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On an unified reduction formula for Srivastava's triple hypergeometric series $F^{(3)}[x,y,z]$

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Abstract: Very recently, by applying the so-called Beta integral method to the Henrici's triple product formula for the generalized hypergeometric series, Choi, et al.[Commun. Korean Math. Soc. 28(2013), No.2, pp. 297-301] have obtained an interesting reduction formula for the Srivastava's triple hypergeometric series $F^{(3)}[x,y,z]$. The aim of this short note is to provide a unified reduction formula for the Srivastava's triple hypergeometric series from which as many new reduction formulas (including the one obtained by Choi, et al.) as desired can be deduced. A few interesing special cases have also been given.

Keywords: Generalized hypergeometric function ${}_pF_q$, Gamma function, Pochhammer symbol, Beta integral, Srivastava's triple hypergeometric series $F^{(3)}[x,y,z]$, Henrici's formula.

1 Introduction

Let $\mathbb{C}$ be the set of *complex* numbers, then for

$$\alpha_j \in \mathbb{C}\,(j=1,\ldots,p) \quad \text{and} \quad \beta_j \in \mathbb{C}\setminus\mathbb{Z}_0^- \quad \left(\mathbb{Z}_0^- := \mathbb{Z}\cup\{0\} = \{0,-1,-2,\ldots\}\right),$$

the *generalized hypergeometric function* ${}_pF_q$ with p numerator parameters $\alpha_1,\ldots,\alpha_p$ and q denominator parameters $\beta_1,\ldots,\beta_q$ is defined by (see, for example, [7, Chapter 4]; see also [11, pp. 71–72]):

$${}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\,;\\ \beta_1,\ldots,\beta_q\,;\end{matrix}\ z\right] = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\alpha_j)_n}{\prod_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!} = {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z) \tag{1}$$

$$\Big(p,q\in\mathbb{N}_0 := \mathbb{N}\cup\{0\} = \{0,1,2,\ldots\};\ p \leqq q+1;\ p\leqq q \text{ and } |z|<\infty;$$
$$p=q+1 \text{ and } |z|<1;\ p=q+1,\ |z|=1 \text{ and } \Re(\omega)>0\Big),$$

where

$$\omega := \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j \quad \left(\alpha_j\in\mathbb{C}\,(j=1,\ldots,p);\ \beta_j\in\mathbb{C}\setminus\mathbb{Z}_0^-\,(j=1,\ldots,q)\right) \tag{2}$$

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and $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$), in terms of the familiar Gamma function Γ, by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N}). \end{cases} \tag{3}$$

On the other hand, in the course of study of triple hypergeometric series, Srivastava(*cf.* [12, p.43]) defined a unification of Lauricella's 14 triple hypergeometric series $F_1, \ldots, F_{14}$ (*cf.* [12, pp. 41–43]) and the additional Srivastava's triple hypergeometric series H_A, H_B, H_C as a general triple hypergeometric series $F^{(3)}[x,y,z]$ (*cf.* [9, p. 428]; see also [12, p. 44–45]) by

$$F^{(3)}[x,y,z] \equiv F^{(3)}\left[\begin{matrix}(a) :: (b); (b'); (b'') : (c); (c'); (c''); \\ (e) :: (g); (g'); (g'') : (h); (h'); (h'');\end{matrix} x,y,z\right] = \sum_{m,n,p=0}^{\infty} \Lambda(m,n,p)\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{4}$$

where, for convenience,

$$\Lambda(m,n,p) = \frac{\prod_{j=1}^{A}(a_j)_{m+n+p}\prod_{j=1}^{B}(b_j)_{m+n}\prod_{j=1}^{B'}(b'_j)_{n+p}\prod_{j=1}^{B''}(b''_j)_{p+m}}{\prod_{j=1}^{E}(e_j)_{m+n+p}\prod_{j=1}^{G}(g_j)_{m+n}\prod_{j=1}^{G'}(g'_j)_{n+p}\prod_{j=1}^{G''}(g''_j)_{p+m}} \cdot \frac{\prod_{j=1}^{C}(c_j)_m\prod_{j=1}^{C'}(c'_j)_n\prod_{j=1}^{C''}(c''_j)_p}{\prod_{j=1}^{H}(h_j)_m\prod_{j=1}^{H'}(h'_j)_n\prod_{j=1}^{H''}(h''_j)_p}, \tag{5}$$

and, (a) abbreviates the array of A parameters $a_1, \ldots, a_A$, with similar interpretations for (b), (b'), (b''), and so on.

Very recently by employing the well known, very useful and interesting Henrici's triple product formula for the hypergeometric series [4] viz.

$${}_0F_1\left[\begin{matrix}-; \\ 6c;\end{matrix} x\right] {}_0F_1\left[\begin{matrix}-; \\ 6c;\end{matrix} \omega x\right] {}_0F_1\left[\begin{matrix}-; \\ 6c;\end{matrix} \omega^2 x\right] = {}_2F_7\left[\begin{matrix}3c-\frac{1}{4},\ 3c+\frac{1}{4}; \\ 6c,\ 2c,\ 2c+\frac{1}{3},\ 2c+\frac{2}{3},\ 4c-\frac{1}{3},\ 4c,\ 4c+\frac{1}{3};\end{matrix} \left(\frac{4x}{9}\right)^3\right] \tag{6}$$

where $\omega = \exp(\frac{2\pi i}{3})$, together with the Beta integral method [5], Choi, et al. [2] estabished the following interesting result for the reducibility of Srivastava's triple hypergeometric series given by

$$F^{(3)}\left[\begin{matrix}e : -; -; - : -; -; -; \\ d : -; -; - : 6c; 6c; 6c;\end{matrix} 1, \omega, \omega^2\right] = {}_5F_{10}\left[\begin{matrix}3c-\frac{1}{4},\ 3c+\frac{1}{4},\ \frac{d}{3},\ \frac{d}{3}+\frac{1}{3},\ \frac{d}{3}+\frac{2}{3}; \\ 6c,\ 2c,\ 2c+\frac{1}{3},\ 2c+\frac{2}{3},\ 4c-\frac{1}{3},\ 4c,\ 4c+\frac{1}{3},\ \frac{e}{3},\ \frac{e}{3}+\frac{1}{3},\ \frac{e}{3}+\frac{2}{3};\end{matrix} \left(\frac{4}{9}\right)^3\right] \tag{7}$$

where $\omega = \exp(\frac{2\pi i}{3})$.

The aim of this short note is to provide a unified reduction formula for the Srivastava's triple hypergeometric series from which as many as new reduction formulas (including the one obtained by Choi, et al.) desired can be obtained. A few interesting special cases are also be given.

2 Main theorem

Theorem 1. For all finite x, the following interesting reduction formula for the Srivastava's triple hypergeometric function $F^{(3)}[x,y,z]$ holds true.

$$F^{(3)}\left[\begin{matrix} e:-;-;-:\ -;\ -;\ -; \\ d:-;-;-:6c\,;\ 6c\,;\ 6c\,; \end{matrix} x,\ \omega x,\ \omega^2 x\right] = {}_5F_{10}\left[\begin{matrix} 3c-\frac{1}{4},\ 3c+\frac{1}{4},\ \frac{d}{3},\ \frac{d}{3}+\frac{1}{3},\ \frac{d}{3}+\frac{2}{3}; \\ 6c,\ 2c,\ 2c+\frac{1}{3},\ 2c+\frac{2}{3},\ 4c-\frac{1}{3},\ 4c,\ 4c+\frac{1}{3},\ \frac{e}{3},\ \frac{e}{3}+\frac{1}{3},\ \frac{e}{3}+\frac{2}{3}; \end{matrix} \left(\frac{4x}{9}\right)^3\right], \tag{8}$$

where $\omega = \exp(\frac{2\pi i}{3})$.

Proof. In order to establish our theorem, we proceed as follows. First of all replacing x by xt in (6) we have

$${}_0F_1\left[\begin{matrix} -\,; \\ 6c\,; \end{matrix} xt\right]{}_0F_1\left[\begin{matrix} -\,; \\ 6c; \end{matrix} \omega xt\right]{}_0F_1\left[\begin{matrix} -\,; \\ 6c; \end{matrix} \omega^2 xt\right] = {}_2F_7\left[\begin{matrix} 3c-\frac{1}{4}, 3c+\frac{1}{4}; \\ 6c, 2c, 2c+\frac{1}{3}, 2c+\frac{2}{3}, 4c-\frac{1}{3}, 4c, 4c+\frac{1}{3}; \end{matrix} \left(\frac{4xt}{9}\right)^3\right]. \tag{9}$$

Now multiplying the left-hand side of (9) by $x^{d-1}(1-x)^{e-d-1}$, where we suppose temporarily that $Re(e) > Re(d) > 0$ and integrating the resulting equation with respect to t from 0 to 1, denoting it by S_1, then expressing each ${}_0F_1$ involved as series, changing the order of integration and summation , which is easily seen to be justified due to the uniform convergence of the involved series, we have

$$S_1 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{x^{m+n+p}\omega^n\omega^{2p}}{(6c)_m(6c)_n(6c)_p m!n!p!} \int_0^1 t^{d+m+n+p-1}(1-t)^{e-d-1}dt.$$

Evaluating the beta integral and interpreting the result thus obtained with the help of the definition of $F^{(3)}[x,y,z]$, we get

$$S_1 = \frac{\Gamma(d)\Gamma(e-d)}{\Gamma(e)} F^{(3)}\left[\begin{matrix} d:\ -;-;-:\ -;\ -;\ -; \\ e:-;-;-:6c\,;\ 6c\,;\ 6c\,; \end{matrix} x,\ \omega x,\ \omega^2 x\right]. \tag{10}$$

Again, multiplying the right-hand side of (9) by $t^{d-1}(1-t)^{e-d-1}$ and as above, integrating the resulting equation with respect to t from 0 to 1, denoting it by S_2, then expressing ${}_2F_7$ as series and proceeding as above, we have, after some simplification

$$S_2 = \frac{\Gamma(d)\Gamma(e-d)}{\Gamma(e)} \times \sum_{n=0}^{\infty} \frac{\left(3c-\frac{1}{4}\right)_n \left(3c+\frac{1}{4}\right)_n x^{3n}}{(6c)_n(2c)_n\left(2c+\frac{1}{3}\right)_n\left(2c+\frac{2}{3}\right)_n\left(4c-\frac{1}{3}\right)_n(4c)_n\left(4c+\frac{1}{3}\right)_n n!} \left(\frac{4}{9}\right)^{3n} \frac{(d)_{3n}}{(e)_{3n}}.$$

Using

$$(\alpha)_{3n} = 3^{3n}\left(\frac{\alpha}{3}\right)_n\left(\frac{\alpha+1}{3}\right)_n\left(\frac{\alpha+2}{3}\right)_n$$

and after some simplification, summing up the series, we have

$$S_2 = \frac{\Gamma(d)\Gamma(e-d)}{\Gamma(e)} \times {}_5F_{10}\left[\begin{matrix} 3c-4, 3c+\frac{1}{4}, \frac{1}{3}d, \frac{1}{3}d+\frac{1}{3}, \frac{1}{3}d+\frac{2}{3}; \\ 6c, 2c, 2c+\frac{1}{3}, 2c+\frac{2}{3}, 4c, 4c+\frac{1}{3}, 4c+\frac{2}{3}, \frac{1}{3}e, \frac{1}{3}e+\frac{1}{3}, \frac{1}{3}e+\frac{2}{3}; \end{matrix} \left(\frac{4x}{9}\right)^3\right]. \tag{11}$$

Finally equating (10) and (11), we get the desired result (8). This completes the proof of our main theorem.

The above restriction on the parameters d and e may now be removed by appeal to analytic continuation.

3 Special cases

In this section, we shall mention a few very interesting special cases of our main result (8).

(1) In (8), if we take $x = 1$, we get a known reduction formula due to Choi, et al.[2, p.299, eq.(2.1)]

(2) In (8), if we take $x = \frac{1}{2}$, we get the following result.

$$F^{(3)}\left[\begin{matrix} d: -;-;- : -; \ -; \ -; \frac{1}{2}, \frac{1}{2}\omega, \frac{1}{2}\omega^2 \\ e: -;-;- : 6c;\ 6c;\ 6c; \end{matrix}\right] = {}_5F_{10}A_1 \tag{12}$$

where

$$A_1 = \left[\begin{matrix} 3c-\frac{1}{4},\ 3c+\frac{1}{4},\ \frac{d}{3},\ \frac{d}{3}+\frac{1}{3},\ \frac{d}{3}+\frac{2}{3}; \\ 6c,\ 2c,\ 2c+\frac{1}{3},\ 2c+\frac{2}{3},\ 4c-\frac{1}{3},\ 4c,\ 4c+\frac{1}{3},\ \frac{e}{3},\ \frac{e}{3}+\frac{1}{3},\ \frac{e}{3}+\frac{2}{3}; \end{matrix}\ \left(\frac{2}{9}\right)^3\right]$$

(3) In (8), if we take $x = \frac{1}{4}$, we get the following result.

$$F^{(3)}\left[\begin{matrix} d:: -;-;- : -; \ -; \ -; \frac{1}{4}, \frac{1}{4}\omega, \frac{1}{4}\omega^2 \\ e:: -;-;- : 6c;\ 6c;\ 6c; \end{matrix}\right] = {}_5F_{10}A_2 \tag{13}$$

where

$$A_2 = \left[\begin{matrix} 3c-\frac{1}{4},\ 3c+\frac{1}{4},\ \frac{d}{3},\ \frac{d}{3}+\frac{1}{3},\ \frac{d}{3}+\frac{2}{3}; \\ 6c,\ 2c,\ 2c+\frac{1}{3},\ 2c+\frac{2}{3},\ 4c-\frac{1}{3},\ 4c,\ 4c+\frac{1}{3},\ \frac{e}{3},\ \frac{e}{3}+\frac{1}{3},\ \frac{e}{3}+\frac{2}{3}; \end{matrix}\ \left(\frac{1}{9}\right)^3\right].$$

Similarly other results can also be obtained.

Remark. For other results, we refer[1,3,5,8,9-13].

Concluding remark

In this paper, we have obtained a unified reduction formula for the Srivastava's triple hypergeometric series, by applying the so - called beta integral method to the Henrici's triple product formula for the generalized hypergeometric series. Since our reduction formula is valid for all finite, therefore, from our main result, we can obtain, as many as reduction formulas as desired. The results presented here may be useful in applied mathematics, physics and engineering.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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