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Some quantum estimates of opial inequality and some of its generalizations

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Abstract: In this paper, we obtain $(p,q)-$analogues of Opial integral inequality, involving $(p,q)-$derivatives of functions. We derive also some of the generalized versions of this integral inequality.

Keywords: Opial inequality; (p,q)-derivatives; (p,q)-integral; Hölder's inequality.

1 Introduction

In 1960 Opial established an integral inequality involving integrals of a function and its derivative which is stated as:

Theorem 1. *[20]Let f be continuous function on $[0,h]$ with $f(0)=f(h)=0$, and $f(t)>0$ in $t\in(0,h)$. Then*

$$\int_0^h \left|f(t)f'(t)\right|dt \le \frac{h}{4}\int_0^h \left[f'(t)\right]^2 dt \tag{1}$$

where the constant $h/4$ is the best possible.

Opial type inequalities play an important role in ordinary and partial differential equations for establishing the existence and uniqueness of initial and boundary value problems. In recent years, there have been numerous study deals with the simple proofs, various generalizations, discrete and fractional analogues of Opial inequality and some of its generalizations. In [29], the authors considered $q-$analogues of some Opial type inequalities and in [25] the authors prove some Opial type inequalities for conformable fractional integrals. See also, [1]-[7], [11]-[23], [26], [30], [31].

The object of this paper is to find $(p,q)-$analogues of Opial inequality and some of its generalized forms. Now, we give some theorems of generalization of Opial inequality which we use in the sequel.

Theorem 2. *[26]Let ω be a nonnegative and continuous function on $[0,a]$. Let u be an absolutely continuous function on $[0,a]$ with $u(0)=u(a)=0$. Then the following inequalities hold:*

$$\int_0^a \omega(t)\left|u(t)\right|^2 dt \le \frac{a}{4}\left(\int_0^a \omega(t)dt\right)\left(\int_0^a \left|f'(t)\right|^2 dt\right) \tag{2}$$

and

$$\int_0^a \omega(t)\left|u(t)u'(t)\right|^2 dt \le \left(\frac{a}{4}\int_0^a \omega^2(t)dt\right)^{\frac{1}{2}}\left(\int_0^a \left|f'(t)\right|^2 dt\right). \tag{3}$$

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2 Preliminaries

In this section, we recall some previously known concepts and basic results. The $(p,q)-$integers were introduced in order to generalize or unify several forms of $q-$oscillator algebras well known in the earlier physics literature related to the representation theory of single parameter quantum algebras. $(p,q)-$calculus is a generalization of $q-$calculus with the rule $0<q<p\leq 1$. The $(p,q)-$integers $[n]_{p,q}$ are defined by

$$[n]_{p,q}=\frac{p^n-q^n}{p-q}.$$

For each $k,n\in\mathbb{N}$, $n\geq k\geq 0$, the $(p,q)-$factorial and $(p,q)-$binomial are defined by

$$[n]_{p,q}!=\prod_{k=1}^{n}[k]_{p,q},\ n\geq 1,\ [0]_{p,q}!=1$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}=\frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!}.$$

Definition 1. *Let $f:\mathbb{R}\to\mathbb{R}$. The $(p,q)-$derivative of the function f is defined as*

$$D_{p,q}f(x)=\frac{f(px)-f(qx)}{(p-q)x},\ x\neq 0 \tag{4}$$

provided that $D_{p,q}f(0)=f'(0)$.

$(p,q)-$derivative of a function is a linear operator. For any constants a and b,

$$D_{p,q}[af(x)+bg(x)]=aD_{p,q}f(x)+bD_{p,q}f(x).$$

The $(p,q)-$derivative of a product is given as

$$\begin{aligned} D_{p,q}[f(x)g(x)] &= f(px)D_{p,q}g(x)+g(qx)D_{p,q}f(x), \\ &= g(px)D_{p,q}f(x)+f(qx)D_{p,q}g(x). \end{aligned}$$

The $(p,q)-$derivative fulfills the following product rules

$$\begin{aligned} D_{p,q}\left[\frac{f(x)}{g(x)}\right] &= \frac{g(qx)D_{p,q}f(x)-f(qx)D_{p,q}g(x)}{g(px)g(qx)} \\ &= \frac{g(px)D_{p,q}f(x)-f(px)D_{p,q}g(x)}{g(px)g(qx)}. \end{aligned}$$

The $(p,q)-$power basis is defined by

$$(x\oplus a)_{p,q}^{n}=(x\oplus a)(px\oplus qa)\left(p^2x\oplus q^2a\right)\cdots\left(p^{n-1}x\oplus q^{n-1}a\right)$$

and

$$(x\ominus a)_{p,q}^{n}=(x\ominus a)(px\ominus qa)\left(p^2x\ominus q^2a\right)\cdots\left(p^{n-1}x\ominus q^{n-1}a\right)$$

The following statements hold true:

$$D_{p,q}(x\ominus a)_{p,q}^{n}=[n]_{p,q}(px\ominus a)_{p,q}^{n-1} \tag{5}$$
$$D_{p,q}(\alpha x\ominus a)_{p,q}^{n}=\alpha [n]_{p,q}(\alpha px\ominus a)_{p,q}^{n-1},\alpha\in\mathbb{C}$$
$$D_{p,q}(a\ominus x)_{p,q}^{n}=-[n]_{p,q}(a\ominus qx)_{p,q}^{n-1}.$$

Definition 2. *Let* $f:C[0,a]\to\mathbb{R}$ $(a>0)$ *then the* $(p,q)-$*integration of* f *defined by*

$$\int_0^a f(t)\,d_{p,q}t=(q-p)a\sum_{n=0}^{\infty}\frac{p^n}{q^{n+1}}f\left(\frac{p^n}{q^{n+1}}a\right)\text{ if }\left|\frac{p}{q}\right|<1 \tag{6}$$
$$\int_0^a f(t)\,d_{p,q}t=(p-q)a\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}a\right)\text{ if }\left|\frac{p}{q}\right|>1. \tag{7}$$

The $(p,q)-$integral on an interval defined as

$$\int_a^b f(t)\,d_{p,q}t=\int_0^b f(t)\,d_{p,q}t-\int_0^a f(t)\,d_{p,q}t. \tag{8}$$

If F is an antiderivative of the function f and f is continuous at $t=0$

$$\int_a^b f(t)\,d_{p,q}t=F(b)-F(a) \tag{9}$$

and for any function of f, we have

$$D_{p,q}\int_a^x f(t)\,d_{p,q}t=f(x). \tag{10}$$

The formula of $(p,q)-$integration by parts is given by

$$\int_a^b f(px)D_{p,q}g(x)\,d_{p,q}t=\ f(x)g(x)|_a^b-\int_a^b g(qx)D_{p,q}f(x)\,d_{p,q}t. \tag{11}$$

For more details, see [4,5,8,9,24,27,28] and [12]-[19]. All notions written above reduce to the $q-$analogs when $p=1$, see [10].

3 Main results

Theorem 3. *Let* f *be an absolutely continuous and non-increasing function on* $[0,a]_{p,q}$ *with* $f(0)=f(a)=0$ *and* $0<q<p\le 1$. *Then*

$$\int_0^a \left|f(pt)D_{p,q}f(t)\right|d_{p,q}t\le\frac{a}{4}\int_0^a\left[D_{p,q}f(t)\right]^2 d_{p,q}t. \tag{12}$$

Proof. Assume that

$$y(t)=\int_0^t\left|D_{p,q}f(s)\right|d_{p,q}s\text{ and }z(t)=\int_t^a\left|D_{p,q}f(s)\right|d_{p,q}s. \tag{13}$$

It is easy to see that

$$D_{p,q}y(t)=\left|D_{p,q}f(t)\right|=-D_{p,q}z(t)$$

and $|f(t)|\le y(t)$, $|f(t)|\le z(t)$. So, we have

$$\int_0^{a/2}\left|f(pt)D_{p,q}f(t)\right|d_{p,q}t\le\int_0^{a/2}y(pt)D_{p,q}y(t)\,d_{p,q}t.$$

Under the assumptions and with integration by parts, we obtain

$$\begin{aligned}\int_0^{a/2} y(pt)D_{p,q}y(t)\,d_{p,q}t &= y^2\left(\frac{a}{2}\right) - \int_0^{a/2} y(qt)D_{p,q}y(t)\,d_{p,q}t \\ &\le y^2\left(\frac{a}{2}\right) - \int_0^{a/2} y(pt)D_{p,q}y(t)\,d_{p,q}t \\ &= \frac{1}{2}y^2\left(\frac{a}{2}\right). \end{aligned} \tag{14}$$

Analogously, we have

$$\begin{aligned}\int_{a/2}^{a} \left|f(pt)D_{p,q}f(t)\right| d_{p,q}t &\le -\int_{a/2}^{a} z(t)D_{p,q}z(t)\,d_{p,q}t \\ &= \frac{1}{2}z^2\left(\frac{a}{2}\right). \end{aligned} \tag{15}$$

If we combine (14) and (15), we get

$$\int_0^{a} \left|f(pt)D_{p,q}f(t)\right| d_{p,q}t \le \frac{1}{2}y^2\left(\frac{a}{2}\right) + \frac{1}{2}z^2\left(\frac{a}{2}\right).$$

From the definition of $y(t)$ and Cauchy-Schwarz integral inequality, we have

$$\begin{aligned} y^2\left(\frac{a}{2}\right) &= \left(\int_0^{a/2} \left|D_{p,q}f(t)\right| d_{p,q}t\right)^2 \le \frac{a}{2}\int_0^{a/2} \left|D_{p,q}f(t)\right|^2 d_{p,q}t \\ z^2\left(\frac{a}{2}\right) &= \left(\int_{a/2}^{a} \left|D_{p,q}f(t)\right| d_{p,q}t\right)^2 \le \frac{a}{2}\int_{a/2}^{a} \left|D_{p,q}f(t)\right|^2 d_{p,q}t. \end{aligned}$$

With adding these inequalities, we complete the proof.

Remark. When $p = 1$, (12) reduces to

$$\int_0^{a} \left|f(t)D_qf(t)\right| d_qt \le \frac{a}{4}\int_0^{a} \left[D_qf(t)\right]^2 d_qt$$

which is stated as Theorem 8 in [29]. If both $p = 1$ and $q \to 1$ (12) reduces to (1).

Theorem 4. *Let $r(t)$ be nonnegative and continuous on $[0,h]_{p,q}$. Let f be $(p,q)-$differentiable on $[0,a]_{p,q}$ with $f(0) = f(a) = 0$ and $0 < q < p \le 1$. Then the following inequalities hold:*

$$\int_0^{a} r(t)\left|f(t)\right|^2 d_{p,q}t \le \frac{a}{4}\left(\int_0^{a} r(t)\,d_{p,q}t\right)\left(\int_0^{a} \left|D_{p,q}f(s)\right|^2 d_{p,q}s\right) \tag{16}$$

$$\int_0^{a} r(t)\left|f(t)D_{p,q}f(t)\right| d_{p,q}t \le \frac{a}{4}\left(\int_0^{a} r^2(t)\,d_{p,q}t\right)^{\frac{1}{2}}\left(\int_0^{a} \left|D_{p,q}f(s)\right|^2 d_{p,q}s\right). \tag{17}$$

Proof. Assume that

$$y(t) = \int_0^{t} \left|D_{p,q}f(s)\right| d_{p,q}s \text{ and } z(t) = \int_t^{a} \left|D_{p,q}f(s)\right| d_{p,q}s. \tag{18}$$

It is easy to see that

$$D_{p,q}y(t) = \left|D_{p,q}f(t)\right| = -D_{p,q}z(t) \tag{19}$$

and

$$\left|f(t)\right| \le y(t), \left|f(t)\right| \le z(t). \tag{20}$$

So, we have
$$f(t)\leq\frac{y(t)+z(t)}{2}=\frac{1}{2}\int_0^a\left|D_{p,q}f(s)\right|d_{p,q}s.$$
From Cauchy-Schwarz inequality, we have
$$\begin{aligned}\int_0^a r(t)\left|f(t)\right|^2 d_{p,q}t &\leq \frac{1}{4}\int_0^a r(t)\left(\int_0^a\left|D_{p,q}f(s)\right|d_{p,q}s\right)^2 d_{p,q}t\\ &\leq\frac{1}{4}\left(\int_0^a r(t)\,d_{p,q}t\right)\left(\int_0^a d_{p,q}t\right)\left(\int_0^a\left|D_{p,q}f(s)\right|^2 d_{p,q}s\right)\\ &=\frac{a}{4}\left(\int_0^a r(t)\,d_{p,q}t\right)\left(\int_0^a\left|D_{p,q}f(s)\right|^2 d_{p,q}s\right)\end{aligned}$$
from which we obtain (16). Now, by using (16) and Cauchy-Schwarz inequality, we have
$$\begin{aligned}\int_0^a r(t)\left|f(t)D_{p,q}f(t)\right|d_{p,q}t &\leq\left(\int_0^a r^2(t)\left|f(t)\right|^2 d_{p,q}t\right)^{\frac{1}{2}}\left(\int_0^a\left|D_{p,q}f(t)\right|^2 d_{p,q}t\right)^{\frac{1}{2}}\\ &\leq\frac{a}{4}\left(\int_0^a r^2(t)\,d_{p,q}t\right)^{\frac{1}{2}}\left(\int_0^a\left|D_{p,q}f(s)\right|^2 d_{p,q}s\right)^{\frac{1}{2}}\left(\int_0^a\left|D_{p,q}f(t)\right|^2 d_{p,q}t\right)^{\frac{1}{2}}\\ &=\frac{a}{4}\left(\int_0^a r^2(t)\,d_{p,q}t\right)^{\frac{1}{2}}\left(\int_0^a\left|D_{p,q}f(s)\right|^2 d_{p,q}s\right).\end{aligned}$$
Thus, the proof is completed.

Corollary 1. *Under the assumptions of Theorem 4, when $p=1$ in (16) and (17), we have*
$$\int_0^a r(t)\left|f(t)\right|^2 d_qt\leq\frac{a}{4}\left(\int_0^a r(t)\,d_qt\right)\left(\int_0^a\left|D_{p,q}f(s)\right|^2 d_qs\right)$$
and
$$\int_0^a r(t)\left|f(t)D_{p,q}f(t)\right|d_qt\leq\frac{a}{4}\left(\int_0^a r^2(t)\,d_qt\right)^{\frac{1}{2}}\left(\int_0^a\left|D_{p,q}f(s)\right|^2 d_qs\right)$$
which is the $q-$analogue of the formulas (16) and (17).

Remark. Also if we take both $p=1$ and $q\to 1$, (16) and (17) reduce to (2) and (3).

Theorem 5. *Let $m\geq 0$, $n\geq 1$ and $\lambda\geq 1$ be real numbers. Let f be $(p,q)-$differentiable on $[0,a]_{p,q}$ with $f(0)=f(a)=0$ and $0<q<p\leq 1$. Then*
$$\int_0^a\left|f(t)\right|^{\lambda(m+n)}d_{p,q}t\leq\left(\left(\frac{p^{m+n}-q^{m+n}}{p-q}\right)^{\lambda}R(\lambda)\right)^n\int_0^a\left|f(s)\right|^{\lambda m}\left|D_{p,q}f(s)\right|^{\lambda n}d_{p,q}s \tag{21}$$
and
$$\int_0^a\left|f(t)\right|^{\lambda(m+n)}d_{p,q}t\leq\left[\left(\frac{p^{m+n}-q^{m+n}}{p-q}\right)^{\lambda}R(\lambda)\right]^{m+n}\int_0^a\left|D_{p,q}f(s)\right|^{\lambda(m+n)}d_{p,q}s \tag{22}$$
where
$$R(\lambda)=\int_0^a\left(t^{1-\lambda}+(a-t)^{1-\lambda}\right)^{-1}d_{p,q}t. \tag{23}$$

Proof. From the Definition of $(p,q)-$integral, we have

$$f^{m+n}(t)=\frac{p^{m+n}-q^{m+n}}{p-q}\int_0^t f^{m+n-1}(s)D_{p,q}f(s)\,d_{p,q}s \tag{24}$$

and

$$f^{m+n}(t)=-\frac{p^{m+n}-q^{m+n}}{p-q}\int_t^a f^{m+n-1}(s)D_{p,q}f(s)\,d_{p,q}s \tag{25}$$

for $t\in[0,a]$. Applying Hölder integral inequality with the indices λ, $\frac{\lambda}{\lambda-1}$ in (24) and (25), we obtain

$$\begin{aligned}|f(t)|^{\lambda(m+n)} &\le \left(\frac{p^{m+n}-q^{m+n}}{p-q}\right)^{\lambda}\left(\int_0^t \left|f^{m+n-1}(s)D_{p,q}f(s)\right|d_{p,q}s\right)^{\lambda} \\ &\le \left(\frac{p^{m+n}-q^{m+n}}{p-q}\right)^{\lambda}\left(\int_0^t d_{p,q}s\right)^{\lambda-1}\left(\int_0^t |f(s)|^{\lambda(m+n-1)}\left|D_{p,q}f(s)\right|^{\lambda}d_{p,q}s\right) \\ &= \left(\frac{p^{m+n}-q^{m+n}}{p-q}\right)^{\lambda} t^{\lambda-1}\int_0^t |f(s)|^{\lambda(m+n-1)}\left|D_{p,q}f(s)\right|^{\lambda}d_{p,q}s.\end{aligned} \tag{26}$$

Similarly, we get

$$\begin{aligned}|f(t)|^{\lambda(m+n)} &\le \left(\frac{p^{m+n}-q^{m+n}}{p-q}\right)^{\lambda}\left(\int_t^a \left|f^{m+n-1}(s)D_{p,q}f(s)\right|d_{p,q}s\right)^{\lambda} \\ &\le \left(\frac{p^{m+n}-q^{m+n}}{p-q}\right)^{\lambda}\left(\int_t^a d_{p,q}s\right)^{\lambda-1}\left(\int_t^a |f(s)|^{\lambda(m+n-1)}\left|D_{p,q}f(s)\right|^{\lambda}d_{p,q}s\right) \\ &= \left(\frac{p^{m+n}-q^{m+n}}{p-q}\right)^{\lambda}(a-t)^{\lambda-1}\int_t^a |f(s)|^{\lambda(m+n-1)}\left|D_{p,q}f(s)\right|^{\lambda}d_{p,q}s.\end{aligned} \tag{27}$$

Multiplying (26) by $t^{1-\lambda}$ and (27) by $(a-t)^{1-\lambda}$ and adding these inequalities, we have

$$\left(t^{1-\lambda}+(a-t)^{1-\lambda}\right)|f(t)|^{\lambda(m+n)}\le\left(\frac{p^{m+n}-q^{m+n}}{p-q}\right)^{\lambda}\int_0^a |f(s)|^{\lambda(m+n-1)}\left|D_{p,q}f(s)\right|^{\lambda}d_{p,q}s.$$

Therefore, we get

$$\begin{aligned}|f(t)|^{\lambda(m+n)} &\le \left(\frac{p^{m+n}-q^{m+n}}{p-q}\right)^{\lambda}\left(t^{1-\lambda}+(a-t)^{1-\lambda}\right)^{-1}\times\int_0^a |f(s)|^{\lambda(m+n-1)}\left|D_{p,q}f(s)\right|^{\lambda}d_{p,q}s \\ &= \left(\frac{p^{m+n}-q^{m+n}}{p-q}\right)^{\lambda}\left(t^{1-\lambda}+(a-t)^{1-\lambda}\right)^{-1}\times\int_0^a |f(s)|^{\lambda m/n}\left|D_{p,q}f(s)\right|^{\lambda}|f(s)|^{\lambda(m+n-1)-\lambda m/n}d_{p,q}s.\end{aligned} \tag{28}$$

Integrating (28) on $[0,a]$, and applying Hölder integral inequality with the indices n, $\frac{n}{n-1}$, we obtain

$$\begin{aligned}\int_0^a |f(t)|^{\lambda(m+n)}d_{p,q}t &\le \left(\frac{p^{m+n}-q^{m+n}}{p-q}\right)^{\lambda}R(\lambda)\times\int_0^a |f(s)|^{\lambda m/n}\left|D_{p,q}f(s)\right|^{\lambda}|f(s)|^{\lambda(m+n-1)-\lambda m/n}d_{p,q}s \\ &\le \left(\frac{p^{m+n}-q^{m+n}}{p-q}\right)^{\lambda}R(\lambda)\times\left(\int_0^a |f(s)|^{\lambda m}\left|D_{p,q}f(s)\right|^{\lambda n}d_{p,q}s\right)^{\frac{1}{n}}\left(\int_0^a |f(s)|^{\lambda(m+n)}d_{p,q}s\right)^{\frac{n-1}{n}}.\end{aligned} \tag{29}$$

(21) is true if $\int_0^a |f(s)|^{\lambda(m+n)}d_{p,q}s=0$. On the other hand if $\int_0^a |f(s)|^{\lambda(m+n)}d_{p,q}s$ is not equal to zero, by dividing both sides of (29) by $\int_0^a |f(s)|^{\lambda(m+n)}d_{p,q}s$ and taking the nth power on both sides of resulting inequality, we get the required

inequality (21).

For proving (22), we apply the Hölder integral inequality to (21) with the indices $\frac{m+n}{n}$, $\frac{m+n}{m}$ and we have

$$\int_0^a |f(t)|^{\lambda(m+n)} d_{p,q}t \le \left[\left(\left(\frac{p^{m+n}-q^{m+n}}{p-q}\right)^{\lambda} R(\lambda)\right]^n \times \left(\int_0^a |f(s)|^{\lambda(m+n)} d_{p,q}s\right)^{\frac{m}{m+n}} \left(\int_0^a |D_{p,q}f(s)|^{\lambda(m+n)} d_{p,q}s\right)^{\frac{n}{m+n}}. \quad (30)$$

Dividing both sides of (30) by $\left(\int_0^a |f(s)|^{\lambda(m+n)} d_{p,q}s\right)^{\frac{m}{m+n}}$, and taking the $\frac{m+n}{n}$th power on both sides, we obtain the required inequality (22).

Theorem 6. *Let $m \ge 0$, $n \ge 1$, $k \ge 0$, $\lambda \ge 1$ be real numbers. Let f be $(p,q)-$differentiable on $[0,a]_{p,q}$ with $f(0) = f(a) = 0$ and $0 < q < p \le 1$. Then*

$$\int_0^a |f(t)|^{\lambda(m+n)} |D_{p,q}f(s)|^{\lambda k} d_{p,q}t \le \left(\left(\frac{p^{m+n}-q^{m+n}}{p-q}\right)^{\lambda} R(\lambda)\right)^n \int_0^a |f(t)|^{\lambda m} |D_{p,q}f(s)|^{\lambda(n+k)} d_{p,q}t \quad (31)$$

and

$$\int_0^a |f(t)|^{\lambda(m+n)} |D_{p,q}f(s)|^{\lambda k} d_{p,q}t \le \left(\left(\frac{p^{m+n}-q^{m+n}}{p-q}\right)^{\lambda} R(\lambda)\right)^{m+n} \int_0^a |D_{p,q}f(s)|^{\lambda(m+n+k)} d_{p,q}t \quad (32)$$

where $R(\lambda)$ defined by (23).

Proof. Applying Hölder inequality with the indices $\frac{n+k}{k}$, $\frac{n+k}{n}$ and by using (21), we have

$$\int_0^a |f(t)|^{\lambda(m+n)} |D_{p,q}f(s)|^{\lambda k} d_{p,q}t = \int_0^a |f(t)|^{\lambda(mk/(n+k))} |D_{p,q}f(s)|^{\lambda k} |f(t)|^{\lambda(m+n)-\lambda(mk/(n+k))} d_{p,q}t$$
$$\le \left(\int_0^a |f(t)|^{\lambda m} |D_{p,q}f(s)|^{\lambda(n+k)} d_{p,q}t\right)^{\frac{k}{n+k}} \left(\int_0^a |f(t)|^{\lambda(m+n+k)} d_{p,q}t\right)^{\frac{n}{n+k}}$$
$$\le \left(\int_0^a |f(t)|^{\lambda m} |D_{p,q}f(s)|^{\lambda(n+k)} d_{p,q}t\right)^{\frac{k}{n+k}} \times \left[\left(\left(\frac{p^{m+n}-q^{m+n}}{p-q}\right)^{\lambda} R(\lambda)\right)^{n+k} \left(\int_0^a |f(t)|^{\lambda m} |D_{p,q}f(s)|^{\lambda(n+k)} d_{p,q}t\right)\right]^{\frac{n}{n+k}}$$
$$= \left(\int_0^a |f(t)|^{\lambda m} |D_{p,q}f(s)|^{\lambda(n+k)} d_{p,q}t\right)^{\frac{k}{n+k}} \times \left(\left(\frac{p^{m+n}-q^{m+n}}{p-q}\right)^{\lambda} R(\lambda)\right)^n \left(\int_0^a |f(t)|^{\lambda m} |D_{p,q}f(s)|^{\lambda(n+k)} d_{p,q}t\right)^{\frac{n}{n+k}}.$$

from which we obtain the desired inequality (31).

In order to prove (32), we apply Hölder integral inequality with the indices $\frac{m+n}{m}$, $\frac{m+n}{n}$ to the integral on the rigth hand side of (31), we obtain

$$\int_0^a |f(t)|^{\lambda(m+n)} |D_{p,q}f(s)|^{\lambda k} d_{p,q}t$$
$$\le \left(\left(\frac{p^{m+n}-q^{m+n}}{p-q}\right)^{\lambda} R(\lambda)\right)^n \times \int_0^a |f(t)|^{\lambda m} |D_{p,q}f(s)|^{\lambda(mk/m+n)} |D_{p,q}f(s)|^{\lambda(n+k)-\lambda(mk/m+n)} d_{p,q}t \quad (33)$$
$$\le \left(\left(\frac{p^{m+n}-q^{m+n}}{p-q}\right)^{\lambda} R(\lambda)\right)^n \left(\int_0^a |f(t)|^{\lambda(m+n)} |D_{p,q}f(s)|^{\lambda k} d_{p,q}t\right)^{\frac{m}{m+n}} \times \left(\int_0^a |D_{p,q}f(s)|^{\lambda(m+n+k)} d_{p,q}t\right)^{\frac{n}{m+n}}.$$

Dividing both sides of (33) by $\left(\int_0^a |f(t)|^{\lambda(m+n)} \left|D_{p,q}f(s)\right|^{\lambda k} d_{p,q}t\right)^{\frac{m}{m+n}}$, and taking the $\frac{m+n}{n}$th power on both sides, we obtain the required inequality (32).

Corollary 2. *Under the assumptions of Theorem 5 and Theorem 6, when $p=1$ in (31) and (32), we have*

$$\int_0^a |f(t)|^{\lambda(m+n)} \left|D_q f(s)\right|^{\lambda k} d_q t \leq \left(\left(\frac{1-q^{m+n}}{1-q}\right)^{\lambda} R(\lambda)\right)^n \int_0^a |f(t)|^{\lambda m} \left|D_q f(s)\right|^{\lambda(n+k)} d_q t \tag{34}$$

and

$$\int_0^a |f(t)|^{\lambda(m+n)} \left|D_q f(s)\right|^{\lambda k} d_q t \leq \left(\left(\frac{1-q^{m+n}}{1-q}\right)^{\lambda} R(\lambda)\right)^{m+n} \int_0^a \left|D_q f(s)\right|^{\lambda(m+n+k)} d_q t \tag{35}$$

which is the $q-$analogue of the formulas (31) and (32).

Remark. Also if we take both $p=1$ and $q \to 1$, (31) and (32) reduce to the results given in [22].

Competing interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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