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Average weakly edge domination number in graphs

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Abstract: Communication is supposed to be continuous in a network design. It is important for a network to be tough so that communication is not interrupted in case any damage. In this paper, it is investigated how to decide which graph model to choose, when a selection is needed to make between different graphs to be used for a network model when all known vulnerability measures are same. We introduce the concept of the average weakly edge domination number of a graph as a new vulnerability measure. We establish relationships between the average weakly edge domination number and some other graph parameters, and the extreme values of given measure among all graphs and average weakly edge domination number for some families of graphs. Also a polynomial time algorithm with complexity $\mathscr{O}(n^3)$ is given.

Keywords: Average weakly edge domination, edge distance, vulnerability.

1 Introduction

Vulnerability is the most important concept in any communication network. The resistance of a network after any disruption is considered as vulnerability value. In a network, this disruption not only can take place on its centers also sometimes on its links. In this case the vulnerability measures given on links (edges) take an important role while constructing communication networks. A communication network can be modeled by a graph whose vertices represent the stations and whose edges represent the lines of communication [4,10,17].

In graph theory, many graph measures have been used widely in the past to describe the stability of a graph. The best known and most useful of the measures of how well a graph is connected is the connectivity, defined to be minimum number of vertices in a set whose deletion results in a disconnected or trivial graph. This measure has been extensively studied As the connectivity is a worst case measure, it does not always reflect what happens throughout the graph. For example a tree and the the graph obtained by appending an end-vertex to a complete graph both have connectivity 1. In this case we need another measure to decide which graph model is more stable than the other. This situation oriented researchers to find new measures. In graph theory, many vulnerability measures have been used widely in the past to describe the stability of a graph. Some of these measures are edge connectivity, Integrity, edge integrity, domination, toughness [3,7,9,18]. Since networks can be modeled by graphs, vulnerability of networks are observed via graphs. For this purpose, there has been great interest on vulnerability in graph theory and different kinds of measures have been defined [1,6,11,12,13,15,16].

In this paper we investigate the average weakly edge domination number as a new measure of global connectedness. Whereas lots of other global measures are computationally NP-hard, the average weakly edge domination number can be computed in polynomial time by using the present algorithms, making it much more attractive for applications. For

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terminology and notation on graph theory not given here, the reader is referred to [5,14]. Throughout this paper, we only consider finite, simple, undirected graphs without isolated vertices. A graph $G = (V,E)$ is a set V of vertices and a subset E of the unordered pairs of vertices, called edges. The rest of the paper is organized as follows: In Section 2, we present the new vulnerability measure called Average Weakly Edge Domination Number and study the properties of this measure under certain graph operations. In Section 3, we compute the values of Average Weakly Edge Domination Number for well-known graph classes. Finally, in section 4, we present an algorithm with time complexity $\mathcal{O}(n^3)$ that computes the Average Weakly Edge Domination Number of any graph.

2 Average weakly edge domination number

In the case of networks are modeled by graphs, some vulnerability measurements of graphs are considered when making decision between two graph design. The reason is to provide the continuity of communication on networks. Since the continuity of communication is related to the robustness of graphs to be used in network models, there are many studies in graph theory on the vulnerability and many measurements such as connectivity, integrity, and tenacity are defined. We need a new vulnerability measurement to preference among graphs when the known vulnerability measurements are the same.

In the case of the edge and vertex integrity values are same for the graphs G_l and G_2 which have the same number of edges and vertices. We need a new measure to determine which graph is preferred for the network model.

Definition 1. *Let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ be two edges of a graph $G = (V,E)$. The number of edges neighbor to these two edges is called Pair Weakly Edge Domination Number ($PWEDN$). If e_l and e_2 have common vertex then $PWEDN$ is added by 1.*

Definition 2. *In a graph G, the sum of all pair weakly edge domination number computed respect to all edge pairs is called Total Pair Weakly Edge Domination Number ($TPWEDN$); i.e.,*

$$TPWEDN = \sum_{e_i, e_j \in E(G)} PWEDN(e_i, e_j).$$

Definition 3. *In a graph G with n vertices, the number obtained via division of $TPWEDN$ by the number of maximum edges paired to n vertices is called Average Weakly Edge Domination Number ($AWEDN$); i.e.,*

$$AWEDN = \frac{TPWEDN}{\binom{m}{2}},$$

where $m = \binom{n}{2}$.

To visualize the efficiency of this measure we present two different graphs G_1 and G_2 with the same number of vertices and edges, and the same values of edge and vertex integrity in Figure 1. We need to compare $AWEDN$ values to make a choice between two graphs. To calculate the $AWEDN$ values, first we need to calculate the $PWEDN$ values of all edge pairs, then find the $TPWEDN$ values. Since graphs G_l and G_2 both have 8 vertices, they are compared to the complete graph with 8 vertices. $AWEDN$ values can be found via the division of $TPWEDN$ values by 378 which is the number of pairs can be constructed by 28 edges since the complete graph with 8 vertices has 28 edges. The $AWEDN$ values of graphs G_l and G_2 are calculated as

$$AWEDN(G_1) = \frac{54}{\binom{28}{2}} \approx 0.14286$$

and

$$AWEDN(G_2) = \frac{64}{\binom{28}{2}} \approx 0.16931.$$

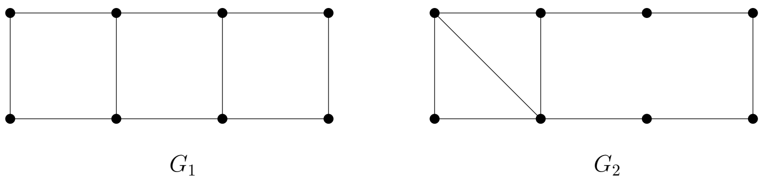

Fig. 1: The graphs G_1 and G_2 with 8 vertices.

In a graph G, the $AWEDN$ being at large shows that relation between vertices is intense, and this leads graph is invulnerable. In the case of a corruption in any line between two vertices, continuity of communication over different lines is more possible in the graph which has greater $AWEDN$ value. Therefore, the graph with greater $AWEDN$ value is more preferable when to make a choice between two graph models. It is enough to set the $PWEDN$ value at least two to increase the resistance of the graph against a possible corruption on edges.

Theorem 1. *Let $G = (V,E)$ be a graph and $u, v \in V$. If we add $e = (u,v)$ to the graph G to get the new graph $G+e$, then the $TPWEDN$ value increases at least the rate of $deg(u)deg(v) + deg(u) + deg(v)$.*

Proof. If the edge $e = (u,v)$ is added to the graph G, then the number of edge pairs in G and $PWEDN$ values in the actual pairs increase. $PWEDN$ values in the actual pairs increase by one or remain stable. Those pairs whose $PWEDN$ values increase are the ones formed by the edges emerging from extreme vertex of the added edge e. Let $deg(u) = m$ and $deg(v) = n$. The number of pairs formed by the edges emerging from u and v is mn. By joining vertices u and v, $PWEDN$ value of mn many pairs is increased by 1. Therefore, $TPWEDN$ value is also increased by 1. The add-on edge $e = (u,v)$ leads new pairs as much as the number of edges in G. Although it is hard to express $PWEDN$ values of these new pairs exactly, it is obvious that the edges which adjacent to end vertex increases $TPWEDN$ the at least $m+n$. As a result, when an edge $e = (u,v)$ is added on a graph G, $TPWEDN$ value increases at least by $deg(u)deg(v) + deg(u) + deg(v)$.

Corollary 1. *To make the increasing value of $TPWEDN$ and $AWEDN$ maximum by adding an edge on a graph $G = (V,E)$, non-edge connected and having the greatest vertex value vertices must be connected.*

Theorem 2. *Let $G = (V,E)$ be a graph. If H is a spanning subgraph of G, then $AWEDN(H) < AWEDN(G)$.*

Proof. If H is a spanning subgraph of G, then by the definition, vertices number of G and H are equal. Since the edge number of H is lesser than edge number of G and by Theorem 1, $TPWEDN(H) < TPWEDN(G) \Rightarrow AWEDN(H) < AWEDN(G)$.

Theorem 3. *Let G_l and G_2 be two graphs with same number of vertices and have same diameter. If $|E(G_l)| > |E(G_2)|$, then $AWEDN(G_l) > AWEDN(G_2)$.*

*Proof.*In the case of G_l and G_2 have the same amount of vertices, $AWEDN$ value determines the relationship intensity amongst the edges. Hence, the proof is straightforward.

Graphs G_1 and G_2 in Figure 2 have the same amount of vertices. Graph G_1 has 4 edges and $TPWEDN(G_1) = 18$ whilst Graph G_2 has 5 edges and $TPWEDN(G_2) = 15$. In spite of G_1 has the lesser amount of edges, it has greater $TPWEDN$ value, there by greater $AWEDN$ value. The reason is that their diameters are not equal. Diameter of G_1 is 2 and diameter of G_2 is 3. Since G_1 has lesser diameter, relation between its vertices is more intense. $TPWEDN$ value is increased by the intensity of the relation between the edges. Edges of two graphs with same amount of vertices and same diameter have similar relationship. In the graph which has more edges has greater $TPWEDN$ value and thereby has greater $AWEDN$ value. Therefore, amongst the two graphs with same amount of vertices and same diameter, the one with more edges has the greater $AWEDN$ value.

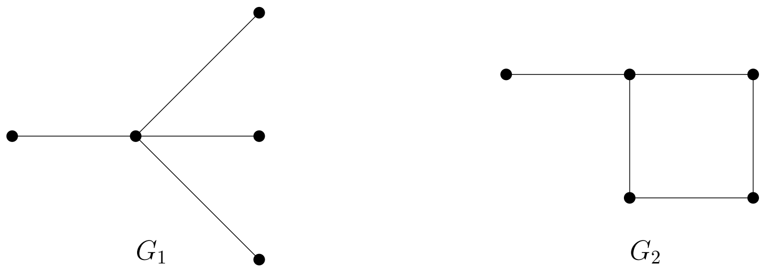

Fig. 2: The graphs G_1 and G_2 with 5 vertices.

Theorem 4. *Let $G = (V,E)$ be a simple, connected graph with n vertices. The $AWEDN$ value of G is bounded by*

$$\frac{1}{3} \leq AWEDN(G) \leq \frac{8(n-2)}{n+1}$$

for $n \geq 3$.

Proof. For $n = 3$, a connected graph with minimum edges is the path graph P_3 and $AWEDN(P_3) = \frac{1}{3}$. $AWEDN$ value reaches to the maximum for the complete graph with $n$vertices. Therefore, for any graph G having more than 2 edges

$$\frac{1}{3} \leq AWEDN(G) \leq \frac{8(n-2)}{n+1}.$$

Theorem 5. *Let G_1 and G_2 be two different graphs with same diameter and same amount of vertices. If average degree of G_1 is greater than the average vertex degree of G_2, then $AWEDN(G_1) > AWEDN(G_2)$.*

Proof. In a graph $G = (V,E)$, all edges has two extreme vertices and while calculating the vertex degree all edges are counted exactly twice. Hence the number of edges can be found by

$$|E| = \frac{1}{2}\sum_{v\in V} deg(v).$$

Since the average vertex degree is

$$d(G) = \frac{1}{|V|}\sum_{v\in V} d(V),$$

the summation $\sum_{v\in V} d(V) = d(G)|V|$. Therefore,

$$|E| = \frac{1}{2}\sum_{v\in V} d(V) = \frac{1}{2}d(G)|V|.$$

Since the graphs G_1 and G_2 have the same amount of vertices and $d(G_1) > d(G_2)$, the number of edges of G_1 is greater than the number of edges of G_2. Therefore, $AWEDN(G_1) > AWEDN(G_2)$.

Theorem 6. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are different two simple connected graphs. If $|V_1| = |V_2|$, $diam(G_1) = diam(G_2)$, and $\delta(G_1) > \delta(G_2)$, then $AWEDN(G_1) > AWEDN(G_2)$.*

*Proof.*In any graph, the sum of vertex degree is equal to 2 times of number of edges. The order in minimum vertex degrees $\delta(G_1) > \delta(G_2)$ implies that the $|E_1| > |E_2|$. Therefore, by the Theorem 3, $AWEDN(G_1) > AWEDN(G_2)$.

Definition 4. [8] *Let $G = (V, E)$ be a connected graph and $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ are two distinct edges of G. The distance between e_1 and e_2 is defined as*

$$ed(e_1, e_2) = \min\{d(u_1, u_2), d(u_1, v_2), d(v_1, u_2), d(v_1, v_2)\}.$$

Definition 5. [8] *$G = (V, E)$ be a connected graph with n vertices. The average edge distance of G can be defined as*

$$\mu'(G) = \frac{\sum_{e_i, e_j \in E} ed(e_i, e_j)}{\binom{n}{2}}.$$

Lemma 1. *Let $G = (V, E)$ be a graph and $e_1 = (u_1, v_1), e_2 = (u_2, v_2) \in E$. If $ed(e_1, e_2) \geq 2$, then $PWEDN(G) = 0$; and if $ed(e_1, e_2) < 2$, then $PWEDN(G) \geq 1$.*

Proof. By the definition of edge distance, there is n dominating edge in the case of $ed(e_1, e_2) \geq 2$, therefore $PWEDN(G) = 0$. If $ed(e_1, e_2) < 2$, then there exists at least one edge that dominating e_1 and e_2. Hence, $PWEDN(G) \geq 1$ in this case.

Theorem 7. *Let G_1 and G_2 be two graphs with same number of vertices and have same diameter. If $\mu'(G_1) < \mu'(G_2)$, then $AWEDN(G_1) > AWEDN(G_2)$.*

Proof. The average edge distance of G_1 is lesser than the average edge distance of G_2 means that the sum of the distances between edges is also lesser. Since the number of vertices are the same, the number of edges of G_1 is greater than the number of edges of G_2. Since these two graphs have the same diameter, by the Theorem 3, we can conclude that $AWEDN(G_1) > AWEDN(G_2)$.

3 Results

In this section, we present the $AWEDN$ value for certain graph classes. First, we consider the basic graph classes like Path Graphs (P_n), Cycle Graphs (C_n), Complete Graphs (K_n), Complete Bipartite Graphs ($K_{m,n}$), and Wheel Graphs ($W_{m,n}$). The basic definitions and theorems on these special classes can be found in [2].

Theorem 8. *For $n \geq 3$, $AWEDN(P_n) = \dfrac{8(2n-5)}{n(n-2)(n^2-1)}$.*

Proof. For $n = 3$, P_3 involves only one edge pair. Since these edges are adjacent, then $AWEDN(P_3) = TPWEDN(P_3) = 1$.

Let us now consider the case $n > 3$. Since P_n has $n-1$ many edges, any edge pair in P_n the $AWEDN(P_n)$ is either 0 or 1. We may obtain this result as

$$AWEDN(P_n) = \begin{cases} 0, \text{ if } ed(e_i, e_j) \geq 2 \\ 1, \text{ otherwise.} \end{cases}$$

by using the Lemma 1.

In P_n there exist two edges that is incident to the end vertices, and there exist two other edges that are one or zero edge distant to those. Hence, the total of the $PWEDN$ of those two including the later two is equal to 4. There exist two adjacent edges of the edges incident to end vertices, and each of them has the edge distance of 0 or 1 to three more edges. For these two edges, the total of $PWEDN$ is equal to 6. $n-5$ many edges other than those four edges have the edge distance of 0 or 1 to four more edges, and those have the total of $PWEDN$ as $4(n-5)$.

If any edge pair is counted, then

$$TPWEDN(P_n) = \frac{4+6+4(n-5)}{2} = 2n-5.$$

Therefore,

$$AWEDN(P_n) = \frac{2n-5}{\binom{m}{2}} = \frac{8(2n-5)}{n(n-2)(n^2-1)},$$

where $m = \binom{n}{2}$.

Theorem 9. *For $n \geq 3$, $AWEDN(C_n) = \dfrac{16}{(n-2)(n^2-1)}$.*

Proof. In C_n there exist four distinct edges that are at the distance of zero or one to any other edges. Hence, for all edge pairs the $PWEDN$ is equal to 4, and the $TPWEDN$ is equal to $4n$. Since the each neighborhood is counted twice,

$$TPWEDN(C_n) = 2n.$$

Therefore,

$$AWEDN(C_n) = \frac{2n}{\binom{m}{2}} = \frac{16}{(n-2)(n^2-1)},$$

where $m = \binom{n}{2}$.

Theorem 10. *For $n \geq 3$, $AWEDN(K_{1,n-1}) = \dfrac{4(n-2)}{n(n+1)}$.*

Proof. In $K_{1,n-1}$, the central vertex is adjacent to the rest. This concludes that all edges are adjacent and at the distance of 0. Since any edge in $K_{1,n-1}$ is adjacent to other $n-2$ edges, the $PWEDN$ is $n-2$ for any edge pairs. Therefore,

$$AWEDN(K_{1,n-1}) = \frac{\binom{n-1}{2}(n-2)}{\binom{m}{2}} = \frac{4(n-2)}{n(n+1)},$$

where $m = \binom{n}{2}$.

Theorem 11. $AWEDN(W_{1,n-1}) = \dfrac{4(n^2+16)}{n(n-2)(n+1)}$.

Proof. The edge pairs in $W_{1,n-1}$ are considered in four cases:

- $PWEDN$ value of the edges in outer cycle:

  Since the outer cycle of $W_{1,n-1}$ involves $n-1$ many edges, the number of edge pairs is equal to $\binom{n-1}{2}$. The $TPWEDN$ value of the outer cycle of $W_{1,n-1}$ can be computed like the $TPWEDN$ of C_n. Moreover, in $W_{1,n-1}$, the central vertex is adjacent to all vertices in the outer cycle. Therefore, the $PWEDN$ value is added by 1 for all edge pairs and the total of the $PWEDN$ is equal to $2(n-1)+(n+1)=3(n-1)$.

•$PWEDN$ value of the edges in C_3:

The number of the edge pairs in C_3 is equal to $2(n-1)$. To obtain C_3 on $W_{1,n-1}$, we need to choose two pairs from the inner part and one pair from the outer cycle. Since there exists $n-1$ many edges on the outer cycle, we can obtain $n-1$ many C_3. There exists only three distinct edge pairs on C_3. Two of them are the pairs that are composed by the edges on the outer cycle, the other one is the pair that is composed by the two edges in the inner part. Since the pairs composed by the edges in inner part is studied in next case, we only consider the former two in this case. Since we have $(n-1)$ many C_3, the number of the edge pairs is $2(n-1)$. For any edge pairs in C_3, the $PWEDN$ value is equal to 3. Hence, the total of the $PWEDN$ value is equal to $6(n-1)$.

•$PWEDN$ value of the edges in the inner part:

There exist $n-1$ many edges adjacent to the inner vertex of $W_{1,n-1}$. Hence, the number of edge pairs in equal to $\binom{n-2}{2}$. Any two of these edges are adjacent to the rest $n-3$ many edges. Since these two edges are also adjacent, the $PWEDN$ value is equal to $n-2$. Since $n-1$ many of the edge pairs are edges of C_3 with an edge on the outer cycle, the total of the $PWEDN$ value is equal to $\binom{n-1}{2}(n-2)+(n-1)$.

•$PWEDN$ value of the vertices that do not share a common edge:

All edges in the inner part of $W_{1,n-1}$ are adjacent. Therefore, the edges of the edge pairs is chosen either both from the outer cycle or one edge from the inner part and the other is from the outer cycle. That is, at least one of the edge pairs that composed by the edges which do not have a common vertex is on the outer cycle. Any edge on the outer cycle is adjacent to two edges on the outer cycle and two edges in the inner part.Hence, any edge on the outer cycle do not have a common vertex with $2(n-1)-5$ many edges. Since we have already studied the $PWEDN$ value of the edges in outer cycle, the number of the edge pairs in this case turns to be equal to $2(n-1)-5-(n-4)=n-3$.

The number of the edge pairs of composed by the one of the $n-1$ many edges on outer cycle and one of the $n-1$ many edges from the inner part that do not have a common vertex is equal to $(n-1)(n-3)$. an edge on the outer cycle is on two distinct C_4 with two distinct edges in the inner part since they do not share common vertices, and the $PWEDN$ value of these edge pairs are equal to 3. The $PWEDN$ value of the edge pairs composed by the rest $n-5$ edges is equal to 2 because one of the vertices of the edge in inner part is adjacent to the vertices on the outer cycle. Therefore, $PWEDN$ value of the vertices that do not share a common edge is equal to $(n-1)(6+2(n-5))=2(n-1)(n-2)$.

There exist $\binom{2(n-1)}{2}$ edge pairs in $W_{1,n-1}$. Hence,

$$AWEDN(W_{1,n-1})=\frac{3(n-1)+6(n-1)+\binom{n-1}{2}(n-2)+(n-1)+2(n-1)(n-2)}{\binom{m}{2}}=\frac{4(n^2+16)}{n(n-2)(n+1)},$$

where $m=\binom{n}{2}$.

Theorem 12. *For $m,n\geq 2$, $AWEDN(K_{m,n})=\dfrac{4mn(m+n-2)}{(m+n)(m+n-1)(m+n+1)}$.*

Proof. In $K_{m,n}=(V_1\sup V_2,E)$, there exist $\binom{mn}{2}$ many edge pairs and we study them in three cases:

•The edge pairs sharing a common vertex in V_1: In this case, we have $m\binom{n}{2}$ many edge pairs. The $PWEDN$ value is equal to $n-1$ for these edge pairs. Any vertex in V_1 is adjacent to all vertices in V_2, hence there exist n many edges adjacent to a vertex in V_1. The edge pair composed by two of these are adjacent to $n-2$ many edges and since these two edges are adjacent the $PWEDN$ value is equal to $n-1$. Therefore, the total of the $PWEDN$ value is equal to $m\binom{n}{2}(n-1)$.

- The edge pairs sharing a common vertex in V_2: In this case, we have $n\binom{m}{2}$ many edge pairs. The total of the $PWEDN$ value can be calculated by following the same fashion of the former case as $n\binom{m}{2}(m-1)$.
- The edge pairs sharing no common vertex: The edge pairs sharing no common vertex are in C_4. There exist $2\binom{m}{2}\binom{n}{2}$ many edge pairs composed by such edges. Hence, the total of the $PWEDN$ value is equal to $4\binom{m}{2}\binom{n}{2}$.

$TPWEDN$ value for these three cases is equal to $m\binom{n}{2}(n-1)+n\binom{m}{2}(m-1)+4\binom{m}{2}\binom{n}{2}$. Hence,

$$
\begin{aligned}
AWEDN(K_{m,n}) &= \frac{m\binom{n}{2}(n-1)+n\binom{m}{2}(m-1)+4\binom{m}{2}\binom{n}{2}}{\binom{k}{2}} \\
&= \frac{\frac{mn}{2}(m+n-2)^2}{\frac{(m+n)(m+n-1)(m+n-2)(m+n+1)}{8}} \\
&= \frac{4mn(m+n-2)}{(m+n)(m+n-1)(m+n+1)},
\end{aligned}
$$

where $k=\binom{m+n}{2}$.

Theorem 13. $AWEDN(K_n)=\dfrac{8(n-2)}{n+1}$.

Proof. In K_n, there exist $\binom{\binom{n}{2}}{2}$ many edge pairs and we study them in two cases:

- Edge pairs composed by adjacent edges: In a complete graph K_n, any edge is adjacent to $2(n-1)$ many edges. The number of the edge pairs composed by the adjacent edges is equal to $\binom{n}{2}(n-2)$. The $PWEDN$ value is equal to $n-1$ for adjacent edges and the total of the $PWEDN$ values for all adjacent edges is equal to $\binom{n}{2}(n-2)(n-1)$.
- Edge pairs composed by nonadjacent edges: If we remove any two vertices of a complete graph K_n, then we obtain the complete graph K_{n-2}. To calculate the number of the nonadjacent edges, we choose any edges and conclude that the edges in the subgraph obtained by vertex removing are nonadjacent to former ones. Hence, the number of the nonadjacent edges is equal to $\dfrac{\binom{n}{2}\binom{n-2}{2}}{2}$. $PWEDN$ value for the edge pairs formed by nonadjacent edges is then equal to $2\binom{n}{2}\binom{n-2}{2}$.

Hence,

$$AWEDN(K_n)=\frac{\binom{n}{2}(n-2)(n-1)+\frac{\binom{n}{2}\binom{n-2}{2}4}{2}}{\binom{\binom{n}{2}}{2}}=\frac{8(n-2)}{(n+1)}.$$

For the rest of this section, we present $AWEDN$ number for $(n,k)-$Banana Tree ($B_{n,k}$) and Binary Trees ($B-$tree).

Theorem 14.

$$AWEDN(B_{n,k})=\frac{n(k-1)+n(n-1)+n(k-2)\binom{k-1}{2}(n-1)\binom{n}{2}}{\binom{m}{2}},$$

where $m=\binom{nk+1}{2}$.

Proof. For the Banana tree $B_{n,k}$ we have five cases for edge pairs:

- $PWEDN$ value of edge pairs composed by the edges in same star graph is equal to 1. Since there exist n many edges coming out the root vertex and there exist $k-1$ many edges in k-star graph, we have $n(k-1)$ many edges pairs in this case. Hence $TPWEDN$ is equal to $n(k-1)$ for this case.
- There exist $n(n-1)(k-1)$ many edge pairs composed by an edge coming out the root vertex and an edge in the different star graph. The distance between the edges in these pairs is equal to 1. Hence, $TPWEDN$ is equal to $n(n-1)$ for this case.
- There exist $n\binom{k-1}{2}$ many edge pairs in each star graphs and the $PWEDN$ value for any edge pairs is equal to $k-2$. Therefore, the $TPWEDN$ value is equal to $n(k-1)\binom{k-1}{2}$.
- There exist $\binom{n}{2}$ many edge pairs composed by edges coming out the root vertex and for any this kind of edge pairs have the $PWEDN$ value as $n-1$. Hence, the $TPWEDN$ is equal to $(n-1)\binom{n}{2}$ for this case.
- There exist $(k-1)^2\binom{n}{2}$ many edge pairs composed by edges belonging to different star graphs. Since the distance is great and equal to 2 for these edge pairs, the $PWEDN$ value is equal to 0. Hence, the $TPWEDN$ is equal to $n(k-1)+n=nk$ for this case.

Hence,

$$AWEDN(B_{n,k})=\frac{n(k-1)+n(n-1)+n(k-2)\binom{k-1}{2}(n-1)\binom{n}{2}}{\binom{m}{2}},$$

where $m=\binom{nk+1}{2}$.

Theorem 15. *For a B-tree with height n,*

$$AWEDN(B-tree)=\frac{10.2^n-23}{\binom{m}{2}},$$

where $m = \binom{2^{n+1}-1}{2}$.

Proof. There exist 2^h many vertices on B-tree at each height h. When we pass to next height, 2^{h+1} many vertex added since we add two vertices to each vertex. Therefore, for a B-tree with height n, there exist $2^{n+1}-2$ edges. For a binary tree, the $PWEDN$ value is equal to 0,1, or 2. The edge pairs in $K_{1,3}$ star graph have the $PWEDN$ value as 2. In each branching, there emerge the vertex number many $K_{1,3}$. Hence, for $h \leq 2$, there exist $2^1+2^2+\ldots 2^{n-1} = 2^n-2$ many $K_{1,3}$. Since the number of edge pairs in this case is equal to $\binom{3}{2}(2^n-2)$, the $TPWEDN$ value is equal to $2\binom{3}{2}(2^n-2)$.

Following the similar way, we may find the number of edge pairs whose $PWEDN$ value is equal to 1 as

$$\begin{aligned}
1+4+2^2 2^2+2^3 2^2+2^4 2^2+\ldots+2^{n-1}2^2 &= 1+4+16+32+\ldots+2^{n+1} \\
&= 5+4\left(1+2+2^2+\ldots+2^{n-3}\right) \\
&= 5+2^4\left(\frac{2^{n-2}-1}{2-1}\right) \\
&= 2^{n+2}-11.
\end{aligned}$$

Therefore, the $TPWEDN$ value is equal to $2\binom{3}{2}(2^n-2)+2^{n+2}-11 = 10.2^n-23$. Hence,

$$AWEDN(B-tree) = \frac{10.2^n-23}{\binom{m}{2}},$$

where $m = \binom{2^{n+1}-1}{2}$.

4 The Algorithm

In this section, we present an algorithm in PASCAL codes to find the $AWEDN$ of a graph G. In the definition block of the algorithm, n is defined as the number of vertices and m is defined as the number of edges. The algorithm firstly construct the distance matrix by using the Warshall-Floyd algorithm, then by using the distance matrix, it calculates the $TPWEDN$ value. At the last step, it divides the $TPWEDN$ value by the number of edge pairs to obtain $AWEDN$ value. The time complexity of the algorithm is $\mathcal{O}(n^3)$.

```
uses crt,graph;
type
koor=record
x,y :longint;
end;
const
n= ; m=;
var
E : array[1..m] of koor;
d ,A : array[1..n,1..n] of longint;
ED : array[1..m,1..m] of longint ;
TPWEDN : longint;
k , i , j ,q,u1,u2,v1,v2,enk,r,t : byte;
```

```
AWEDN ,c ,t1 :real;
begin
q:=0;
for i:=1 to n do begin
for j:=1 to n do begin
readln(A[i,j]);
if( A[i,j]=1) and (i<j) then begin
q:=q+1;
E[q].x:=i;
E[q].y:=j;
end;
end;
end;
\{ Warshall - Floyd \}
for i:=1 to n do
for j:=1 to n do
if A[i,j]>0 then d[i,j]:=A[i,j]
else if i=j then d[i,j]:=0 else d[i,j]:=10000;
for k:=1 to n do begin
for i:=1 to n do begin
for j:=1 to n do begin
if d[i,j]>d[i,k]+d[k,j] then d[i,j]:=d[i,k]+d[k,j];
end;
end;
end;
for r:=1 to q do begin
ED[r,r]:=50000;
u1:=E[r].x;
v1:=E[r].y;
for t:=r+1 to q do begin
u2:=E[t].x;
v2:=E[t].y;
enk:=D[u1,u2];
if (u1=u2) or (u1=v2) or (v1=u2) or (v1=v2) then begin
ED[r,t]:=0 ; ED[t,r]:=0;
end;
if (ED[r,t]<>50000) or (Ed[r,t]<>0) then begin
if enk >= D[u1,v2] then enk:=D[u1,v2];
if enk >= D[v1,u2] then enk:=D[v1,u2] ;
if enk >= D[v1,v2] then enk:=D[v1,v2];
ED[r,t]:=enk; ED[t,r]:=enk;
end;
end;
end;
TPWEDN:=0;
for i:=1 to (q-1) do begin
```

```
for j:=i+1 to q do begin
if ED[i,j]=0 then TPWEDN:=TPWEDN+1;
for k:=1 to q do begin
if (ED[i,k]=0) and (ED[j,k]=0) then TPWEDN:= TPWEDN +1;
end;
end;
end;
writeln(TPWEDN);
t1:=(n*(n-1))/2; c:=(t1*(t1-1))/2;
AWEDN:= TPWEDN /c;
writeln(Average Edge Domination Number=,AWEDN);
end.
```

5 Conclusion

In this study we present a new measure for vulnerability instead of well known measures. The main advantage of this measure is that it can be computed in a polynomial runtime. In any graph, the least $PWEDN$ value indicates the graph has the highest rate of resistance to corruption on the edge pairs. In the case of $PWEDN$ value is 2, even though one of the edges is off-line, shows that there is an alternative line; therefore it increases the continuity of the communication amongst the edges. Hence, $AWEDN$ value can be used to make a choice between two graph models as to be used to design an invulnerable graph model.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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