New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2018.290

On inequalities for strongly M_{φ} A-S- convex functions

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Received: 1 February 2018, Accepted: 10 April 2018
Published online: 20 July 2018.

Abstract: In this paper, it is given a new concept which is a generalization of the concepts s-convexity, M_{φ}A-convexity, M_{φ}A-s-convexity and obtained some theorems for Hermite-Hadamard type inequalities for this class of functions. Some natural applications to special means of real numbers are also given.

Keywords: M_{φ}A-s-convex function, Hermite-Hadamard type inequality.

1 Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave. For some results which generalize, improve and extend the inequalities (1) we refer the reader to the recent papers (see [1, 4, 6, 7]).

For $r \in \mathbb{R}$ the power mean $M_r(a,b)$ of order r of two positive numbers a and b is defined by

$$M_r = M_r(a,b) = \begin{cases} \left(\frac{a^r+b^r}{2}\right)^{1/r}, & r \neq 0 \\ \sqrt{ab}, & r = 0 \end{cases}.$$

It is well-known that $M_r(a,b)$ is continuous and strictly increasing with respect to $r \in \mathrm{R}$ for fixed $a, b > 0$ with $a \neq b$. Let

$$L = L(a,b) = (b-a)/(\ln b - \ln a), I = I(a,b) = \frac{1}{e}\left(a^a/b^b\right)^{1/a-b},$$

$A = A(a,b) = (a+b)/2, G = G(a,b) = \sqrt{ab}$ *and* $H = H(a,b) = 2ab/(a+b)$ be the logarithmic, identric, arithmetic, geometric, and harmonic means of two positive real numbers a and b with $a \neq b$, respectively. Then

$$\min\{a,b\} < H(a,b) = M_{-1}(a,b) < G(a,b) = M_0(a,b) < L(a,b) < I(a,b) < A(a,b) = M_1(a,b) < \max\{a,b\}.$$

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Let $\mathtt{M}$ be the family of all mean values of two numbers in $\mathbb{R}_+ = (0,\infty)$. Given $M,N \in \mathtt{M}$, we say that a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is (M,N)-convex if $f(M(x,y)) \leq N(f(x), f(y))$ for all $x,y \in \mathbb{R}_+$. The concept of (M,N)-convexity has been studied extensively in the literature from various points of view (see e.g. [2, 3, 5,]). Let

$$A(a,b;t) = ta + (1-t)b, G(a,b;t) = a^t b^{1-t}, H(a,b;t) = ab/(ta + (1-t)b)$$

and

$$M_p(a,b;t) = (ta^p + (1-t)b^p)^{1/p}$$

be the weighted arithmetic, geometric, harmonic, power of order p means of two positive real numbers a and b with $a \neq b$ for $t \in [0,1]$, respectively.

The most used class of means is quasi-arithmetic mean, which are associated to a continuous and strictly monotonic function $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ by the formula

$$M_\varphi(x,y) = \varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right), \quad \text{for } x,y \in I.$$

Weighted quasi-arithmetic mean is given by the formula

$$M_\varphi(x,y;t) = \varphi^{-1}(t\varphi(x) + (1-t)\varphi(y)), \text{ for } x,y \in I, t \in [0,1],$$

where $t \in (0,1)$ and $x < y$ always implies $x < M_\varphi(x,y;t) < y$. The function φ is called *Kolmogoroff-Naguma function of* M. The special interest are the power means M_p on $\mathbb{R}_+$, defined by

$$\varphi_p(x) := \begin{cases} x^p, & p \neq 0 \\ \ln x, & p = 0 \end{cases}.$$

For $p = 1$, we get the arithmetic mean $A = M_1$, for $p = 0$, we get the geometric mean $G = M_0$ and for $p = -1$, we get the harmonic mean $H = M_{-1}$.

For any two quasi-arithmetic means M,N (with *Kolmogoroff-Naguma function* φ,ψ defined on intervals I,J, respectively), a function $f : I \to J$ can be called (M_φ, M_ψ)-convex if it satisfies

$$f(M_\varphi(x,y;t)) \leq M_\psi(f(x), f(y);t) \quad for\ \ all\ \ x,y \in I\ \ and\ \ t \in [0,1]. \tag{2}$$

If the inequality in (2) is reversed, then f is said to be (M_φ, M_ψ)-concave. If $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$ $\psi(x) = x$, (i.e., $M_\psi(f(x), f(y);t) = A(a,b;t)$), then we just say that f is $M_\varphi A$-convex.

Let f be a $M_\varphi A$-convex. In this case

(i) If we take $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$, $\varphi(x) = x$, then $M_\varphi A$-convexity deduce usual convexity.
(ii) If we take $\varphi : I \subseteq (0,\infty) \to \mathbb{R}$ $\varphi(x) = \ln x$, then $M_\varphi A$-convexity deduce GA-convexity. (see [13,14])
(iii) If we take $\varphi : I \subseteq (0,\infty) \to \mathbb{R}$ $\varphi(x) = x^{-1}$, then $M_\varphi A$-convexity deduce harmonically convexity. (see [7])
(iv) If we take $\varphi : I \subseteq (0,\infty) \to \mathbb{R}$, $\phi(x) = x^p$, $p \in \mathrm{R} \setminus \{0\}$, then $M_\varphi A$-convexity deduce p-convexity. (see [8]).

The theory of (M_φ, M_ψ)-convex functions can be deduced from the theory of usual convex functions.

Lemma 1. [9], *If φ and ψ are two continuous and strictly monotonic functions (on intervals I and J respectively) and ψ is increasing then a function $f : I \to J$ is $\left(M_{\varphi}, M_{\psi}\right)$-convex if and only if $\psi \circ f \circ \varphi^{-1}$ is convex on $\varphi(I)$ in the usual sense.*

Definition 1. *Let $0 < s \leq 1$. A function $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ where $\mathbb{R}_0 = [0, \infty)$, is said to be s-convex in the first sense if*

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in I$ and $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$. We denote this class of real functions by K_s^1.

In [5], Hudzik and Maligranda considered the following class of functions.

Definition 2. *A function $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ where $\mathbb{R}_0 = [0, \infty)$, is said to be s-convex in the second sense if*

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in I$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and s fixed in $(0, 1]$. They denoted this by K_s^2.

It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.

In [4], Dragomir and Fitzpatrick proved a variant of Hermite-Hadamard inequality which holds for the s-convex functions.

Theorem 1. *Suppose that $f : \mathbb{R}_0 \to \mathbb{R}_0$ is an s-convex function in the second sense, where $s \in (0, 1]$ and let $a, b \in 0, \infty)$, $a < b$. If $f \in L[a, b]$, then the following inequalities hold*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{s+1}. \tag{3}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (3).

The main purpose of this paper is to introduce the concepts $M_{\varphi}A$-s-convex function in the first sense and the second sense and give the Hermite-Hadamard's inequality for these classes of functions. Morever, in this paper we establish a new identity and a consequence of the identity is that we obtain some new general integral inequalities.

2 Definitions of $M_{\varphi}A$-s-convex functions

Definition 3. *Let I be a real interval, $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous and strictly monotonic function and $s \in (0, 1]$.*

(i) *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be $M_{\varphi}A$-s-convex in the first sense, if*

$$f\left(\varphi^{-1}\left(t\varphi(x) + (1-t)\varphi(y)\right)\right) \leq t^s f(x) + (1 - t^s) f(y) \tag{4}$$

for all $x, y \in I$ and $t \in 0, 1]$. If the inequality in (4) is reversed, then f is said to be $M_{\varphi}A$-s-concave in the first sense.

(ii) *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be $M_{\varphi}A$-s-convex in the second sense, if*

$$f\left(\varphi^{-1}\left(t\varphi(x) + (1-t)\varphi(y)\right)\right) \leq t^s f(x) + (1-t)^s f(y) \tag{5}$$

for all $x, y \in I$ and $t \in 0, 1]$. If the inequality in (5) is reversed, then f is said to be $M_{\varphi}A$-s-concave in the second sense.

It can be easily seen that:

(i) For $\varphi : I \subseteq (0,\infty) \to \mathbb{R}$, $\varphi(x) = mx+n$, $m \in \mathbb{R}\setminus 0$, $n \in \mathbb{R}$, $M_{\varphi}A$-s-convexity (in the first sense or second sense) reduces to ordinary s convexity on I.

(ii) For $\varphi : I \subseteq (0,\infty) \to \mathbb{R}, \varphi(x) = \ln x$, then $M_{\varphi}A$-s-convexity deduce GA-s-convexity.

(iii) For $\varphi : I \subseteq (0,\infty) \to \mathbb{R}$, $\varphi(x) = x^{-1}$, then $M_{\varphi}A$-s-convexity deduce harmonically s-convexity.

(iv) For $\varphi : I \subseteq (0,\infty) \to \mathbb{R}$, $\varphi(x) = x^{p}$, $p \in \mathbb{R}\setminus 0$ then $M_{\varphi}A$-s-convexity deduce (p,s)-convexity.

Lemma 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° and $a,b \in I$ with $a < b$ and $\varphi^{-1} : \varphi(I^{\circ}) \to I^{\circ}$ is continously diferentiable. If $f^{'} \in L[a,b]$, then the following equality holds:*

$$\begin{aligned}
&\frac{(\varphi(x)-\varphi(a))f(a)+(\varphi(b)-\varphi(x))f(b)}{\varphi(b)-\varphi(a)}-\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(u)\varphi^{'}(u)\,du \\
&=\frac{(\varphi(x)-\varphi(a))^2}{\varphi(b)-\varphi(a)}\int_0^1 (t-1)(\varphi^{-1})'(t\varphi(x)+(1-t)\varphi(a))f'(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(a)))dt \\
&+\frac{(\varphi(b)-\varphi(x))^2}{\varphi(b)-\varphi(a)}\int_0^1 (1-t)(\varphi^{-1})'(t\varphi(x)+(1-t)\varphi(b))f'(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(b)))dt.
\end{aligned}$$

Proof. Let us define I_1 and I_2 as follows:

$$\begin{aligned}
I_1 &= \int_0^1 (t-1)\,d(f(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(a)))) \\
&= (t-1)f\left(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(a))\right)\Big|_0^1 - \int_0^1 f\left(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(a))\right)dt \\
&= f(a) - \frac{1}{\varphi(x)-\varphi(a)}\int_a^x f(u)\varphi^{'}(u)\,du,
\end{aligned}$$

$$\begin{aligned}
I_2 &= \int_0^1 (1-t)\,d(f(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(b)))) \\
&= (1-t)f\left(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(b))\right)\Big|_0^1 - \int_0^1 f\left(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(b))\right)dt \\
&= -f(b) - \frac{1}{\varphi(x)-\varphi(b)}\int_b^x f(u)\varphi^{'}(u)\,du.
\end{aligned}$$

Then we can write

$$\begin{aligned}
&(\varphi(x)-\varphi(a))\int_0^1 (t-1)(\varphi^{-1})'(t\varphi(x)+(1-t)\varphi(a))f'(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(a)))dt \\
&= f(a) - \frac{1}{\varphi(x)-\varphi(a)}\int_a^x f(u)\varphi^{'}(u)\,du, \\
&(\varphi(x)-\varphi(b))\int_0^1 (1-t)(\varphi^{-1})'(t\varphi(x)+(1-t)\varphi(b))f'(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(b)))dt \\
&= -f(b) + \frac{1}{\varphi(x)-\varphi(b)}\int_x^b f(u)\varphi^{'}(u)du
\end{aligned}$$

and so we have

$$
\begin{aligned}
&(\varphi(x)-\varphi(a))^2\int_0^1 (t-1)(\varphi^{-1})'(t\varphi(x)+(1-t)\varphi(a))f'(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(a)))dt\\
&=(\varphi(x)-\varphi(a))f(a)-\int_a^x f(u)\varphi'(u)du,\\
&(\varphi(b)-\varphi(x))^2\int_0^1 (1-t)(\varphi^{-1})'(t\varphi(x)+(1-t)\varphi(a))f'(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(a)))dt\\
&=(\varphi(b)-\varphi(x))f(b)-\int_x^b f(u)\varphi'(u)du.
\end{aligned}
$$

By multiplying with $\frac{1}{\varphi(b)-\varphi(a)}$ both of sides these equalities and adding side by to side we have

$$
\begin{aligned}
&\frac{(\varphi(x)-\varphi(a))f(a)+(\varphi(b)-\varphi(x))f(b)}{\varphi(b)-\varphi(a)}-\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(u)\varphi'(u)du\\
&=\frac{(\varphi(x)-\varphi(a))^2}{\varphi(b)-\varphi(a)}\int_0^1 (t-1)(\varphi^{-1})'(t\varphi(x)+(1-t)\varphi(a))f'(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(a)))dt\\
&+\frac{(\varphi(b)-\varphi(x))^2}{\varphi(b)-\varphi(a)}\int_0^1 (1-t)(\varphi^{-1})'(t\varphi(x)+(1-t)\varphi(b))f'(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(b)))dt.
\end{aligned}
$$

as desired. Thus the Lemma is proved.

Theorem 2. *Let $f: I\subset[0,\infty]\to\mathbb{R}$ be differentiable on I° and $a<b$, $\varphi: I\to\mathbb{R}$ be continuous and strictly monotonic function such that $\varphi^{-1}:\varphi(I^\circ)\to I^\circ$ is continuously differentiable and $f'\in L[a,b]$. If $\left|f'\right|$ strongly $M_\varphi-A-s$ convex function, we have,*

$$
\begin{aligned}
&\left|\frac{(\varphi(x)-\varphi(a))f(a)+(\varphi(b)-\varphi(x))f(b)}{\varphi(b)-\varphi(a)}-\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx\right|\\
&\le\frac{(\varphi(x)-\varphi(a))^2}{\varphi(b)-\varphi(a)}\left[A_1\left|f'(x)\right|+B_1\left|f'(a)\right|-C_1\left|((\varphi(x)-\varphi(a))^2\right|\right]\\
&+\frac{(\varphi(b)-\varphi(x))^2}{\varphi(b)-\varphi(a)}\left[A_2\left|f'(x)\right|+B_2\left|f'(a)\right|-C_2\left|((\varphi(b)-\varphi(x))^2\right|\right]
\end{aligned}
\tag{6}
$$

where

$$A_1=\int_0^1(1-t)t^s\left|\left(\varphi^{-1}\right)'(t\varphi(x)+(1-t)\varphi(a))\right|dt$$

$$B_1=\int_0^1(1-t)^{s+1}\left|\left(\varphi^{-1}\right)'(t\varphi(x)+(1-t)\varphi(a))\right|dt$$

$$C_1=\int_0^1 ct(1-t)^2\left|\left(\varphi^{-1}\right)'(t\varphi(x)+(1-t)\varphi(a))\right|dt$$

$$A_2=\int_0^1(1-t)t^s\left|\left(\varphi^{-1}\right)'(t\varphi(x)+(1-t)\varphi(a))\right|dt$$

$$B_2=\int_0^1(1-t)^{s+1}\left|\left(\varphi^{-1}\right)'(t\varphi(x)+(1-t)\varphi(a))\right|dt$$

$$C_2=\int_0^1 ct(1-t)^2\left|\left(\varphi^{-1}\right)'(t\varphi(x)+(1-t)\varphi(a))\right|dt$$

Proof. From Above Lemma and strongly $M_\varphi - A - s$ convexity of $|f'|$, we have

$$\begin{aligned}
&\Bigg|\frac{(\varphi(x)-\varphi(a))^2}{\varphi(b)-\varphi(a)}\int_0^1 (t-1)(\varphi^{-1})'(t\varphi(x)+(1-t)\varphi(a))f'(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(a)))dt \\
&+\frac{(\varphi(b)-\varphi(x))^2}{\varphi(b)-\varphi(a)}\int_0^1 (1-t)(\varphi^{-1})'(t\varphi(x)+(1-t)\varphi(b))f'(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(b)))dt\Bigg| \\
&\leq \frac{(\varphi(x)-\varphi(a))^2}{\varphi(b)-\varphi(a)}\int_0^1 (1-t)\left|\left(\varphi^{-1}\right)'(t\varphi(x)+(1-t)\varphi(a))\right|\left|f'\left(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(a))\right)\right|dt \\
&+\frac{(\varphi(b)-\varphi(x))^2}{\varphi(b)-\varphi(a)}\int_0^1 (1-t)\left|\left(\varphi^{-1}\right)'(t\varphi(x)+(1-t)\varphi(b))\right|\left|f'\left(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(b))\right)\right|dt \\
&\leq \int_0^1 (1-t)\left|\left(\varphi^{-1}\right)'(t\varphi(x)+(1-t)\varphi(a))\right|\left[t^s f'(x)+(1-t)^s f'(a)-ct(1-t)(\varphi(x)-\varphi(a))^2\right]dt \\
&+\frac{(\varphi(b)-\varphi(x))^2}{\varphi(b)-\varphi(a)}\int_0^1 (1-t)\left|\left(\varphi^{-1}\right)'(t\varphi(x)+(1-t)\varphi(a))\right| \\
&\left[t^s f'(x)+(1-t)^s f'(b)-ct(1-t)(\varphi(x)-\varphi(b))^2\right]dt \\
&=\frac{(\varphi(x)-\varphi(a))^2}{\varphi(b)-\varphi(a)}\left[\left(\int_0^1 (1-t)t^s\left|\left(\varphi^{-1}\right)'(t\varphi(x)+(1-t)\varphi(a))\right|dt\right)\left|f'0(x)\right|\right. \\
&+\left(\int_0^1 (1-t)^{s+1}\left|\left(\varphi^{-1}\right)'(t\varphi(x)+(1-t)\varphi(a))\right|dt\right)\left|f'(a)\right| \\
&\left.-\left(c\int_0^1 t(1-t)^2\left|\left(\varphi^{-1}\right)'(t\varphi(x)+(1-t)\varphi(a))\right|dt\right)\left|((\varphi(x)-\varphi(a))^2\right|\right] \\
&+\frac{(\varphi(b)-\varphi(x))^2}{\varphi(b)-\varphi(a)}\left[\left(\int_0^1 (1-t)t^s\left|\left(\varphi^{-1}\right)'(t\varphi(x)+(1-t)\varphi(b))\right|dt\right)\left|f'(x)\right|\right. \\
&+\left(\int_0^1 (1-t)^{s+1}\left|\left(\varphi^{-1}\right)'(t\varphi(x)+(1-t)\varphi(b))\right|dt\right)\left|f'(b)\right| \\
&\left.-\left(c\int_0^1 t(1-t)^2\left|\left(\varphi^{-1}\right)'(t\varphi(x)+(1-t)\varphi(b))\right|dt\right)\left|((\varphi(b)-\varphi(x))^2\right|\right]
\end{aligned}$$

Thus the proof is completed.

Corollary 1. *If we take $\varphi(x)=x$ in above Teorem, we get*

$$\begin{aligned}
&\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(x-a)^2}{b-a}\left[\frac{1}{(s+1)(s+2)}\left|f'(x)\right|\right. \\
&\left.+\frac{1}{(s+2)}\left|f'(a)\right|-\frac{c}{12}|x-a|^2\right]+\frac{(b-x)^2}{b-a}\left[\frac{1}{(s+1)(s+2)}\left|f'(x)\right|+\frac{1}{(s+2)}\left|f'(a)\right|-\frac{c}{12}|b-x|^2\right].
\end{aligned}$$

Remark. From above Corollary, if we take the limit as $c\to 0$, then we get

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \left[\frac{(x-a)^2+(b-x)^2}{(s+1)(s+2)(b-a)}\right]\left|f'(x)\right|+\frac{(x-a)^2\left|f'(a)\right|+(b-x)^2\left|f'(b)\right|}{(b-a)(s+2)}.$$

Theorem 3. *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be differentiable on I° and $a,b\in I^\circ$ with $a<b$, $a,b\in I^\circ$, $\varphi: I\to\mathbb{R}$ be continuous and strictly monotonic funcion such that $\varphi^{-1}:\varphi(I^\circ)\to I^\circ$ is continuously differentiable and $f'\in L[a,b]$, for some fixed*

$s\in(0,1]$ and $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $|f'|^q$ is strongly $M_\varphi-A-s$ convex, we get

$$\begin{aligned}
&\left|\frac{(\varphi(x)-\varphi(a))f(a)+(\varphi(b)-\varphi(x))f(b)}{\varphi(b)-\varphi(a)}-\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)\,dx\right| \\
&\leq \frac{(\varphi(x)-\varphi(a))^2}{\varphi(b)-\varphi(a)}D_1^{1/p}\left(\int_0^1 (t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(a)\right|^q-ct(1-t)(\varphi(x)-\varphi(a))^2)dt\right)^{1/q} \\
&\leq \frac{(\varphi(b)-\varphi(x))^2}{\varphi(b)-\varphi(a)}D_2^{1/p}\left(\int_0^1 (t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(b)\right|^q-ct(1-t)(\varphi(b)-\varphi(x))^2)dt\right)^{1/q},
\end{aligned} \tag{7}$$

where

$$D_1=\int_0^1(1-t)^p\left|\left(\varphi^{-1}\right)'(t\varphi(x)+(1-t)\varphi(a))\right|^p dt,$$

$$D_2=\int_0^1(1-t)^p\left|\left(\varphi^{-1}\right)'(t\varphi(x)+(1-t)\varphi(b))\right|^p dt.$$

Proof. By using above Lemma and Hölder's İnequality, we have

$$\begin{aligned}
&\left|\frac{(\varphi(x)-\varphi(a))f(a)+(\varphi(b)-\varphi(x))f(b)}{\varphi(b)-\varphi(a)}-\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)\,dx\right| \leq \frac{(\varphi(x)-\varphi(a))^2}{\varphi(b)-\varphi(a)} \\
&\int_0^1(1-t)\left|\left(\varphi^{-1}\right)'(t\varphi(x)+(1-t)\varphi(a))\right|\left|f'\left(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(a))\right)\right|dt+\frac{(\varphi(b)-\varphi(x))^2}{\varphi(b)-\varphi(a)} \\
&\int_0^1(1-t)\left|\left(\varphi^{-1}\right)'(t\varphi(x)+(1-t)\varphi(b))\right|\left|f'\left(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(b))\right)\right|dt \\
&\leq \frac{(\varphi(x)-\varphi(a))^2}{\varphi(b)-\varphi(a)}\left(\int_0^1(1-t)^p\left|\left(\varphi^{-1}\right)'(t\varphi(x)+(1-t)\varphi(a))\right|^p dt\right)^{1/p} \\
&\left(\int_0^1\left|f'\left(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(b))\right)\right|^q dt\right)^{1/q}.
\end{aligned}$$

In the last inequality, if we consider that $|f'|^q$ is strongly $M_\varphi-A-s$ convex function, then we get

$$\begin{aligned}
&\leq \frac{(\varphi(x)-\varphi(a))^2}{\varphi(b)-\varphi(a)}[(\int_0^1(1-t)^p\left|\left(\varphi^{-1}\right)'(t\varphi(x)+(1-t)\varphi(a))\right|^p dt)^{1/p} \\
&(\int_0^1 t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(a)\right|^q-ct(1-t)(\varphi(x)-\varphi(a))^2dt)^{1/q}] \\
&+\frac{(\varphi(b)-\varphi(x))^2}{\varphi(b)-\varphi(a)}[(\int_0^1(1-t)^p\left|\left(\varphi^{-1}\right)'(t\varphi(x)+(1-t)\varphi(b))\right|^p dt)^{1/p} \\
&(\int_0^1 t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(b)\right|^q-ct(1-t)(\varphi(b)-\varphi(x))^2dt)^{1/q}].
\end{aligned}$$

Thus the proof of the Theorem 3.1 is completed.

Corollary 2. *If we take $\varphi(x)=x$ in above Theorem, then we get*

$$\begin{aligned}
&\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(x-a)^2}{b-a}[(\int_0^1(1-t)^p dt)^{1/p} \\
&(\int_0^1 t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(a)\right|^q-ct(1-t)(x-a)^2dt)^{1/q}]+\frac{(b-x)^2}{b-a}[(\int_0^1(1-t)^p dt)^{1/p}
\end{aligned}$$

$$
\begin{aligned}
&(\int_0^1 t^s \left|f'(x)\right|^q + (1-t)^s \left|f'(b)\right|^q - ct(1-t)(b-x)^2 dt)^{1/q}] \\
&\leq \frac{(x-a)^2}{b-a}\left[\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\left|f'(x)\right|^q + \frac{1}{s+1}\left|f'(a)\right|^q - \frac{c}{6}(x-a)^2\right)^{1/q}\right] \\
&+ \frac{(b-x)^2}{b-a}\left[\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\left|f'(x)\right|^q + \frac{1}{s+1}\left|f'(b)\right|^q - \frac{c}{6}(b-x)^2\right)^{1/q}\right].
\end{aligned}
$$

Corollary 3. *If we take $\varphi(x) = x$ in above Theorem, then we get*

$$
\begin{aligned}
&\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(x-a)^2}{(b-a)(p+1)^{1/p}}\left[\left(\frac{1}{s+1}\left(\left|f'(x)\right|^q + \left|f'(a)\right|^q\right) - \frac{c(x-a)^2}{6}\right)^{1/q}\right] \\
&+ \frac{(b-x)^2}{(b-a)(p+1)^{1/p}}\left[\left(\frac{1}{s+1}\left(\left|f'(x)\right|^q + \left|f'(b)\right|^q\right) - \frac{c(x-a)^2}{6}\right)^{1/q}\right].
\end{aligned}
$$

Remark. From above Corollary if we take the limit as $c \to 0$, then we get

$$
\begin{aligned}
\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{(x-a)^2}{b-a}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{\left|f'(x)\right|^q + \left|f'(a)\right|^q}{s+1}\right)^{1/q} \\
&\leq \frac{(b-x)^2}{b-a}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{\left|f'(x)\right|^q + \left|f'(b)\right|^q}{s+1}\right)^{1/q}.
\end{aligned}
$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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