchers in the disciplines of applied mathematics, mathematical physics and engineering.

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