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A new iterative approach for solving nonlinear programming problem

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Abstract: In this paper, we proposed a new iterative approach for solving the nonlinear programming (NLP) problem having n nonlinear (or linear) algebraic equality constraints with nonlinear (or linear) algebraic objective function in $n+1$ variables. The advantage of this developed iterative approach is to construct different optimization problems corresponding to the parameter related with arbitrary points which are chosen satisfying the constraints. Solution(s) obtained from constructed optimization problem(s) satisfies the constraints oversensitively. Several numerical examples are given to illustrate the proposed approach.

Keywords: Taylor Series, linearization algorithm, linear programming.

1 Introduction

Optimization theory deals with the minimization (maximization) of an objective function subject to a set of constraints. Mathematical programming is a branch of optimization theory with minimized (maximized) single (multiple) valued objective function f in n real variables $x_1,x_2,...,x_n$ subject to a finite number of constraints which are written as inequalities or equalities. Optimization problems can be classified regarding as the nature of the objective function and constraints. If these are both linear, then the problem is called linear programming (LP) problem using restricted resources in order to make optimum. Profitability and analysis of mathematical modeling has made LP an important tool for solving problems in diverse fields. The operational research modelling as integer programming and stochastic programming, etc. is based on the LP approach.

The real life problems can be determined as a mathematical model to effectively find the optimal strategies. In these realistic problems, routing problems in traffic and chemical industries, applications in structural optimization, economics, marketing and business applications, etc., cannot be adequately represented or approximated as a LP problem. NLP problems appear from these viewpoints [14]. NLP refers to the defined problem in which the objective function becomes nonlinear, or one or more constraints have nonlinear or both.

There is a wide variety of approaches for solving the NLP problems in various fields of the real life. The first approach was based on the idea of iterative descent within the confines of the constraint set. A second approach was based on the possibility of solving the system of equations and inequalities which constitute necessary conditions for optimality for the optimization problem. A third approach was based on elimination of constraints through the use of penalty functions. Computational methods for solving these problems became the subject of investigation during the late fifties.

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The methods for constrained optimization can be divided into two categories as deterministic and stochastic methods. Reduced gradient methods and sequential quadratic programming (SQP) methods can be considered as the popular deterministic local optimization methods to solve having nonlinear constrained problems [13]. The issues arising in SQP is given by Boggs and Tolle [3]. The stochastic global optimization methods having probabilistic functions, such as genetic algorithms, tabu search, etc., do not require gradient information unlike deterministic methods. The augmented Lagragian function as a penalty function was first proposed by Hestenes [10], its properties are described by Fletcher [6]. Penalty and barrier methods are based on minimizing the Lagrangian function while attempting to maintain feasibility. When inequality constraints are present, these methods generalize the simplex method. They solve a sequence of sub problems until a solution to the original constrained problem is found. There are some disadvantages to this approach. First, as the number of constraints increases, the number of sub problems increases. Second, satisfying the constraints exactly can be achieved easily in the case of linear constraints, however it is much more difficult to accomplish in the case of nonlinear constraints. Logarithmic barrier methods were introduced by Frisch [7] and developed by Fiacco and McCormich [5]. The logarithmic barriers penalty function is best suited to problems which only have inequality constraints. Nash and Sofer have written a paper [15], using a log-barrier function for inequality constrained problems. Conjugate-gradient methods (CG) are used to solve large-dimensional problems that arise in computational linear or nonlinear optimization problem. The linear CG method for solving the system of linear n equations in n unknowns was developed in [11]. The method did not compete with direct method, Gauss elimination, but it is used in real-world applications. The nonlinear CG method extends the linear CG approach to the problem of minimizing a smooth nonlinear function $f(x)$ where $x \in R^n$ and n can be large. It was developed in 1964 by R. Fletcher and C. Reeves.

The decomposition method was first introduced by Adomian who gives the solution as an infinite series converging to an accurate solution for solving wide range of problems whose mathematical models yield algebraic, differential, integral equation or system of equation since the beginning of the 1980s. Abbasbandy [1] is presented some efficient numerical algorithms for solving a system of two nonlinear equations based on Newtons method. The efficient modifications are proposed for the standard Adomian decomposition method. Golbabai [8] is used effective homotopy perturbation method for solving system of nonlinear algebraic equations. Biazar [2] built up an efficient iterative method based on the Gauss-Seidel method for solving systems of nonlinear equations, which is also known for solving systems of linear equations. Jafari [12] discussed a powerful iterative method for solving systems of nonlinear equations. Vahidi [17] applied restarted Adomian decomposition method, based standard Adomian method for solving system of nonlinear algebraic equations. In Vahidi [18], the restarted Adomian decomposition method and the standard Adomian method are applied to find an approximate solution for system of nonlinear equations. In Wang [16], two-multi-step derivative-free iterative methods, which has high computational efficiency and low computational cost, are presented for solving systems of nonlinear equations.

Our focus is on nonlinear optimization problems with n nonlinear (or linear) algebraic equality constraints and nonlinear (or linear) algebraic objective function in $n+1$ variables. If all the constraints are linear, maintaining feasibility is straightforward. When nonlinear constraints are present, then more elaborate procedures are required. From this point of view, for solving the constructed NLP problem related with parameter, we proposed a new iterative approach. Obtained solution(s) satisfies the constraints oversensitively. This proposed approach leads to the novel methods.

This paper is organized as follows: Section 2 presents brief required information used in this work. In Section 3, the proposed method is handled. Section 4 and Section 5 consist of our numerical examples and conclusions, respectively.

2 Preliminaries

In this section, required information is presented.

Definition 1. *A general constrained NLP problem can be defined as follows:*

$$\begin{aligned} Min f(x) \\ s.t. \\ g_e(x) = 0, e = 1,2,...,l \\ g_i(x) \leq 0, i = l+1,...,m \end{aligned} \tag{1}$$

where $x = [x_1, x_2, ..., x_n] \in R^n$ is a vector, $g_e : R^n \to R, (e = 1,2,...,l), g_i : R^n \to R, (i = l+1,...,m)$, and $m \leq n$. A NLP problem can be defined as a maximization problem with the inequality constraints in the form $g_i(x) \geq 0, (i = l+1,...,m)$.

Definition 2. [4] *Any point x satisfying the constraints is called the feasible point. The set of all feasible points is called the feasible set such that $X = \{x \in R^n : g_e(x) = 0, (e = 1,2,...,l), g_i(x) \leq 0, (i = l+1,...,m)\}$.*

Definition 3. *An optimal solution x^* to a LP problem is a feasible solution with the smallest objective function value for a minimization problem.*

Theorem 1. [4] *If $f : R^n \to R$ is differentiable, then the function ∇f is defined by*

$$\nabla \mathbf{f}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{pmatrix}$$

is called gradient of f. If ∇f is differentiable, we say that f is twice differentiable, and we write the derivatives of ∇f as

$$\mathbf{H}(\mathbf{x}) = \begin{pmatrix} h_{11} & h_{12} & \dots & h_{1n} \\ h_{21} & h_{22} & \dots & h_{2n} \\ \dots & \dots & \dots & \dots \\ h_{n1} & h_{n2} & \dots & h_{nn} \end{pmatrix}$$

where $h_{i_j} = [\frac{\partial^2 f(x)}{\partial x_i \partial x_j}]$. The matrix $H(x)$ is called Hessian matrix of f at x. The leading principle minors of $H(x)$ are as follows:

$$\Delta_1 = |h_{11}|, \Delta_2 = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix}, ..., \Delta_n = \begin{vmatrix} h_{11} & h_{12} & \dots & h_{1n} \\ h_{21} & h_{22} & \dots & h_{2n} \\ \dots & \dots & \dots & \dots \\ h_{n1} & h_{n2} & \dots & h_{nn} \end{vmatrix} = |H(x)|.$$

Theorem 2. [4] *$H(x)$ is the Hessian matrix of function $f(x)$ and $\Delta_i, (i = 1,2,...,n)$ are the leading principle minors of $H(x)$.*

- *$H(x)$ is positive definite at x if and only if all leading principle minors are positive, i.e. $\Delta_i > 0, (i = 1,2,...,n)$,*
- *$H(x)$ is negative definite at x if and only if $\Delta_1 < 0$ and remaining $\Delta_i, (i = 2,3,...,n)$ alternate in sign,*

- *$H(x)$ is indefinite if it is neither positive definite nor negative definite.*

Definition 4. *A point x in the feasible set X is said to be an interior point if X contains some neighborhood of x.*

Theorem 3. *Let $f \in C^2$ be defined on a region in which x^* is an interior point. If*

- $\nabla f(x^*) = 0$,
- *$H(x)$ is positive definite at x^*, i.e. $H(x^*) > 0$, then x^* is called a strict local minimizer of f. x^* is called a strict local maximizer of f while satisfying the following conditions:*
- $\nabla f(x^*) = 0$,
- *$H(x)$ is negative definite at x^*, i.e. $H(x^*) < 0$.*

Definition 5. *A solution of a system of equations $g_1(x), g_2(x), ..., g_n(x)$ in n variables is a point $(a_1, a_2, ..., a_n) \in R^n$ such that $g_1(a_1, a_2, ..., a_n) = ... = g_n(a_1, a_2, ..., a_n) = 0$.*

Definition 6. *An iterative method is a procedure that is repeated over and over again to find the solution of a system of equations.*

3 Our proposed approach

Our proposed approach solves NLP problem having n nonlinear (or linear) algebraic equality constraints and nonlinear (or linear) algebraic objective function in $n+1$ variables. Some of the possible cases of our proposed approach are given as follows:

Case 1.
For solving a NLP problem having n linear algebraic equality constraints and a nonlinear algebraic objective function in $n+1$ variables is considered as

$$
\begin{aligned}
Opt\, f(x_1, x_2, ..., x_{n+1}) \\
s.t. \\
g_1(x_1, x_2, ..., x_{n+1}) = 0 \\
\vdots \\
g_n(x_1, x_2, ..., x_{n+1}) = 0
\end{aligned} \tag{2}
$$

Since there are $n+1$ variables and n linear algebraic equality constraints, the solution of the linear system,i.e. constraints of (2), depends on one parameter. These parametric variables are written in the nonlinear objective function and the parametric nonlinear objective function is obtained. First derivative of this function is taken and set equal to zero. By substituting these obtained parameter value(s) in the parametric variables, the solution(s) of the linear system is determined. These solution(s) are substituted in the second order derivative of the parametric objective function. Each obtained parameter value determines whether the solution makes the objective function of (2) maximize or minimize. If this is not possible, the second derivative for the vicinity of the parameter value(s) is checked. The process is terminated whether making the problem optimize is not possible.

Case 2.
The problem (2) is considered as having n nonlinear algebraic equality constraints and a nonlinear algebraic objective function in $n+1$ variables. Initial arbitrary points satisfying the equations,i.e. constraints of (2), individually are chosen.

Each constraint is expanded to Taylor series at chosen point and then the same solution process continues as Case 1. The Hessian matrix of the objective function $f(x_1,x_2,...,x_{n+1})$ is constructed. The leading principal minors of Hessian matrix, $\Delta_j,(j=1,2,...,n+1)$ are determined such that the objective function of (2) will be optimized. The following new variables are generated:

$$x_j=\bar{x}_j+u_j-v_j, j=1,2,...,n+1 \quad (3)$$

where $\bar{x}_j,(j=1,2,...,n+1)$ values of variables obtained by considering the parameter values, u_j and $v_j,(j=1,2,...,n+1)$ are new balancing variables.

Substituting the generated new variables $(x_1,x_2,...,x_{n+1})$ in the constraints of (2) and considering the leading principal minors, the following new nonlinear system is constructed:

$$\begin{aligned} g_1(x_1,x_2,...,x_{n+1})&=0\\ &:\\ g_n(x_1,x_2,...,x_{n+1})&=0\\ \Delta_j(\leq,\geq)0, j=1,2,...,n&+1 \end{aligned} \quad (4)$$

Each equation in (4) is expanded to Maclaurin Series and the following linear system (5) is constructed:

$$\begin{aligned} g_{1L}(u_j,v_j)&=0\\ &:\\ g_{nL}(u_j,v_j)&=0\\ \Delta_{jL}(\leq,\geq)0, j=1,2,...,n&+1 \end{aligned} \quad (5)$$

By adding new variables $u_k,v_k,(k=n+2,...,2n+1)$ and $u_p,v_p,(p=2n+2,...,3n+2)$ to (5), the following LP problem is obtained:

$$\begin{aligned} Min\sum_{j=1}^{n+1}(u_j+v_j)+\sum_{k=n+2}^{2n+1}(u_k+v_k)+\sum_{p=2n+2}^{3n+2}(u_p+v_p)&\\ s.t&\\ g_{iL}(u_j,v_j)+u_k-v_k=0, i=1,2,...,n; k=n+2,...,2n+1&\\ \Delta_{jL}(u_j,v_j)+u_p-v_p(\leq,\geq)0, j=1,2,...,n+1; p=2n+2,...,3n+2.& \end{aligned} \quad (6)$$

For all $u_j,v_j,(j=1,2,...,n+1)$, $u_k,v_k,(k=n+2,...,2n+1)$ and $u_p,v_p,(p=2n+2,...,3n+2)$ the problem (6) is solved. If all $u_j,v_j,(j=1,2,...,n+1)$ are not zero, considering $u_j,v_j,(j=1,2,...,n+1)$ in (3), the new variables $x_1,x_2,...,x_{n+1}$ are found. Corresponding $x_1,x_2,...,x_{n+1}$ to $\bar{x}_1,\bar{x}_2,...,\bar{x}_{n+1}$, respectively, the process from the beginning of (3) to the end of (6) is applied until all $u_j,v_j,(j=1,2,...,n+1)$ become zero. At the end of the process, a solution $(x_1,x_2,...,x_{n+1})$ is found and the objective value is determined for the general NLP problem (2). The flow chart of finding solution of NLP problem is given in Fig. 1.

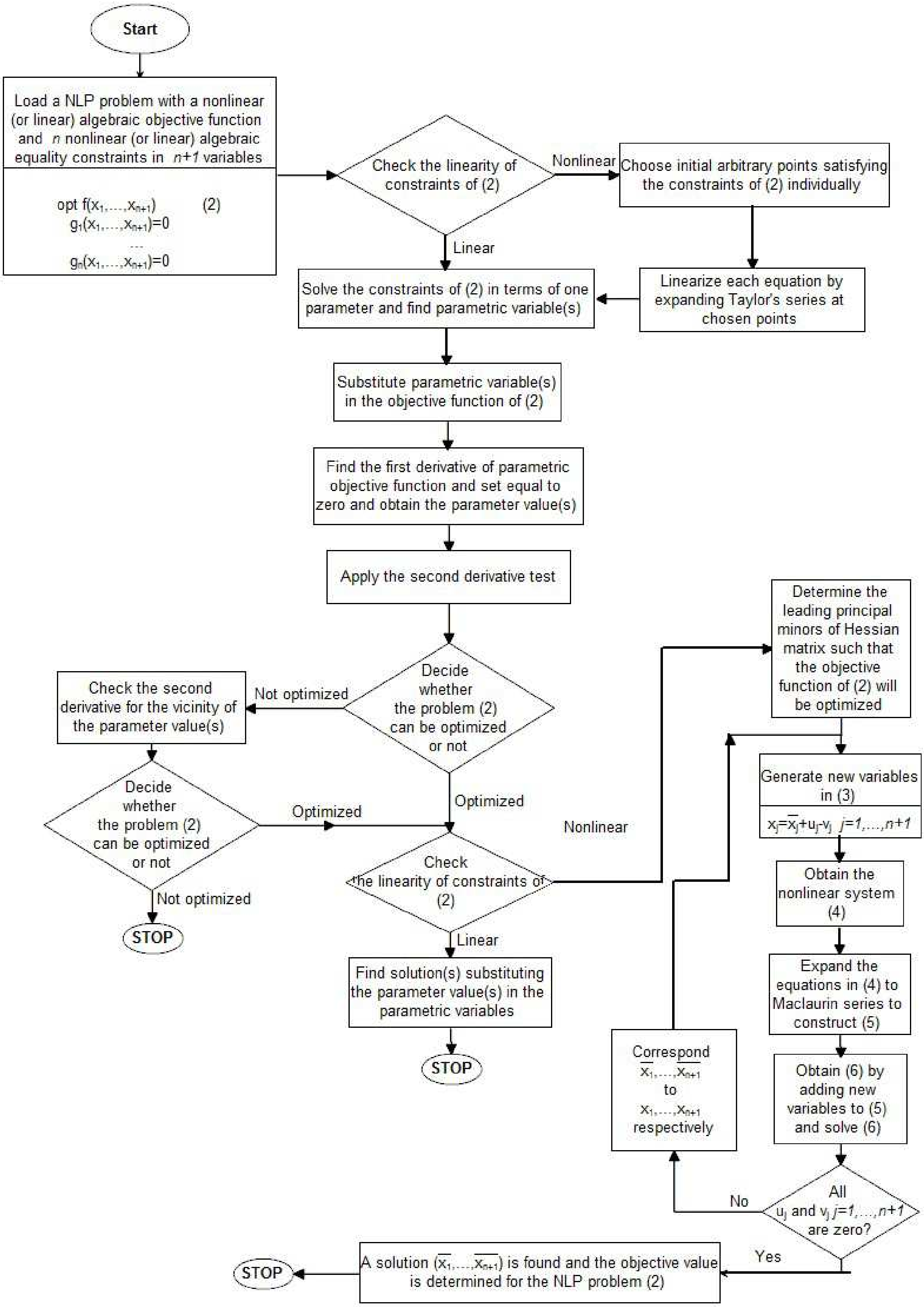

Fig. 1: The flow chart of finding solution of NLP problem.

4 Numerical experiments

Example 1. Consider the following NLP problem with two linear algebraic equality constraints and a nonlinear algebraic objective function in three variables

$$\begin{aligned} z &= x_1^2 + x_2^2 + x_3^2 \\ g_1(x) &= x_1 + x_2 + 3x_3 - 2 = 0 \\ g_2(x) &= 5x_1 + 2x_2 + x_3 - 5 = 0 \end{aligned} \tag{7}$$

Solving the constraints of (7), $x_1 = \frac{1}{3} + \frac{5t}{3}$, $x_2 = \frac{5}{3} - \frac{14t}{3}$, $x_3 = t$ are determined. Substituting the variables in the objective function of (7), taking the first derivative of the parametric objective function, setting equal to zero, $t = 0.2826$ is found. Considering this parameter value, the second derivative of the parametric objective function is tested. According the result of the test, the optimization problem (7) is found as a minimization problem. Thus, using this parameter value, the solution $x_1 = 0.8044$, $x_2 = 0.3478$ and $x_3 = 0.2826$ is obtained and the objective value of the NLP problem (7) is 0.8479.

Example 2. Consider the following NLP problem with two nonlinear algebraic equality constraints and a nonlinear algebraic objective function in three variables

$$\begin{aligned} z &= x_1^2 + 3x_2^2 + 5x_1x_3^2 \\ g_1(x) &= x_1x_3 + 2x_2 + x_2^2 - 11 = 0 \\ g_2(x) &= x_1^2 + 2x_1x_2 + x_3^2 - 14 = 0 \end{aligned} \tag{8}$$

Firstly, the constraints of (8) are made linear by expanding Taylor series at chosen arbitrary points $(1,1,8),(1,6,1)$, respectively. Thus, the following equations are obtained:

$$\begin{aligned} 8x_1 + 4x_2 + x_3 &= 20 \\ 7x_1 + x_2 + x_3 &= 14 \end{aligned} \tag{9}$$

By solving (9), $x_1 = \frac{36-3t}{20}$, $x_2 = \frac{28+t}{20}$, $x_3 = t$ are found. Substituting the parametric variables in the objective function of (8), by taking the first derivative of the parametric objective function and setting equal to zero, the parameter values $t_1 = 0.0067$ and $t_2 = 8.02$ are found. Considering $t_1 = 0.0067$, the second derivative of the parametric objective function is tested. According to the result of the test, the optimization problem (8) will be solved as a maximization problem. For solving (8), the leading principal minors of Hessian matrix of objective function of (8) are determined such that (8) will be optimized.

For the parameter value $t_1 = 0.0067$, the initial solution obtained as $x_1 = 1.799$, $x_2 = 1.4003$ and $x_3 = 0.0067$. By means of the initial solution $(1.799, 1.4003, 0.0067)$, the following new variables are generated:

$$\begin{aligned} x_1 &= 1.799 + u_1 - v_1 \\ x_2 &= 1.4003 + u_2 - v_2 \\ x_3 &= 0.0067 + u_3 - v_3 \end{aligned} \tag{10}$$

Table 1: Results and Analysis of Example 2.

	Initial	1st iteration	2nd iteration	3rd iteration
x_1	1.799	1.9646	2.0114	2.0169
x_2	1.4003	2.6971	2.4696	2.4622
x_3	0.0067	0.0067	0.0067	0.0067
Constraint 1	6.2265	-1.6817	-0.0516	-0.0003
Constraint 2	5.7253	-0.4571	0.0195	0
Objective	9.1193	25.6831	22.343	22.2556

where $u_j, v_j, (j = 1,2,3)$ are new balancing variables. Substituting the generated new variables (x_1, x_2, x_3) in the constraints of (8) and considering the leading principal minors, the following new nonlinear system is constructed:

$$\begin{aligned} g_1(x) &= 0 \\ g_2(x) &= 0 \\ \Delta_1 \leq 0, \Delta_2 \geq 0, \Delta_3 &\leq 0 \end{aligned} \tag{11}$$

Each equation in (11) is expanded to Maclaurin Series and the following linear system (12) is obtained:

$$\begin{aligned} g_{1L}(u_j, v_j) &= 0 \\ g_{2L}(u_j, v_j) &= 0 \\ \Delta_{1L}(u_j, v_j) &\leq 0 \\ \Delta_{2L}(u_j, v_j) &\geq 0 \\ \Delta_{3L}(u_j, v_j) &\leq 0, j = 1,2,3. \end{aligned} \tag{12}$$

By adding new variables $u_k, v_k, (k = 4,5)$ and $u_p, v_p, (k = 6,7,8)$ to (12), the following LP problem (13) is determined:

$$\begin{aligned} Min \sum_{j=1}^{3}(u_j + v_j) + \sum_{k=4}^{5}(u_k + v_k) + \sum_{p=6}^{8}(u_p + v_p) \\ s.t \\ 0.0067(u_1 - v_1) + 4.8006(u_2 - v_2) + 1.799(u_3 - v_3) + 4.7735 + u_4 - v_4 = 6.2265 \\ 6.3986(u_1 - v_1) + 3.5980(u_2 - v_2) + 0.0134(u_3 - v_3) + 8.2747 + u_5 - v_5 = 5.7253 \\ 2 + u_6 - v_6 \leq 0 \\ 12 + u_7 - v_7 \geq 0 \\ u_1 - v_1 - 0.067(u_3 - v_3) + 1.7988 + u_8 - v_8 \leq 0 \end{aligned} \tag{13}$$

For all $u_j, v_j, (j = 1,2,3)$,$u_k, v_k, (k = 4,5)$ and $u_p, v_p, (k = 6,7,8)$, the problem (13) is solved and $u_1 = 0.1656, v_1 = 0$, $u_2 = 1.2968, v_2 = 0$, $u_3 = 0, v_3 = 0$ are found. All $u_j, v_j, (j = 1,2,3)$ are found zero at the third iteration. Thus, the solution and objective value for the NLP problem (8) are found as $(x_1, x_2, x_3) = (2.0169, 2.4622, 0.0067)$ and $z = 22.2556$, respectively. Satisfying the constraints of these variables and the objective value for each iteration are given in Table 1. Considering $t_2 = 8.02$, the second derivative of the parametric objective function is tested. According to the result of the test, the optimization problem (8) will be solved as a minimization problem. For solving (8), the leading principal minors of Hessian matrix of the objective function (8) are determined such that (8) will be optimized and using the same process, the solution and objective value for the NLP problem (8) are found as $(x_1, x_2, x_3) = (7.7216, -3.0222, 1.0245)$ and $z = 127.5472$, respectively.

5 Conclusion

Our proposed approach generates a point in each iteration for finding better approximation to obtain a solution for solving NLP problems having n nonlinear (or linear) algebraic equality constraints with nonlinear (or linear) algebraic objective function in $n+1$ variables. Obtained solution by using proposed approach satisfies the constraints, oversensitively. Because of the algorithm is based on parametric solutions of the constraints, it enables how the problem can be optimized for each considered parameter. The algorithm based on the LP sub problems makes clear and easy to solve the considered NLP problems.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] Abbasbandy, S. (2005). Extended Newtons Method for a System of Nonlinear Equations by Modified Adomian Decomposition Method. Applied Mathematics and Computation, 170(1), 648-656.

[2] Biazar, J. and Ghanbary, B. (2008). A New Technique for Solving Systems of Nonlinear Equations. Applied Mathematical Sciences, 2(55), 2699-2703.

[3] Boggs, P. T. and Tolle, J. W. (1995). Sequential Quadratic Programming. Acta Numerica, 4, 1-51.

[4] Chong, E. K. and Zak, S. H. (2013). An Introduction to Optimization (Vol. 76). John Wiley & Sons.

[5] Fiacco, A. V. and McCormick, G. P. (1968). Nonlinear programming: SUMT. Mgmt Sci, 10, 19.

[6] Fletcher, R. (1987). Practical Methods of Optimization, volume 1st vol. of Wiley-interscience publication. Chichester [etc.]: John Wiley & Sons, United Kingdom, USA, 2nd edition edition.

[7] Frisch, K. R. (1955). The Logarithmic Potential Method of Convex Programming. Memorandum, University Institute of Economics, Oslo, 5(6).

[8] Golbabai, A. and Javidi, M. (2007). A New Family of Iterative Methods for Solving System of Nonlinear Algebraic Equations. Applied mathematics and computation,190(2), 1717-1722.

[9] Hartree D.R. 1958, Numerical Analysis, Oxford University Press, London.

[10] Hestenes, M. R. (1969). Multiplier and Gradient Methods. Journal of Optimization Theory and Applications, 4(5), 303-320.

[11] Hestenes, M. R. and Stiefel, E. (1952). Methods of Conjugate Gradients for Solving Linear Systems (Vol. 49, p. 1). NBS.

[12] Jafari, H. and Seyfi, S. (2008). An Iterative Method for Solving a System of Nonlinear Algebraic Equations.

[13] Kao, C. (1998). Performance of Several Nonlinear Programming Software Packages on Microcomputers. Computers and Operations Research, 25(10), 807-816.

[14] Luenberger D.G. and Ye Y.(2008). Linear and Nonlinear Programming, Third Edition.

[15] Nash, S. G. and Sofer, A. (1993). A Barrier Method for Large-Scale Constrained Optimization. ORSA Journal on Computing, 5(1), 40-53.

[16] Wang, X. and Fan, X. (2016). Two Efficient Derivative-Free Iterative Methods for Solving Nonlinear Systems. Algorithms, 9(1), 14.

[17] Vahidi, A. R., Babobian, E., Cordshooli, G. A. and Mirzaie, M. (2009). Restarted Adomian Decomposition Method to Systems of Nonlinear Algebraic Equations. Applied Mathematical Sciences, 18, 883-889.

[18] Vahidi, A. R., Javadi, S. and Khorasani, S. M. (2012). Solving System of Nonlinear Equations by Restarted Adomain's Method. Journal of Applied Mathematical and Computation, 6, 509-516.