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General evaluation of suborbital graphs

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Abstract: In this paper, we aim to review suborbital graphs and also give an example to an extension of directed graphs.

Keywords: Bianchi groups, suborbital graphs, circuits, paths.

1 Introduction

It is known that when graph topics are investigated, it is seen that there is a lot of study in literature. One of them is suborbital graphs. The first study to give the relation between graphs and permutation groups is in [1]. That is, the connection between transitive groups and graphs is introduced and used to give new insight into some known results. In [1], the structures of directed and undirected graphs were analyzed and applied to primitive groups. Later in [2] suborbital graphs were gave for action of Γ on extended rational number set $\mathbb{Q}\cup\{\infty\}$. Authors explained imprimitive action and also showed generalized Farey graph and connectedness. In [3], authors mentioned permutation groups, transitivity, primitivity and applications to graph theory.

In these papers $[4-14]$, authors examined some properties of suborbital graphs for the normalizer of $\Gamma_0(N)$ in $PSL(2,\mathbb{R})$ and also discussed the structure and signature of the normalizer. They found a certain part of the total order of ramification of $\Gamma_0(N)$ over its normalizer. They characterized all circuits in the suborbital graph for the normalizer of $\Gamma_0(N)$. Edge and circuit conditions on graphs were obtained. Moreover, the results are quite successful. They considered the action of a permutation group on a set in the spirit of the theory of permutation groups, and graph arising from this action in hyperbolic geometric terms. In addition that authors examined some relations among elliptic elements, circuits in graph for the normalizer of $\Gamma_0(N)$ in $PSL(2,\mathbb{R})$ and the congruence equations arising from related group action. Especially in $[15-20]$, they chose $N=2^{\alpha}p^2$, $N=3^{\beta}p^2$ and $p>3$ prime number where $\alpha=0,1,..8$, $\beta=0,1,2,3$. Hence authors gave the conditions to be a forest for normalizer. Consequently, for $N=2^{\alpha}3^{\beta}p^2$ the following result has been reached:

α	β	*Circuits*	*Conditions*
$0,2,4,6$	$0,2$	*triangle*	$p\equiv 1\,(mod\,3)$
$1,3,5,7$	$0,2$	*quadrilateral*	$p\equiv 1\,(mod\,4)$
$0,2,4,6$	$1,3$	*hexagon*	$p\equiv 1\,(mod\,3)$

We can say that in [21] authors studied on the simple group known as Monster and gave final form elements of normalizer. Indeed normalizer is a Fuchsian group whose fundamental domain has finite area, so it has a signature consisting of the geometric invariants. The signature on the working group is extremely important in terms of revealing invariants. This signature problem is in a way the identity of discrete group. The main purpose in these studies, is to set

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the foundations of a new method which would help to identify the normalizer much better, which have been subject to many studies and gaining particular importance since 1970s and to reveal how the producing elements of the normalizer can be gained by graph method. Therefore, it is by this way that the signature problem was transferred to the suborbital graphs and a new approach was tried to be achieved.

In $[22-38]$ authors examined some properties of suborbital graphs for the modular group, congruence subgroups, extended modular group, invariance group, Fricke group, Hurwitz group, the simple groups $PSL(2,q)$, Atkin-Lehner group, Picard group, Γ^2 and Γ^3 which are the group generated by the second and third powers of the elements of the modular group, respectively. Furthermore in [28] the dimensional of the graphs has been increased by selecting group $SL(3,\mathbb{Z})$. In short, in these studies obtained circuit and forest. Besides necessary and sufficient conditions for being self paired edge were provided. They investigated connectedness of suborbital graphs and studied combinatorial structures of some Fuchsian groups and search genus for these groups. Of course, all calculate was made in upper half plane or Poincare disc model.

In $[39-41]$ authors emphasized some properties of directed graphs for the Hecke groups. They said that the regular maps corresponding to the principal congruence subgroups of Hecke groups. Additionally, they related the sizes of the Petrie polygons on these maps and used Fibonacci numbers.

As the last word in these studies $[42-45]$ digraphs give rise to a special continued fraction which are relate to a continued fraction representation of any rational number. It is obviously that authors described a new kind of continued fraction. The fraction arises from a subgroup of the Farey graph. And they also studied the analogues of certain properties of regular continued fractions in the context. Specially, in [45] there is the chromatic numbers of the suborbital graphs for the modular group and the extended modular group. They verify that the chromatic numbers of the graphs are 2 and 3.

Using general ideas in the study of [1], this paper is an extension of suborbital graphs on 3-dimensional upper half space.

2 Bianchi Groups and their congruence subgroups

Definition 1. *Let d be a square-free natural number. Consider the imaginary quadratic number field $\mathbb{Q}(\sqrt{d})$, $d<0$ and let O_d be its ring of integers. The groups $\Gamma_d := PSL(2,O_d) = SL(2,O_d)/\{\pm I\}$ are called Bianchi groups.*

This class of groups is of interest in many different areas. In number theory they naturally come up in the study of L-functions and elliptic curves. Bianchi groups can be considered as the generalization of the classical modular group $\Gamma_1 := PSL(2,O_1) = PSL(2,\mathbb{Z})$. For $d \in \{-1,-2,-3,-7,-11\}$ the rings O_d are Euclidean rings and the corresponding Bianchi groups are called Euclidean Bianchi groups. Euclidean Bianchi groups have similar properties to the modular group. The structure of the modular group is well understood. For example Γ_1 is isomorphic to the free product of the cyclic groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$, and has a presentation $\Gamma_1 = \langle x,y | x^2 = (xy)^3 = 1\rangle$, where $x : z \longrightarrow z+1$ and $y : z \longrightarrow -\frac{1}{z}$. In addition that Picard was the first one who studied the group $\Gamma_{-1} = PSL(2,O_{-1}) = PSL(2,\mathbb{Z}[i])$ where $\mathbb{Z}[i]$ Gaussian integer, in 1883 and this group is known as the Picard group. As is to be expected, these are much closer in properties to the modular group than in the non-Euclidean cases.

Bianchi groups are discrete subgroups of $PSL(2,\mathbb{C})$. The elements of $PSL(2,\mathbb{C})$ act via linear fractional transformation on the extended complex plane and hence, using the Poincare extension, on upper half 3-space $\mathbb{H}^3 = \{(z,t) \in \mathbb{C}\times\mathbb{R} \mid t>0\}$. We know that $PSL(2,\mathbb{C}) = Isom^+(\mathbb{H}^3)$, the orientation preserving subgroup of the full isometry group of $\mathbb{H}^3$. The quotients of $\mathbb{H}^3$ by the actions of the groups $PSL(2,O_d)$ are then hyperbolic orbifolds of finite hyperbolic volume the Bianchi orbifolds.

Bianchi groups have attracted a great deal of attention both for their intrinsic interest as discrete groups and also for their applications in hyperbolic geometry, topology and number theory.

In number theory they have been used to study the zeta functions of binary Hermitian forms over the rings O_d. They are of interest in the theory of Fuchsian groups and the related theory of Riemann surfaces. The Bianchi groups can be considered as the natural algebraic generalization of the classical modular group $PSL(2,\mathbb{Z})$.

Now we will give two lemmas.

Lemma 1. *The Bianchi group Γ_d is a finitely presented group.*

The number of conjugacy classes of finite subgroups of Γ_d is finite. It is in fact possible to compute a set of representatives for the Γ_d conjugacy classes of finite subgroups.

Lemma 2. (i) *Γ_{-1} contains all possible types of finite subgroups,*
(ii) *Γ_{-2} contains $\mathbb{Z}_2$, $\mathbb{Z}_3$, D_2, A_4 but no S_3,*
(iii) *Γ_{-3} contains $\mathbb{Z}_2$, $\mathbb{Z}_3$, S_3, A_4 but no D_2,*
(iv) *Γ_{-7} contains $\mathbb{Z}_2$, $\mathbb{Z}_3$, S_3 but no A_4 and D_2,*
(v) *Γ_{-11} contains $\mathbb{Z}_2$, $\mathbb{Z}_3$, A_4 but no S_3 and D_2,*
where dihedral group D_2, symmetric group S_3, alternating group A_4 and cyclic groups $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$ order m.

It is well known from in literature that the above lemmas are very important. Because they give also some information about torsion free subgroups of finite index of Γ_d.

Number theoretic interest in the Bianchi groups has centered primarily on the congruence subgroups and the congruence subgroup property. There has also been a considerable amount of work on quadratic forms with entries in O_d and their relation to Γ_d. If σ is an ideal in O_d then the principal congruence subgroup $mod\sigma$, $\Gamma_d(\sigma)$ consists of those transformations in Γ_d corresponding to matrices in $SL(2,O_d)$ congruent to $\pm I mod \sigma$.

$$\Gamma_d(\sigma) = \{\pm T : T \in SL(2,O_d), T \equiv I mod \sigma\}.$$

$\Gamma_d(\sigma)$ can also be described as the kernel of the natural this map $\psi : SL(2,O_d) \longrightarrow SL(2,O_d/\sigma)$ modulo $\pm I$. Thus each principal congruence subgroup is normal and of finite index. A congruence subgroup is a subgroup which contains a principal congruence subgroup. Notice that in the Euclidean cases each ideal is principal and a formula in M. Newman allows us to compute the index of a principal congruence subgroup. Namely if $\alpha \in O_d$, $d \in \{-1,-2,-3,-7,-11\}$ then $|\Gamma_d : \Gamma_d(\alpha)| = \rho|\alpha|^3 \prod_{p|\alpha}(1-\frac{1}{p^2})$ where p runs over the primes dividing α and $\rho = 1$ if $\alpha|2$ or $\rho = \frac{1}{2}$ otherwise.

We can give applications for $PSL(2,O_{-1})$, so we will obtain an extension of suborbital graphs.

Theorem 1. *The action of $PSL(2,O_{-1})$ on $\Pi := \mathbb{Q}(\sqrt{-1}) \cup \{\infty\}$ is transitive.*

Proof. We can show that the orbit containing 0 is Π. If $\frac{x}{y} \in \Pi$, then as $(x,y) = 1$, there exist $\alpha, \beta \in \mathbb{Z}[i]$ with $\alpha y - \beta x = 1$. Then the element $\begin{pmatrix} \alpha & x \\ \beta & y \end{pmatrix}$ of $PSL(2,O_{-1})$ sends 0 to $\frac{x}{y}$.

Lemma 3. *The stabilizer of* 0 *in* $PSL(2,O_{-1})$ *is the set*

$$PSL(2,O_{-1})_0 := \Gamma_{-1,0} = \left\{ \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \ \mid \lambda \in \mathbb{Z}[i] \right\}.$$

Proof. The stabilizer of a point in Π is a infinite cyclic group. As the action is transitive, stabilizer of any two points are conjugate. Hence it is enough to examine the stabilizer of 0 in $PSL(2,O_{-1})$.

$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and so $\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then $b=0$, $d=1$ and as $detT=1$, $a=1$. Therefore $c=\lambda \in \mathbb{Z}[i]$. Thus $T = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$. Indeed this shows that the set $PSL(2,O_{-1})_0$ is equal to $\left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$.

Definition 2. $PSL(2,O_{-1})^0(N) := \Gamma_{-1}^0(N) = \{T \in PSL(2,O_{-1}) \ \mid b \equiv 0\,(modN), N \in \mathbb{Z}[i]\}$.

It is clear that $\Gamma_{-1,0} < \Gamma_{-1}^0(N) < \Gamma_{-1}$. We may use an equivalence relation $\approx$ induced on Π by Γ_{-1}. Now let $\frac{r}{s}, \frac{x}{y} \in \Pi$. Corresponding to these, there are two matrices

$$T_1 := \begin{pmatrix} \star & r \\ \star & s \end{pmatrix}, \quad T_2 := \begin{pmatrix} \star & x \\ \star & y \end{pmatrix}$$

in Γ_{-1} for which $T_1(0) = \frac{r}{s}$, $T_2(0) = \frac{x}{y}$. Therefore $\frac{r}{s} \approx \frac{x}{y}$ if and only if

$$T_1^{-1}T_2 = \begin{pmatrix} s & -r \\ \star & \star \end{pmatrix}\begin{pmatrix} \star & x \\ \star & y \end{pmatrix} = \begin{pmatrix} \star & sx-ry \\ \star & \star \end{pmatrix} \in \Gamma_{-1}^0(N).$$

Therefore $sx - ry \equiv 0\,(modN)$ and then $ry - sx \equiv 0\,(modN)$.

3 Directed graphs

Definition 3. *Let* $\mathbb{G}$ *be a graph and a sequence* $v_1, v_2, \ldots, v_k$ *of different vertices. Then form* $v_1 \longrightarrow v_2 \longrightarrow \ldots \longrightarrow v_k \longrightarrow v_1$, *where* $k>2$ *and* k *positive integer, is called a directed circuit in* $\mathbb{G}$.

Definition 4. *Let* (Γ_{-1}, Π) *be transitive permutation group. Then* Γ_{-1} *acts on* $\Pi \times \Pi$ *by* $\Theta : \Gamma_{-1} \times (\Pi \times \Pi) \longrightarrow \Pi \times \Pi$, $\Theta(T,(\alpha_1,\alpha_2)) = (T(\alpha_1), T(\alpha_2))$, *where* $T \in \Gamma_{-1}$ *and* $\alpha_1, \alpha_2 \in \Pi$. *The orbits of this action are called suborbitals of* Γ_{-1}.

Now we investigate the suborbital digraphs for the action Γ_{-1} on Π. We say that the subgraph of vertices form the block

$$[0] := \left[\frac{0}{1}\right] = \left\{ \frac{x}{y} \in \Pi \ \mid \ x \equiv 0(modN), y \equiv 1(modN) \right\}$$

is denoted by $Z_{N,u} := Z(\frac{0}{1}, \frac{N}{u})$ where $(u,N) = 1$.

Theorem 2. *There is an edge* $\frac{r}{s} \longrightarrow \frac{x}{y}$ *in* $Z_{N,u}$ *if and only if there exists a unit* $\kappa \in \mathbb{Z}[i]$ *such that* $x \equiv \pm\kappa ur\,(modN)$, $y \equiv \pm\kappa us\,(modN)$ *and* $ry - sx = \kappa N$.

Proof. Suppose that there exists an edge $\frac{r}{s} \longrightarrow \frac{x}{y} \in Z_{N,u}$. Hence there exist some $T \in \Gamma_{-1}$ such that sends the pair $(0, \frac{N}{u})$ to the pair $(\frac{r}{s}, \frac{x}{y})$. Clearly $T(0) = \frac{r}{s}$ and $T(\frac{N}{u}) = \frac{x}{y}$. For $T(z) = \frac{az+b}{cz+d}$ we have that $\frac{b}{d} = \frac{r}{s}$ and $\frac{aN+bu}{cN+du} = \frac{x}{y}$. Then there exist the units $\kappa_0, \kappa_1 \in \mathbb{Z}[i]$ such that $b = \kappa_0 r$, $d = \kappa_0 s$ and $aN + bu = \kappa_1 x$, $cN + du = \kappa_1 y$. So, we can write $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & N \\ 1 & u \end{pmatrix} = \begin{pmatrix} \kappa_0 r & \kappa_1 x \\ \kappa_0 s & \kappa_1 y \end{pmatrix}$. Finally, taking with $\kappa = \kappa_0 \kappa_1$, we get that $x \equiv \pm \kappa ur \,(modN)$, $y \equiv \pm \kappa us \,(modN)$. And also from the determinant $ry - sx = \kappa N$ is achieved.

Conversely, we can take the plus sign. Therefore there exist $a, c \in \mathbb{Z}[i]$ such that $x = \kappa ur + aN$, $y = \kappa us + cN$. If we choose $b = \kappa r$ and $d = \kappa s$, then we find $x = ub + aN$ and $y = ud + cN$. So $T(0, \frac{N}{u}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & N \\ 1 & u \end{pmatrix} = \begin{pmatrix} \kappa r & x \\ \kappa s & y \end{pmatrix}$. Since $-\kappa(ry - sx) = N$ we get $ad - bc = 1$ and $T \in \Gamma_{-1}$. Hence $\frac{r}{s} \longrightarrow \frac{x}{y}$ in $Z_{N,u}$. Similarly, minus sign case may shown.

Theorem 3. *$Z_{N,u}$ contains directed hyperbolic triangles if and only if there exists a unit $\kappa \in \mathbb{Z}[i]$ such that $\kappa^2 u^2 - \kappa u + 1 \equiv 0\,(modN)$ and $\kappa^2 u^2 + \kappa u + 1 \equiv 0\,(modN)$.*

Proof. We suppose that $Z_{N,u}$ contains a directed hyperbolic triangle. Since the transitive action, the form of the directed hyperbolic triangle can written like this

$$\frac{0}{1} \longrightarrow \frac{N}{u} \longrightarrow \frac{N}{s} \longrightarrow \frac{0}{1}.$$

As the edge condition in above the theorem, we have to be provided for the second edge $\frac{N}{u} \longrightarrow \frac{N}{s}$, that is $s - u = \kappa$ and $s \equiv \pm \kappa u^2\,(modN)$. Then we have $\kappa s \equiv \pm \kappa^2 u^2\,(modN)$. Therefore $\mp \kappa^2 u^2 + \kappa s \equiv 0\,(modN)$ and $\kappa s - \kappa u = \kappa^2 = \pm 1$ are obtained. Hence there are two cases. The first case is $-\kappa^2 u^2 + \kappa s \equiv 0\,(modN)$ and $\kappa s = \kappa u - 1$. The second case is $\kappa^2 u^2 + \kappa s \equiv 0\,(modN)$ and $\kappa s = \kappa u + 1$. Finally we may say that for $\kappa \in \mathbb{Z}[i]$ these equations $\kappa^2 u^2 - \kappa u + 1 \equiv 0\,(modN)$ and $\kappa^2 u^2 + \kappa u + 1 \equiv 0\,(modN)$ are satisfied.

Conversely, we can solve these equations only with special conditions. Let $\kappa \in \mathbb{Z}[i]$ be a unit such that $\kappa^2 u^2 - \kappa u + 1 \equiv 0\,(modN)$. Above the theorem implies that there is a directed hyperbolic triangle

$$\frac{0}{1} \longrightarrow \frac{N}{u} \longrightarrow \frac{N}{u - \frac{1}{\kappa}} \longrightarrow \frac{0}{1}$$

in $Z_{N,u}$. Similarly, we can find another directed hyperbolic triangle for $\kappa^2 u^2 + \kappa u + 1 \equiv 0\,(modN)$, in this case

$$\frac{0}{1} \longrightarrow \frac{N}{u} \longrightarrow \frac{N}{u + \frac{1}{\kappa}} \longrightarrow \frac{0}{1}$$

in $Z_{N,u}$.

Now we get example for this equation $\kappa^2 u^2 + \kappa u + 1 \equiv 0\,(modN)$.

Example 1. Let $N = 2 + 3i$. If we take $u = 1 - i$, then this equation $-2i\kappa^2 + (1 - i)\kappa + 1 \equiv 0\,mod(2 + 3i)$ is achieved for $\kappa = i$. Hence we get this directed hyperbolic triangle as

$$\frac{0}{1} \longrightarrow \frac{2+3i}{1-i} \longrightarrow \frac{2+3i}{1-2i} \longrightarrow \frac{0}{1}$$

in $Z_{2+3i,1-i}$. And also, it is clear that $(2 + 3i, 1 - i) = 1$ and $ry - sx = \kappa N$.

Similarly, we take another equation. Let $N=13$. Again we choose $u=-3$, then $9\kappa^2+3\kappa+1\equiv 0\,mod(13)$ is held if and only if $\kappa=1$. So, directed triangle is

$$\frac{0}{1}\longrightarrow -\frac{13}{3}\longrightarrow -\frac{13}{4}\longrightarrow \frac{0}{1}$$

in $Z_{13,-3}$.

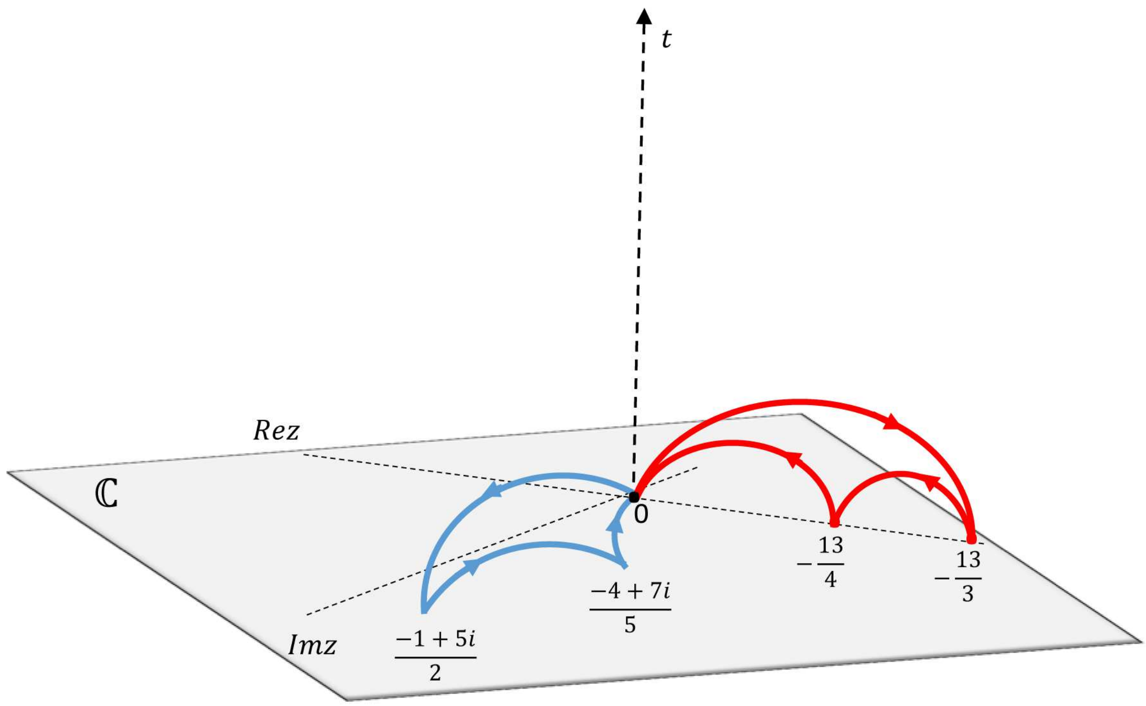

Fig. 1: Circuits in $Z_{2+3i,1-i}$ and $Z_{13,-3}$.

Corollary 1. *The transformation* $\psi := \begin{pmatrix} \kappa u\mp 1 & -\kappa N \\ \frac{\kappa^2u^2\mp\kappa u+1}{\kappa N} & -\kappa u \end{pmatrix}$ *which is defined by means of the congruence* $\kappa^2u^2\mp\kappa u+1\equiv 0\,(modN)$ *is an elliptic element of order 3. Obviously that* $det\,\psi=1$, $\psi^3=I$ *and* $tr(\psi)=\mp 1$. *Moreover, it is easily seen that* $\psi\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}N\\u\end{pmatrix}$, $\psi^2\begin{pmatrix}0\\1\end{pmatrix}:=\psi\begin{pmatrix}N\\u\end{pmatrix}=\begin{pmatrix}N\\u\mp\frac{1}{\kappa}\end{pmatrix}$, $\psi^3\begin{pmatrix}0\\1\end{pmatrix}:=\psi\begin{pmatrix}N\\u\mp\frac{1}{\kappa}\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$.

Corollary 2.*The transformation* $\eta := \begin{pmatrix} \kappa u\pm 1 & -\kappa N \\ \frac{\kappa^2u^2\pm\kappa u-1}{\kappa N} & -\kappa u \end{pmatrix}$ *has* $det\,\eta=-1$ *and* $tr(\eta)=\pm 1$. *It is clearly that the map* η *is not elliptic element. Besides,* $\eta\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}N\\u\end{pmatrix}$, $\eta\begin{pmatrix}N\\u\end{pmatrix}=\begin{pmatrix}N\\u\mp\frac{1}{\kappa}\end{pmatrix}$, $\eta^2\begin{pmatrix}N\\u\end{pmatrix}=\begin{pmatrix}N\\u\mp\frac{1}{2\kappa}\end{pmatrix}$, $\eta^3\begin{pmatrix}N\\u\end{pmatrix}=\begin{pmatrix}N\\u\mp\frac{2}{3\kappa}\end{pmatrix}$, $\eta^4\begin{pmatrix}N\\u\end{pmatrix}=\begin{pmatrix}N\\u\mp\frac{3}{5\kappa}\end{pmatrix}$,..., $\eta^n\begin{pmatrix}N\\u\end{pmatrix}=\begin{pmatrix}N\\u\mp\frac{F_n}{F_{n+1}\kappa}\end{pmatrix}$, *where* $n\geq 0$ *positive integer and* F_n *Fibonacci numbers. It is known that* $F_0=0, F_1=1, F_{n+2}=F_n+F_{n+1}$ *and also* $\lim_{n\to\infty}\frac{F_n}{F_{n+1}}=\frac{1-\sqrt{5}}{2}$. *Hence, two directed paths in hyperbolic 3-dimensional* *upper half space as follows*

$$\frac{0}{1} \longrightarrow \frac{N}{u} \longrightarrow \frac{N}{u \mp \frac{1}{\kappa}} \longrightarrow \frac{N}{u \mp \frac{1}{2\kappa}} \longrightarrow \frac{N}{u \mp \frac{2}{3\kappa}} \longrightarrow \frac{N}{u \mp \frac{3}{5\kappa}} \cdots \longrightarrow \frac{N}{u \mp \frac{F_n}{F_{n+1}\kappa}}$$

are obtained.

Example 2. Let $N = 155$ and $u = 13$. We consider first case. If we take $\kappa = -1$ then we have this directed path

$$\frac{0}{1} \longrightarrow \frac{155}{13} \longrightarrow \frac{155}{13+1} \longrightarrow \frac{155}{13+\frac{1}{2}} \longrightarrow \frac{155}{13+\frac{2}{3}} \longrightarrow \frac{155}{13+\frac{3}{5}} \cdots \longrightarrow \frac{155}{13+\frac{\sqrt{5}-1}{2}}$$

and if we take $N = 2-i$ and $u = 1$ then $\kappa = i$. So we obtain this directed another path as follow

$$\frac{0}{1} \longrightarrow \frac{2-i}{1} \longrightarrow \frac{2-i}{1-\frac{1}{i}} \longrightarrow \frac{2-i}{1-\frac{1}{2i}} \longrightarrow \frac{2-i}{1-\frac{2}{3i}} \longrightarrow \frac{2-i}{1-\frac{3}{5i}} \cdots \longrightarrow \frac{2-i}{1-\frac{\sqrt{5}-1}{2i}}.$$

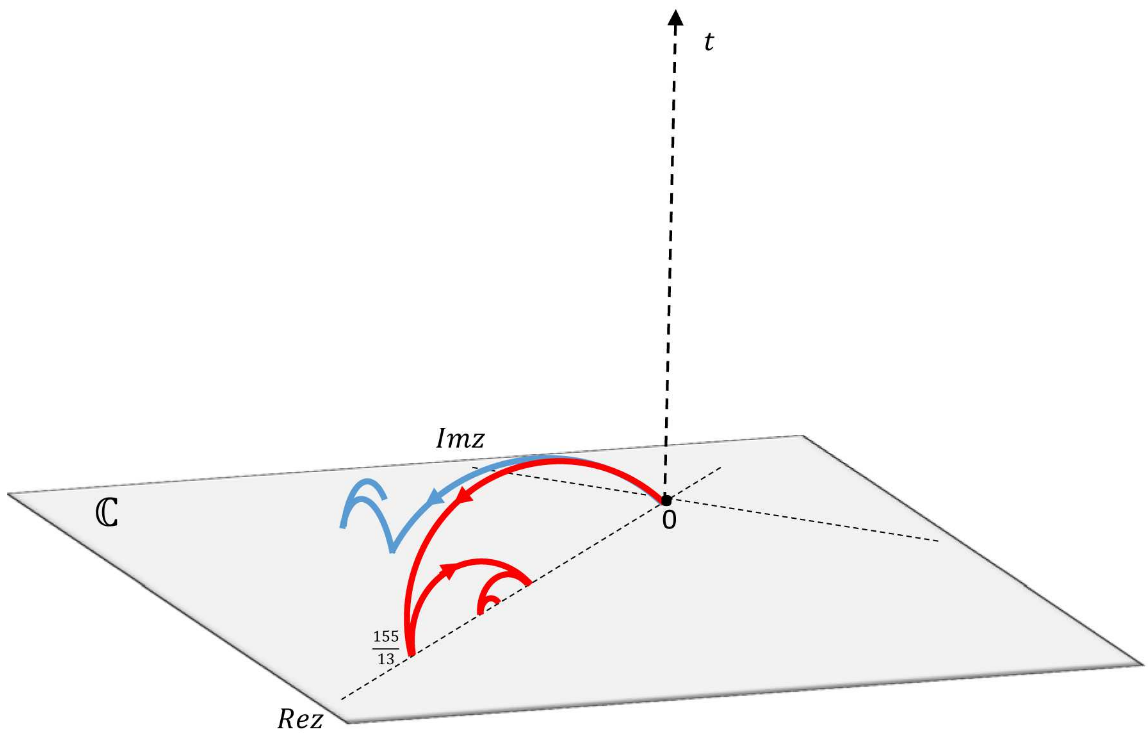

Fig. 2: Directed paths in $Z_{155,13}$ and $Z_{2-i,1}$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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