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Analysis of inverse parabolic problem with non-local boundary condition

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Abstract: Aim of the paper is to investigate solution of inverse parabolic problem with non-local boundary condition. Under some natural regularity and consistency conditions on the input data the existence, uniqueness and continuous dependence upon the data of solution are shown by using the generalized Fourier method. Also, an iteration algorithm for the numerical solution of this problem is constructed and examined numerical solution by using linearization and Crank-Nicolson difference scheme.

Keywords: Inverse problem, quasilinear parabolic equation, Crank-Nicolson difference scheme.

1 Introduction

Recently, there have been a lot of recent problems with inverse problems that have a lot of applications like chemical diffusion, applications in heat conduction, population dynamics, thermoelasticity, medical science, electrochemistry, engineering, wide scope, chemical engineering.The inverse problem of determining unknown coefficient in a quasi-linear parabolic equation has generated an increasing amount of interest from engineers and scientist [3,4,5,6]. Nonlocal boundary conditions have been a lot of many important role in heat transfer, termoelasticity, control theory, mathematical biology [1,2].

Let's take the following problem with unknowns (q,u)

$$u_t = u_{xx} + q(t)g(x,t,u),\ (x,t) \in \Gamma, \tag{1}$$

$$u(x,0) = \theta(x),\ x \in [0,1], \tag{2}$$

$$u(0,t) = 0,\ u_x(0,t) = u_x(1,t),\ 0 \le t \le T, \tag{3}$$

$$h(t) = u(1,t), 0 \le t \le T. \tag{4}$$

Here $\Gamma := \{0 < x < 1,\ 0 < t < T\}$, $\theta(x) \in [0,1]$ and $g(x,t,u) \in \overline{\Gamma} \times (-\infty,\infty)$.

Definition 1. $\{g(t),u(x,t)\} \in C[0,T] \times (C^{2,1}(\Gamma) \cap C^{1,0}(\overline{\Gamma}))$ *is called the classical solution.*

2 Solution of the inverse problem

Let assume the following conditions are ensured.
(C1) $h(t) \in C^1[0,T]$, $q(t) \in C[0,T]$,

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(C2) $\theta(x) \in C^3[0,1]$,
$\theta(x)|_{x=0} = 0,\ \theta_x(x)|_{x=0} = \theta_x(x)|_{x=1}$, (C3) $g(x,t,u)$ is provided the following conditions (1)

$$\left|\frac{\partial^{(n)} g(x,t,u)}{\partial x^n} - \frac{\partial^{(n)} g(x,t,\tilde{u})}{\partial x^n}\right| \le b(x,t)\,|u-\tilde{u}|\,,\ n = 0,1,2,$$

where $b(x,t) \in L_2(\Gamma),\ b(x,t) \ge 0$,
(2) $g(x,t,u) \in C^2[0,1],\ t \in [0,T]$,
(3) $g(x,t,u)|_{x=0} = 0,\ g_x(x,t,u)|_{x=0} = g_x(x,t,u)|_{x=1}$,
(4) $4g_0(t,u) + 16\sum_{k=1}^{\infty} g_{2k-1}(t,u) \neq 0$. By Fourier method,

$$u(x,t) = 2u_0(t) + 4\sum_{k=1}^{\infty}\left[u_{2k}(t)\cos 2\pi kx + u_{2k-1}(t)\sin 2\pi kx\right],$$

$$\begin{aligned} u(x,t) = {} & 2\left[\theta_0 + \int_0^t q(\tau)g_0(\tau,u)d\tau\right] \\ & + 4\sum_{k=1}^{\infty} x\cos 2\pi kx\,(\theta_{2k-1} - 4\pi kt\ \theta_{2k})\,e^{-(2\pi k)^2 t} \\ & + 16\sum_{k=1}^{\infty} x\cos 2\pi kx \int_0^t q(\tau)g_{2k-1}(\tau,u)\ e^{-(2\pi k)^2(t-\tau)}d\tau \\ & - 16\pi\sum_{k=1}^{\infty} k(t-\tau)x\cos 2\pi kx \int_0^t q(\tau)g_{2k}(\tau,u)\ e^{-(2\pi k)^2(t-\tau)}d\tau \\ & + 4\sum_{k=1}^{\infty} \sin 2\pi kx\ \theta_{2k}\, e^{-(2\pi k)^2 t} \\ & + 4\sum_{k=1}^{\infty} \sin 2\pi kx \int_0^t q(\tau)g_{2k}(\tau,u)\ e^{-(2\pi k)^2(t-\tau)}d\tau. \end{aligned} \tag{5}$$

Under the condition (A1)-(A3), differentiating (4), we obtain

$$u_t(1,t)dx = h'(t), 0 \le t \le T, \tag{6}$$

(5), (6) yield

$$q(t) = \frac{h'(t) + 4\sum_{k=1}^{\infty}(2\pi k)^2 e^{-(2\pi k)^2 t}(\theta_{2k-1} - 4\pi kt\theta_{2k}) + 16\sum_{k=1}^{\infty}\int_0^t q(\tau)g_{2k-1}(\tau,u)\ e^{-(2\pi k)^2(t-\tau)}d\tau}{4g_0(t,u) + 16\sum_{k=1}^{\infty} g_{2k-1}(t,u)}. \tag{7}$$

Definition 2. *Let* $\{u(t)\} = \{u_0(t), u_{2k}(t), u_{2k-1}(t), k = 1,...,n\}$ *is satisfied that*
$\max_{0\le t\le T}\frac{|u_0(t)|}{2} + \sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|u_{2k}(t)| + \max_{0\le t\le T}|u_{2k-1}(t)|\right) < \infty$, *by* $\mathbf{B}_1$.
$\|u(t)\| = \max_{0\le t\le T}\frac{|u_0(t)|}{2} + \sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|u_{2k}(t)| + \max_{0\le t\le T}|u_{2k-1}(t)|\right)$, *be the norm where* $\mathbf{B}_1$ *is Banach space.*

Theorem 1. *If the assumptions (C1)-(C3) be provided then the problem (1), (4) has a unique solution.*

Proof. Using iteration to equation (5)

$$u_0^{(N+1)}(t) = u_0^{(0)}(t) + \int_0^t \int_0^1 q^{(N)}(\beta) g\left(\alpha,\beta,u^{(N)}(\alpha,\beta)\right) d\alpha d\beta,$$

$$u_{2k-1}^{(N+1)}(t) = u_{2k-1}^{(0)}(t) + 4\int_0^t \int_0^1 q^{(N)}(\beta)\, g(\alpha,\beta,u^{(N)}(\alpha,\beta)) \cos 2\pi k\alpha\, e^{-(2\pi k)^2(t-\beta)} d\alpha d\beta,$$

$$u_{2k}^{(N+1)}(t) = u_{2k}^{(0)}(t) + 4\int_0^t \int_0^1 q^{(N)}(\beta)\, g(\alpha,\beta,u^{(N)}(\alpha,\beta))(1-\alpha) \sin 2\pi k\alpha\, e^{-(2\pi k)^2(t-\beta)} d\alpha d\beta \tag{8}$$

$$-4\pi k \int_0^t \int_0^1 (t-\beta) q^{(N)}(\beta) g(\alpha,\beta,u^{(N)}(\alpha,\beta)) \cos 2\pi k\alpha\, e^{-(2\pi k)^2(t-\beta)} d\alpha d\beta,$$

$$u_0^{(0)}(t) = \theta_0,\ u_{2k}^{(0)}(t) = (\theta_{2k-1} - 4\pi kt\ \theta_{2k})\, e^{-(2\pi k)^2 t},\ u_{2k-1}^{(0)}(t) = \theta_{2k} e^{-(2\pi k)^2 t}.$$

$$q^{(N+1)}(t) = \frac{h'(t) + 4\sum\limits_{k=1}^{\infty} (2\pi k)^2 e^{-(2\pi k)^2 t} (\theta_{2k-1} - 4\pi kt\ \theta_{2k})}{4g_0(t,u^{(N)}) + 16\sum\limits_{k=1}^{\infty} g_{2k-1}(t,u^{(N)})} + \frac{+16\sum\limits_{k=1}^{\infty}\int\limits_0^t q^{(N)}(\tau) g_{2k-1}(\beta,u^{(N)})\, e^{-(2\pi k)^2(t-\tau)} d\tau}{4g_0(t,u^{(N)})\ + 16\sum\limits_{k=1}^{\infty} g_{2k-1}(t,u^{(N)})}.$$

From the theorem, we find $u^{(0)}(t) \in \mathbf{B}_1$, $t \in [0,T]$.

For $N = 0$,

$$u_0^{(1)}(t) = u_0^{(0)}(t) + \int_0^t \int_0^1 q^{(0)}(\beta)\, g\left(\alpha,\beta,u^{(0)}(\alpha,\beta)\right) d\xi d\beta.$$

Adding and subtracting $\int\limits_0^t\int\limits_0^1 q^{(0)}(\tau) f(\xi,\tau,0) d\xi d\tau$, we find

$$u_0^{(1)}(t) = \theta_0(t) + \int_0^t \int_0^1 q^{(0)}(\beta)[g(\alpha,\beta,u^{(0)}(\xi,\tau)) - g(\alpha,\beta,0)] d\alpha d\beta + \int_0^t \int_0^1 q^{(0)}(\beta) g(\alpha,\beta,0) d\alpha d\beta.$$

Applying Cauchy inequality,

$$\left|u_0^{(1)}(t)\right| \leq |\theta_0| + \left(\int_0^t d\beta\right)^{\frac{1}{2}} \left(\int_0^t \left\{\int_0^1 \left|q^{(0)}(\beta)\right| [g(\alpha,\beta,u^{(0)}(\alpha,\beta)) - g(\alpha,\beta,0)] d\alpha\right\}^2 d\beta\right)^{\frac{1}{2}}$$

$$+ \left(\int_0^t d\beta\right)^{\frac{1}{2}} \left(\int_0^t \left\{\int_0^1 \left|q^{(0)}(\beta)\right| |g(\alpha,\beta,0)| d\alpha\right\}^2 d\beta\right)^{\frac{1}{2}},$$

and using Lipschitzs condition, we obtain

$$\left|u_0^{(1)}(t)\right| \le |\theta_0| + \sqrt{t}\left(\int_0^t \left\{\int_0^1 \left|q^{(0)}(\beta)\right| b(\alpha,\beta)\left|u^{(0)}(\alpha,\beta)\right| d\alpha\right\}^2 d\beta\right)^{\frac{1}{2}}$$
$$+\sqrt{t}\left(\int_0^t \left\{\int_0^1 \left|q^{(0)}(\beta)\right| |g(\alpha,\beta,0)|\, d\alpha\right\}^2 d\beta\right)^{\frac{1}{2}},$$

and taking maximum, we find

$$\max_{0\le t\le T}\left|u_0^{(1)}(t)\right| \le |\theta_0| + \sqrt{T}\,\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(0)}(t)\right\|_{B_1}\left\|q^{(0)}(t)\right\|_{C[0,T]}$$
$$+\sqrt{T}\left\|q^{(0)}(t)\right\|_{C[0,T]}\|g(x,t,0)\|_{L_2(\Gamma)},$$

using the same estimations and Hölder, Bessel inequality and taking maximum

$$\sum_{k=1}^{\infty}\max_{0\le t\le T}\left|u_{2k-1}^{(1)}(t)\right| \le \sum_{k=1}^{\infty}|\theta_{2k-1}| + \frac{\sqrt{3}}{12}\|b(x,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\|_{B_1}\left\|q^{(0)}(t)\right\|_{C[0,T]}$$
$$+\frac{\sqrt{3}}{12}\left\|q^{(0)}(t)\right\|_{C[0,T]}\|g(x,t,0)\|_{L_2(\Gamma)},$$

and applying the same estimations, we obtain

$$\sum_{k=1}^{\infty}\max_{0\le t\le T}\left|u_{2k}^{(1)}(t)\right| \le \sum_{k=1}^{\infty}|\theta_{2k-1}| + 4\pi|T|\sum_{k=1}^{\infty}\left|\theta_{2k}'\right|$$
$$+\frac{\sqrt{3}}{12}\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(0)}(t)\right\|_{B_1}\left\|q^{(0)}(t)\right\|_{C[0,T]}$$
$$+\frac{\sqrt{3}}{12}\left\|q^{(0)}(t)\right\|_{C[0,T]}\|g(x,t,0)\|_{L_2(\Gamma)}$$
$$+\frac{\sqrt{2}|T|}{4\pi}\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(0)}(t)\right\|_{B_1}\left\|q^{(0)}(t)\right\|_{C[0,T]}$$
$$+\frac{\sqrt{2}|T|}{4\pi}\left\|q^{(0)}(t)\right\|_{C[0,T]}\|g(x,t,0)\|_{L_2(\Gamma)},$$

and then, we find

$$\left\|u^{(1)}(t)\right\|_{\mathbf{B}_1}$$
$$\le 2|\theta_0| + 8\sum_{k=1}^{\infty}|\theta_{2k-1}| + 16\pi|T|\sum_{k=1}^{\infty}\left|\theta_{2k}'\right|$$
$$+\left(2\sqrt{T}+\frac{2\sqrt{3}}{3}+\frac{\sqrt{2}|T|}{\pi}\right)\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(0)}(t)\right\|_{B_1}\left\|q^{(0)}(t)\right\|_{C[0,T]}$$
$$+\left(2\sqrt{T}+\frac{2\sqrt{3}}{3}+\frac{\sqrt{2}|T|}{\pi}\right)\left\|q^{(0)}(t)\right\|_{C[0,T]}\|g(x,t,0)\|_{L_2(\Gamma)}.$$

$u^{(1)}(t) \in \mathbf{B}_1$. Same estimations for N,

$$\begin{aligned}
&\left\|u^{(N+1)}(t)\right\|_{\mathbf{B}_1} \\
&\leq 2|\theta_0| + 8\sum_{k=1}^{\infty}|\theta_{2k-1}| + 16\pi|T|\sum_{k=1}^{\infty}\left|\theta_{2k}^{'}\right| \\
&+\left(2\sqrt{T}+\frac{2\sqrt{3}}{3}+\frac{\sqrt{2}|T|}{\pi}\right)\|b(x,t)\|_{L_2(D)}\left\|u^{(N)}(t)\right\|_{B_1}\left\|q^{(N)}(t)\right\|_{C[0,T]} \\
&+\left(2\sqrt{T}+\frac{2\sqrt{3}}{3}+\frac{\sqrt{2}|T|}{\pi}\right)\left\|q^{(N)}(t)\right\|_{C[0,T]}\|g(x,t,0)\|_{L_2(D)}.
\end{aligned}$$

According to $u^{(N)}(t) \in \mathbf{B}_1$ and theorem, $u^{(N+1)}(t) \in \mathbf{B}_1$,

$$\{u(t)\} = \{u_0(t), u_{2k}(t), u_{2k-1}(t),\ k=1,2,...\} \in \mathbf{B}_1.$$

If we used with same estimations, we obtain

$$\begin{aligned}
\left\|q^{(N+1)}\right\|_{B_2} \leq & \frac{1}{M}\left\|h^{'}(t)\right\| \\
&+\frac{4}{M}\sum_{k=1}^{\infty}\left(\left\|\theta_{2k-1}^{''}\right\|+2\left\|\theta_{2k}^{'''}\right\|\right) \\
&+\frac{4\sqrt{3}}{3M}\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(N)}(t)\right\|_{B_1}\left\|q^{(N)}(t)\right\|_{C[0,T]} \\
&+\frac{4\sqrt{3}}{3M}\left\|q^{(N)}(t)\right\|_{C[0,T]}\|g(x,t,0)\|_{L_2(\Gamma)}.
\end{aligned}$$

We show that the iterations $u^{(N+1)}(t)$, $q^{(N+1)}$ converge $\mathbf{B}_1$ and $C[0,T]$, respectively for $N \to \infty$.

$$\begin{aligned}
&u^{(1)}(t) - u^{(0)}(t) \\
&= 2\left(\int_0^t\int_0^1 q^{(0)}(\beta)\left[g(\alpha,\beta,u^{(0)}(\alpha,\beta)) - g(\alpha,\beta,0)\right]d\alpha d\beta\right) \\
&+4\sum_{k=1}^{\infty}\int_0^t\int_0^1 q^{(0)}(\tau)\left[g(\alpha,\beta,u^{(0)}(\alpha,\beta)) - g(\alpha,\beta,0)\right]\cos 2\pi k\alpha e^{-(2\pi k)^2(t-\beta)}d\alpha d\beta \\
&+4\sum_{k=1}^{\infty}\int_0^t\int_0^1 q^{(0)}(\tau)\ \left[g(\alpha,\beta,u^{(0)}(\alpha,\beta)) - g(\alpha,\beta,0)\right](1-\alpha)\sin 2\pi k\alpha\ e^{-(2\pi k)^2(t-\beta)}d\alpha d\beta \\
&-4\pi k\int_0^t\int_0^1 (t-\beta)q^{(0)}(\tau)\left[g(\alpha,\beta,u^{(0)}(\alpha,\beta)) - g(\alpha,\beta,0)\right]\cos 2\pi k\alpha\ e^{-(2\pi k)^2(t-\beta)}d\alpha d\beta \\
&+2\left(\int_0^t\int_0^1 q^{(0)}(\beta)g(\alpha,\beta,0)d\alpha d\beta\right)+4\sum_{k=1}^{\infty}\int_0^t\int_0^1 q^{(0)}(\beta)g(\alpha,\beta,0)e^{-(2\pi k)^2(t-\beta)}\cos 2\pi k\alpha d\alpha d\beta
\end{aligned}$$

$$+4\sum_{k=1}^{\infty}\int_0^t\int_0^1 q^{(0)}(\beta)g(\alpha,\beta,0)(1-\alpha)\sin 2\pi k\alpha\, e^{-(2\pi k)^2(t-\beta)}d\alpha d\beta$$
$$-4\pi k\int_0^t\int_0^1 (t-\beta)q^{(0)}(\beta)\, g(\alpha,\beta,0)\cos 2\pi k\alpha\, e^{-(2\pi k)^2(t-\beta)}d\alpha d\beta.$$

Using Cauchy, Bessel, Hölder inequality, Lipschitzs condition and taking maximum of both side of the last inequality, we obtain

$$\left\|u^{(1)}(t)-u^{(0)}(t)\right\|_{B_1}\le\left(2\sqrt{T}+\frac{2\sqrt{3}}{3}+\frac{\sqrt{2}\,|T|}{\pi}\right)\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(0)}(t)\right\|_{B_1}\left\|q^{(0)}t)\right\|_{C[0,T]},$$
$$+\left(2\sqrt{T}+\frac{2\sqrt{3}}{3}+\frac{\sqrt{2}\,|T|}{\pi}\right)\left\|q^{(0)}(t)\right\|_{C[0,T]}\|g(x,t,0)\|_{L_2(\Gamma)}.$$

$$A=\left(2\sqrt{T}+\frac{2\sqrt{3}}{3}+\frac{\sqrt{2}\,|T|}{\pi}\right)\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(0)}(t)\right\|_{B_1}\left\|q^{(0)}(t)\right\|_{C[0,T]},$$
$$+\left(2\sqrt{T}+\frac{2\sqrt{3}}{3}+\frac{\sqrt{2}\,|T|}{\pi}\right)\left\|q^{(0)}(t)\right\|_{C[0,T]}\|g(x,t,0)\|_{L_2(\Gamma)}.$$

$$\left\|q^{(1)}(t)-q^{(0)}(t)\right\|_{C[0,T]}\le\frac{4\sqrt{3}}{3M}\left\|q^{(1)}(t)\right\|_{C[0,T]}\left\|u^{(1)}(t)-u^{(0)}(t)\right\|_{B_1}\|b(x,t)\|_{L_2(\Gamma)},$$

$$\left\|u^{(2)}(t)-u^{(1)}(t)\right\|_{B_1}\le\left(2\sqrt{T}+\frac{2\sqrt{3}}{3}+\frac{\sqrt{2}\,|T|}{\pi}\right)\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(1)}(t)-u^{(0)}(t)\right\|_{B_1}\left\|q^{(1)}(t)\right\|_{C[0,T]}$$
$$+\left(2\sqrt{T}+\frac{2\sqrt{3}}{3}+\frac{\sqrt{2}\,|T|}{\pi}\right)\left(\frac{4\sqrt{3}}{3M}\left\|q^{(1)}(t)\right\|_{C[0,T]}\left\|u^{(1)}(t)-u^{(0)}(t)\right\|_{B_1}\|b(x,t)\|_{L_2(\Gamma)}\right).$$

$$\left\|u^{(2)}(t)-u^{(1)}(t)\right\|_{B_1}\le\left\{\left(2\sqrt{T}+\frac{2\sqrt{3}}{3}+\frac{\sqrt{2}\,|T|}{\pi}\right)\left(1+\frac{4\sqrt{3}}{3M}\right)\right\}A\,\|b(x,t)\|_{L_2(\Gamma)}\left\|q^{(1)}(t)\right\|_{C[0,T]},$$

$$\left\|q^{(2)}(t)-q^{(0)}(t)\right\|_{C[0,T]}\le\frac{4\sqrt{3}}{3M-4\sqrt{3}M}\left\|q^{(2)}(t)\right\|_{C[0,T]}\left\|u^{(2)}(t)-u^{(1)}(t)\right\|_{B_1}\|b(x,t)\|_{L_2(\Gamma)},$$
$$\left\|u^{(3)}(t)-u^{(2)}(t)\right\|_{B_1}\le\left\{\left(2\sqrt{T}+\frac{2\sqrt{3}}{3}+\frac{\sqrt{2}\,|T|}{\pi}\right)\left(\frac{3}{3-4\sqrt{3}}\right)\right\}^2\frac{A}{\sqrt{2!}}\|b(x,t)\|_{L_2(\Gamma)}\left\|q^{(1)}(\tau)\right\|_{C[0,T]},$$

For N :

$$\left\|q^{(N+1)}(t)-q^{(N)}(t)\right\|_{C[0,T]}\le\frac{4\sqrt{3}}{3M-4\sqrt{3}M}\left\|q^{(N+1)}(t)\right\|_{C[0,T]}\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\|_{B_1}\|b(x,t)\|_{L_2(\Gamma)},$$

$$\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\|_{B_1}\le\frac{A}{\sqrt{N!}}\left\{\left(2\sqrt{T}+\frac{2\sqrt{3}}{3}+\frac{\sqrt{2}\,|T|}{\pi}\right)\left(\frac{3}{3-4\sqrt{3}}\right)\right\}^N\times$$
$$\left\|q^{(1)}(t)\right\|_{C[0,T]}\left\|q^{(2)}(t)\right\|_{C[0,T]}\cdots\left\|q^{(N)}(t)\right\|_{C[0,T]}\|b(x,t)\|^N_{L_2(\Gamma)}. \tag{9}$$

For $N\to\infty$, $u^{(N+1)}(t), q^{(N+1)}$ are converged. Let show that there exists u and q such that

$$\lim_{N\to\infty} u^{(N+1)}(t)=u(t),\ \lim_{N\to\infty} q^{(N+1)}(t)=q(t).$$

$$\begin{aligned}
\left|u-u^{(N+1)}\right| \le\; & 2\left|\int_0^t\int_0^1 q(\beta)\left[g(\alpha,\beta,u(\alpha,\beta))-g(\alpha,\beta,u^{(N+1)}(\alpha,\beta))\right]d\alpha d\beta\right| \\
& 2\left|\int_0^t\int_0^\pi q(\beta)\left[g(\alpha,\beta,u^{(N+1)}(\alpha,\beta))-g(\alpha,\beta,u^{(N)}(\alpha,\beta))\right]d\alpha d\beta\right| \\
& +\left|\int_0^t\int_0^1 \left(q(\beta)-q^{(N)}(\beta)\right)g(\alpha,\beta,u^{(N)}(\alpha,\beta))d\alpha d\beta\right| \\
& +4\sum_{k=1}^{\infty}\left|\int_0^t\int_0^1 q(\beta)\left[g(\alpha,\beta,u(\alpha,\beta))-g(\alpha,\beta,u^{(N+1)}(\alpha,\beta))\right]\cos 2\pi k\alpha\ e^{-(2\pi k)^2(t-\tau)}d\alpha d\beta\right| \\
& +4\sum_{k=1}^{\infty}\left|\int_0^t\int_0^1 q(\beta)\left[g(\alpha,\beta,u^{(N+1)}(\alpha,\beta))-g(\alpha,\beta,u^{(N)}(\alpha,\beta))\right]\cos 2\pi k\xi\ e^{-(2\pi k)^2(t-\tau)}d\xi d\tau\right| \\
& +4\sum_{k=1}^{\infty}\left|\int_0^t\int_0^1 q(\beta)g(\alpha,\beta,u^{(N)}(\alpha,\beta))\cos 2\pi k\alpha\ e^{-(2\pi k)^2(t-\beta)}d\alpha d\beta\right| \\
& +4\sum_{k=1}^{\infty}\left|\int_0^t\int_0^1 q(\beta)\ \left[g(\alpha,\beta,u(\alpha,\beta))-g(\alpha,\beta,u^{(N+1)}(\alpha,\beta))\right](1-\alpha)\sin 2\pi k\alpha\ e^{-(2\pi k)^2(t-\beta)}d\alpha d\beta\right| \\
& +4\sum_{k=1}^{\infty}\left|\int_0^t\int_0^1 q(\beta)\left[g(\alpha,\beta,u^{(N+1)}(\alpha,\beta))-f(\alpha,\beta,u^{(N)}(\alpha,\beta))\right](1-\alpha)\sin 2\pi k\alpha\ e^{-(2\pi k)^2(t-\beta)}d\alpha d\beta\right| \\
& +4\sum_{k=1}^{\infty}\left|\int_0^t\int_0^1 q(\alpha)g(\alpha,\beta,u^{(N)}(\alpha,\beta))(1-\alpha)\sin 2\pi k\alpha\ e^{-(2\pi k)^2(t-\beta)}d\alpha d\beta\right| \\
& +4\pi\sum_{k=1}^{\infty}k\left|\int_0^t\int_0^1 (t-\beta)q(\beta)\ \left[g(\alpha,\beta,u(\alpha,\beta))-g(\alpha,\beta,u^{(N+1)}(\alpha,\beta))\right]\cos 2\pi k\beta\ e^{-(2\pi k)^2(t-\beta)}d\alpha d\beta\right| \\
& +4\pi\sum_{k=1}^{\infty}k\left|\int_0^t\int_0^1 (t-\beta)q(\beta)\ \left[g(\alpha,\beta,u^{(N+1)}(\alpha,\beta))-g(\alpha,\beta,u^{(N)}(\alpha,\beta))\right]\cos 2\pi k\alpha\ e^{-(2\pi k)^2(t-\beta)}d\alpha d\beta\right| \\
& +4\pi\sum_{k=1}^{\infty}k\left|\int_0^t\int_0^1 (t-\beta)q(\beta)\ g(\alpha,\beta,u^{(N)}(\alpha,\beta))\cos 2\pi k\alpha\ e^{-(2\pi k)^2(t-\beta)}d\alpha d\beta\right|.
\end{aligned}$$

Using same inequality and Gronwall's inequality ,we obtain

$$\left\|u(t)-u^{(N+1)}(t)\right\|_{\mathbf{B}_1}^2 \leq$$

$$2\left[\frac{A}{\sqrt{N!}}\left\{\left(2\sqrt{T}+\frac{2\sqrt{3}}{3}+\frac{\sqrt{2}|T|}{\pi}\right)\left(\frac{3}{3-4\sqrt{3}}\right)\right\}^N \|q(\tau)\|_{C[0,T]}\|Q(\tau)\|_{C[0,T]}\|b(x,t)\|_{L_2(\Gamma)}^N\right]^2 \tag{10}$$

$$\times\exp\left\{2\left(2\sqrt{T}+\frac{2\sqrt{3}}{3}+\frac{\sqrt{2}|T|}{\pi}\right)\left(\frac{3}{3-4\sqrt{3}}\right)\right\}^2 \|q(\tau)\|_{C[0,T]}^2\|b(x,t))\|_{L_2(\Gamma)}^2.$$

$$\left\|q(t)-q^{(N+1)}(t)\right\|_{C[0,T]} \leq \frac{4\sqrt{3}}{3M-4\sqrt{3}M}\|q(t)\|_{C[0,T]}\left\|u(t)-u^{(N+1)}(t)\right\|_{B_1}\|b(x,t)\|_{L_2(\Gamma)},$$

where $\|Q(t)\|_{C[0,T]}=\left\|q^{(1)}(t)\right\|_{C[0,T]}\left\|q^{(2)}(t)\right\|_{C[0,T]}\cdots\left\|q^{(N)}(t\right\|_{C[0,T]}$.
We obtain $u^{(N+1)}\to u, q^{(N+1)}\to q,\, N\to\infty$.

For the uniqueness, let (u,g), (v,h) are two solution of (1), (4). After applying Cauchy, Bessel, Lipschitz, Hölder inequality to $|u(t)-v(t)|$ and $|r(t)-q(t)|$, we obtain

$$\|u(t)-v(t)\|_{\mathbf{B}_1} \leq \left[\left(2\sqrt{T}+\frac{2\sqrt{3}}{3}+\frac{\sqrt{2}|T|}{\pi}\right)\left(\frac{3}{3-4\sqrt{3}}\right)\right]\left(\int_0^t q^2(\beta)b^2(\alpha,\beta)\,|u(\beta)-v(\beta)|^2\,d\alpha d\beta\right)^{\frac{1}{2}}, \tag{11}$$

here, $u(t)=v(t)$ and then $r(t)=q(t)$.

The proof is over.

3 Stability of problem

Theorem 2. *Assumption (C1)-(C3) the solution (q,u) of the problem (1), (4) depends continuously upon the data θ, h.*

Proof. Let $\Phi=\{\theta,h,f\}$ and $\overline{\Phi}=\left\{\overline{\theta},\,\overline{h},f\right\}$ be two sets of the data, which satisfy the assumptions $(C_1)-(C_3)$. Let us denote $\|\Phi\|=(\|h\|_{C^1[0,T]}+\|\theta\|_{C^3[0,1]}+\|f\|_{C^{3,0}(\overline{\Gamma})})$. Let (q,u) and $(\overline{q},\overline{u})$ be solutions of problems (1), (4).

$$\begin{aligned}
u-\overline{u} &= 2(\theta_0-\overline{\theta_0})+4\sum_{k=1}^{\infty}(\theta_{2k}-\overline{\theta_{2k}})\sin 2\pi kxe^{-(2\pi k)^2 t}\\
&+4\sum_{k=1}^{\infty}(\theta_{2k-1}-\overline{\theta_{2k-1}})x\cos 2\pi kxe^{-(2\pi k)^2 t}\\
&-16\pi\sum_{k=1}^{\infty}kt(\theta_{2k}-\overline{\theta_{2k}})x\cos 2\pi kxe^{-(2\pi k)^2 t}\\
&+2\left(\int_0^t\int_0^1 q(\beta)\left[g(\alpha,\beta,u(\alpha,\beta))-g(\alpha,\beta,\overline{u}(\alpha,\beta))\right]d\alpha d\beta\right)\\
&+2\left(\int_0^t\int_0^1 (q(\beta)-\overline{q}(\beta))\,g(\alpha,\beta,\overline{u}(\alpha,\beta))d\alpha d\beta\right)\\
&+4\sum_{k=1}^{\infty}\int_0^t\int_0^1 q(\beta)\left[g(\alpha,\beta,u(\alpha,\beta))-g(\alpha,\beta,\overline{u}(\alpha,\beta))\right]\cos 2\pi k\alpha\, e^{-(2\pi k)^2(t-\beta)}d\alpha d\beta
\end{aligned}$$

$$+4\sum_{k=1}^{\infty}\int_0^t\int_0^1 (q(\beta)-\overline{q}(\beta))\,g(\alpha,\beta,\overline{u}(\alpha,\beta))\cos 2\pi k\alpha e^{-(2\pi k)^2(t-\beta)}d\alpha d\beta$$

$$+4\sum_{k=1}^{\infty}\int_0^t\int_0^1 q(\beta)\left[g(\alpha,\beta,u(\alpha,\beta))-g(\alpha,\beta,\overline{u}(\alpha,\beta))\right](1-\alpha)\sin 2\pi k\alpha e^{-(2\pi k)^2(t-\beta)}d\alpha d\beta$$

$$+4\sum_{k=1}^{\infty}\int_0^t\int_0^1 (q(\beta)-\overline{q}(\beta))\,g(\alpha,\beta,\overline{u}(\alpha,\beta))(1-\alpha)\sin 2\pi k\alpha e^{-(2\pi k)^2(t-\beta)}d\alpha d\beta$$

$$-4\pi\sum_{k=1}^{\infty}kt\int_0^t\int_0^1 (t-\beta)q(\beta)\left[g(\alpha,\beta,u(\alpha,\beta))-g(\alpha,\beta,\overline{u}(\alpha,\beta))\right]\cos 2\pi k\alpha\, e^{-(2\pi k)^2(t-\beta)}d\alpha d\beta$$

$$-4\pi\sum_{k=1}^{\infty}kt\int_0^t\int_0^1 (t-\beta)\,(q(\beta)-\overline{q}(\beta))\,g(\alpha,\beta,\overline{u}(\alpha,\beta))\cos 2\pi k\alpha\, e^{-(2\pi k)^2(t-\beta)}d\alpha d\beta$$

By using same estimations, we obtain

$$\|u-\overline{u}\|_{B_1}\le\left\|\theta-\overline{\theta}\right\| + \left(2\sqrt{T}+\frac{2\sqrt{3}}{3}+\frac{\sqrt{2}\,|T|}{\pi}\right)\left(\int_0^t\int_0^1 q^2(\beta)b^2(\alpha,\beta)\,|u(\beta)-\overline{u}(\beta)|^2\,d\alpha d\beta\right)^{\frac{1}{2}}, \tag{12}$$

where, let $\left\|\theta-\overline{\theta}\right\|\le 2\,\left\|\theta_0-\overline{\theta_0}\right\|+4\sum_{k=1}^{\infty}(\left\|\theta_{2k-1}-\overline{\theta_{2k-1}}\right\|+\left\|\theta_{2k}-\overline{\theta_{2k}}\right\|)+4\pi\sum_{k=1}^{\infty}\left\|\theta_{2k}'-\overline{\theta_{2k}'}\right\|$.

$$\|q-\overline{q}\|_{B_2}\le\frac{1}{M}\left\|h'(t)-\overline{h'(t)}\right\| +\frac{4}{M}\sum_{k=1}^{\infty}\left(\left\|\theta_{2k-1}''-\overline{\theta_{2k-1}''}\right\|+2\left\|\theta_{2k}'''-\overline{\theta_{2k}'''}\right\|\right) +\frac{4\sqrt{3}}{3M}\left(\int_0^t\int_0^1 q^2(\beta)b^2(\alpha,\beta)\,|u(\beta)-\overline{u}(\beta)|^2\,d\alpha d\beta\right)^{\frac{1}{2}} +\frac{4\sqrt{3}}{3}\left(\int_0^t\int_0^1 q^2(\beta)d\alpha d\beta\right)^{\frac{1}{2}},$$

applying Gronwall's inequality to (12), we obtain

$$\|u-\overline{u}\|_{B_1}\le 2\left\|\Phi-\overline{\Phi}\right\|^2 \times\exp\left(2\int_0^t\int_0^1 q^2(\beta)b^2(\alpha,\beta)d\alpha d\beta\right),$$

where, let $\left\|\Phi-\overline{\Phi}\right\|\le\left\|\theta-\overline{\theta}\right\|+\frac{1}{M}\left\|h'(t)-\overline{h'(t)}\right\|+\frac{4}{M}\sum_{k=1}^{\infty}\left(\left\|\theta_{2k-1}''-\overline{\theta_{2k-1}''}\right\|+2\left\|\theta_{2k}'''-\overline{\theta_{2k}'''}\right\|\right)$. For $\Phi\to\overline{\Phi}$ then $u\to\overline{u}$. Hence $q\to\overline{q}$.

4 Numerical Procedure for the nonlinear problem

An iteration algorithm for the linearization of the problem (1), (4)

$$\frac{\partial u^{(n)}}{\partial t} = \frac{\partial^2 u^{(n)}}{\partial x^2} + q(t)g(x,t,u^{(n-1)}), \qquad (x,t) \in D \tag{13}$$

$$u^{(n)}(x,0) = \theta(x)\,, \qquad x \in [0,1]. \tag{14}$$

$$u^{(n)}(0,t) = 0, \qquad t \in [0,T] \tag{15}$$

$$u_x^{(n)}(0,t) = u_x^{(n)}(1,t),\ t \in [0,T]. \tag{16}$$

Let $u^{(n)}(x,t) = v(x,t)$ and $g(x,t,u^{(n-1)}) = \widetilde{g}(x,t)$ then a new linear problem

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + q(t)\widetilde{g}(x,t) \qquad (x,t) \in D \tag{17}$$

$$v(0,t) = 0, \qquad t \in [0,T] \tag{18}$$

$$v_x(0,t) = v_x(1,t), \qquad t \in [0,T] \tag{19}$$

$$v(x,0) = \theta(x), \qquad x \in [0,1]. \tag{20}$$

we use the method of the linearization and the finite difference method to solve (17), (18), (19),(20). We subdivide the intervals $[0,1]$ and $[0,T]$ into subintervals N_x and N_t of equal lengths $h = \frac{1}{N_x}$ and $\tau = \frac{T}{N_t}$, respectively. We use the Crank-Nicolson scheme which is absolutely stable and has a second-order accuracy in h and a first-order accuracy in τ. The Crank-Nicolson scheme for (17), (18), (19), (20) is as follows

$$\frac{1}{\tau}\left(v_i^{j+1} - v_i^j\right) = \frac{1}{2h^2}\left(v_{i-1}^{j+1} - 2v_i^{j+1} + v_{i+1}^{j+1}\right) + \frac{1}{2h^2}\left(v_{i-1}^j - 2v_i^j + v_{i+1}^j\right) + \frac{1}{4}\left(q^{j+1} + q^j\right)\left(\widetilde{g}_i^{j+1} + \widetilde{g}_i^j\right),$$

$$v_i^0 = \theta_i, \tag{21}$$

$$v_0^j = 0, \tag{22}$$

$$v_{N_x+1}^j = v_1^j + v_{N_x}^j, \tag{23}$$

where $1 \le i \le N_x$ and $0 \le j \le N_t$ are the indices for the spatial and time steps respectively, $v_i^j = v(x_i,t_j)$, $\theta_i = \theta(x_i)$, $q^j = q(t_j)$, $\widetilde{g}_i^j = \widetilde{g}(x_i,t_j)$, $x_i = ih$, $t_j = j\tau$. Now, let us construct the predicting-correcting mechanism. We obtain

$$q(t) = \frac{h'(t) - v_{xx}(1,t)}{\widetilde{g}(1,t)}. \tag{24}$$

The finite difference approximation of (23) is

$$q^j = \frac{\left(\left(h^{j+1} - h^j\right)/\tau\right) - \left(v_{N_x+1}^j - 2v_{N_x}^j + v_{N_x-1}^j\right)/h}{\widetilde{g}_{N_x}^j},$$

where $h^j = h(t_j)$, $j = 0,1,...,N_t$.

We denote the values of q^j, v_i^j at the s-th iteration step $q^{j(s)}$, $v_i^{j(s)}$, respectively. In numerical computation, since the time step is very small, we can take $q^{j+1(0)} = q^j$, $v_i^{j+1(0)} = v_i^j$, $j = 0,1,2,....N_t$, $i = 1,2,...,N_x$. At each $(s+1)$-th iteration step we first determine $r^{j(s+1)}$ from the formula

$$q^{j(s+1)} = \frac{\left(\left(h^{j+1} - h^j\right)/\tau\right) - \left(v_{N_x+1}^{j(s)} - 2v_{N_x}^{j(s)} + v_{N_x-1}^{j(s)}\right)/h}{\widetilde{g}_{N_x}^j}.$$

Then from (21), (21), (22) we obtain

$$\frac{1}{\tau}\left(v_i^{j(s+1)}-v_i^{j(s)}\right)=\frac{1}{2h^2}\left[\left(v_{i-1}^{j(s+1)}-2v_i^{j(s+1)}+v_{i+1}^{j(s+1)}\right)+\left(v_{i-1}^{j(s)}-2v_i^{j(s)}+v_{i+1}^{j(s)}\right)\right]+\frac{1}{4}\left(q^{j(s+1)}+q^{j(s)}\right)\left(\tilde{g}_i^{j+1}+\tilde{g}_i^{j}\right),| \tag{25}$$

$$v_0^{j(s)}=0, \tag{26}$$

$$v_{N_x+1}^{j(s)}=v_1^{j(s)}+v_{N_x}^{j(s)}. \tag{27}$$

The system of equations (24), (25), (26) can be solved by the Gauss elimination method and $v_i^{j+1(s+1)}$ is determined. If the difference of values between two iterations reaches the prescribed tolerance, the iteration is stopped and we accept the corresponding values $q^{j(s+1)}$, $v_i^{j(s+1)}(i,j=1,2,...,N_x)$ as q^j, v_i^j $(i=1,2,...,N_x)$, on the $(j+1)$-th time step, respectively. In virtue of this iteration, we can move from level j to level $j+1$.

5 Numerical example

Example 1. If we consider the inverse problem (1), (2), (3), (4), with

$$g(x,t,u)=\left(1+(2\pi)^2\right)u\exp(-2t)+4\pi\sin 2\pi x\exp(t),$$
$$\theta(x)=x\cos(2\pi x),h(t)=\exp(t),\ x\in[0,1],\ t\in[0,T].$$

It is easy to check that the analytical solution of this problem is $\{q(t),\ u(x,t)\}=\{\exp(2t),\ x\cos(2\pi x)\exp(t)\}$. Let us apply the scheme which was explained in the previous section for the step sizes $h=0.005$, $\tau=0.005$. In the case when $T=1$ the comparisons between the analytical solution and the numerical finite difference solution are shown in Figures 1 and 2.

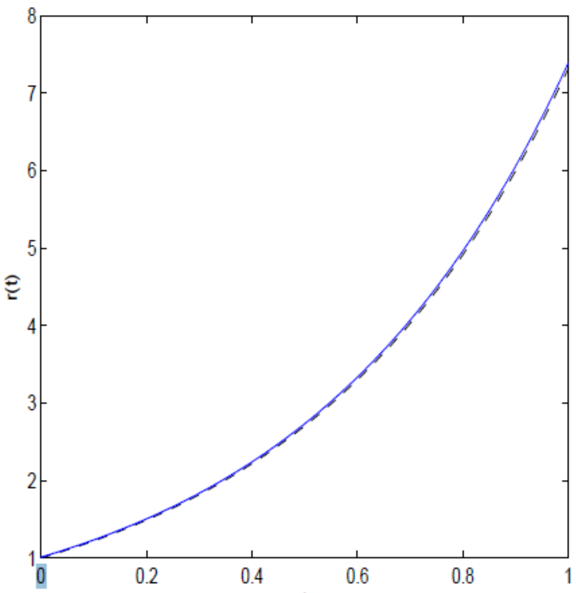

Fig. 1: Exact and approximate q(t) when T=1.

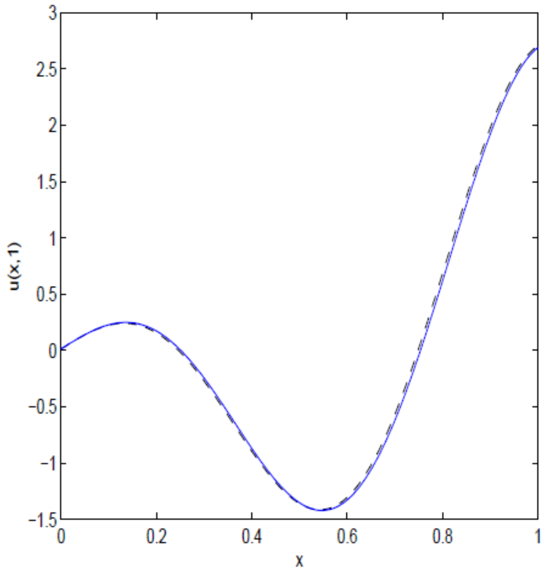

Fig. 2: Exact and approximate solutions of u(x,t) at the T=1

6 Discussions

The inverse problem regarding the simultaneously identification of the time-dependent coefficient and the temperature distribution in inverse parabolic problem with non-local boundary condition has been considered. This inverse problem has been investigated from both theoretical and numerical points of view. In the theoretical part of the article, the conditions for the existence, uniqueness and continuous dependence upon the data of the problem have been established. In the numerical part, the finite difference method by the Crank-Nicolson difference scheme with an iteration are presented. This work advances our understanding of the use of the Fourier method of separation of variables and the finite difference methods in the investigation of inverse problems for inverse quasilinear parabolic equations. The authors plan to consider various inverse problems in future studies, since the method discussed has a wide range of applications. Also, the implicit monotone difference schemes can be examined. The convergence of these numerical methods may be studied.

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