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Asymptotics of eigenvalues for Sturm-Liouville problem including eigenparameter-dependent boundary conditions with integrable potential

Elif Baskaya

Department of Mathematics, Karadeniz Technical University, Trabzon, Turkey

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Abstract: In this paper, we obtain asymptotic estimates of eigenvalues for regular Sturm-Liouville problems having the eigenvalue parameter in all boundary conditions with integrable potential.

Keywords: Sturm-Liouville problems, integrable potential, eigenvalues, asymptotics.

1 Introduction

In this paper, we consider the boundary value problem

$$y''(t)+\{\lambda-q(t)\}y(t)=0,\ \ t\in[a,b], \tag{1}$$

$$a_1y(a)+a_2y'(a)=\lambda\left[a_1'y(a)+a_2'y'(a)\right], \tag{2}$$

$$b_1y(b)+b_2y'(b)=\lambda\left[b_1'y(b)+b_2'y'(b)\right], \tag{3}$$

where λ is a real parameter; $q(t)$ is a real-valued function; $a_i,\ a_i',\ b_i,\ b_i'\in\mathbb{R},\ i=0,1$. Also we assume that $q(t)$ is integrable. This problem differs from the usual regular Sturm-Liouville problem in the sense that eigenvalue parameter λ is contained in the boundary condition at a. Problems of this type arise from the method of separation of variables applied to mathematical models for certain physical problems including heat conduction and wave propagation, etc. [8]. It is shown by Walter [15] that this problem is self-adjoint problem. The purpose of this paper is to obtain asymptotic approximations for the eigenvalues of (1)-(3).

Approximations of this type have been derived before. We mention in particular [7,8] and [2]. Fulton's approach in [7] is based on an iteration of the usual Volterra integral equation, producing an asymptotic expansion of the solution in higher powers of $1/\lambda^{1/2}$ as $\lambda\to\infty$ and in [8] is based on the analysis of [14] for regular Sturm-Liouville problems on a finite closed interval and involves some operator-theoretical results of [15]. The approach used in [2] is based on an iterative procedure solving the associated Riccati equation and producing an asymptotic expansion of the solution in the higher powers of $1/\lambda^{1/2}$ as $\lambda\to\infty$ for smooth $q(t)$. There is also a vast amount of literature dealing with asymptotic estimates of eigenvalues for standart Sturm-Liouville problems with regular endpoints [3,4,5,6,9,10,11,13,14]. Here we follow the similar approach in [4,10,12]. We assume without loss of generality, that $q(t)$ has mean value zero. That is $\int_a^b q(t)\,dt=0$.

* Corresponding author e-mail: elifbekar@ktu.edu.tr

2 Conclusion

Theorem 1. *The eigenvalues λ_n of (1)-(3) satisfy as $\lambda \to \infty$,*

(i) *if $a_2' \neq 0$ and $b_2' \neq 0$,*

$$\lambda_n^{1/2} = \frac{(n+1)\pi}{(b-a)} + \frac{1}{(n+1)\pi}\left\{\frac{a_2' b_1' - a_1' b_2'}{a_2' b_2'} + \frac{1}{2}\int_a^b \left(\cos\frac{2(n+1)\pi(x-a)}{b-a}\right) q(x)\,dx\right\}$$
$$+ O\left(n^{-2}\eta(n)\right) + O\left(n^{-1}\eta^2(n)\right),$$

(ii) *if $a_2' \neq 0$ and $b_2' = 0$,*

$$\lambda_n^{1/2} = \frac{(2n+3)\pi}{2(b-a)} + \frac{2}{(2n+3)\pi}\left\{\frac{a_2' b_2 - a_1' b_1'}{a_2' b_1'} + \frac{1}{2}\int_a^b \left(\cos\frac{(2n+3)\pi(x-a)}{b-a}\right) q(x)\,dx\right\}$$
$$+ O\left(n^{-2}\eta(n)\right) + O\left(n^{-1}\eta^2(n)\right).$$

Theorem 2. *The eigenvalues λ_n of (1)-(3) satisfy as $\lambda \to \infty$,*

(i) *if $a_2' = 0$ and $b_2' \neq 0$,*

$$\lambda_n^{1/2} = \frac{(2n+3)\pi}{2(b-a)} + \frac{2}{(2n+3)\pi}\left\{\frac{a_1' b_1' - a_2 b_2'}{a_1' b_2'} - \frac{1}{2}\int_a^b \left(\cos\frac{(2n+3)\pi(x-a)}{b-a}\right) q(x)\,dx\right\}$$
$$+ O\left(n^{-2}\eta(n)\right) + O\left(n^{-1}\eta^2(n)\right),$$

(ii) *if $a_2' = 0$ and $b_2' = 0$,*

$$\lambda_n^{1/2} = \frac{(n+2)\pi}{(b-a)} + \frac{1}{(n+2)\pi}\left\{\frac{a_1' b_2 - a_2 b_1'}{a_1' b_1'} - \frac{1}{2}\int_a^b \left(\cos\frac{2(n+2)\pi(x-a)}{b-a}\right) q(x)\,dx\right\}$$
$$+ O\left(n^{-2}\eta(n)\right) + O\left(n^{-1}\eta^2(n)\right).$$

3 The method

We associate with (1) the Riccati equation

$$v'(t,\lambda) = -\lambda + q - v^2.$$

We define

$$S(t,\lambda) = Re\left[v(t,\lambda)\right], \tag{4}$$
$$T(t,\lambda) = Im\left[v(t,\lambda)\right]. \tag{5}$$

It is shown in [3] that any real-valued solution of (1) is in the form

$$y(t,\lambda) = R(t,\lambda)\cos\theta(t,\lambda) \tag{6}$$

with

$$S(t,\lambda) = \frac{R'(t,\lambda)}{R(t,\lambda)}, \tag{7}$$

$$T(t,\lambda) = \theta'(t,\lambda). \tag{8}$$

Our approach to calculating λ_n is to approximate those λ which are such that

$$\theta(b,\lambda)-\theta(a,\lambda)=\int_a^b T(x,\lambda)\,dx. \tag{9}$$

We suppose that there exist functions $A(t)$ and $\eta(\lambda)$ so that

$$\left|\int_t^b e^{2i\lambda^{1/2}x}q(x)\,dx\right|\le A(t)\,\eta(\lambda),\quad t\in[a,b] \tag{10}$$

where

(i) $A(t):=\int_t^b |q(x)|\,dx$ is a decreasing function of t,
(ii) $A(.)\in L[a,b]$,
(iii) $\eta(\lambda)\to 0$ as $\lambda^{1/2}\to\infty$.

For $q\in L[a,b]$ the existence of the A and η functions may be established for λ positive as follows. We note that, avoiding the trivial case $\int_t^b |q(x)|\,dx=0$. $\left|\int_t^b e^{2i\lambda^{1/2}x}q(x)\,dx\right|\le\int_t^b |q(x)|\,dx<\infty$ so, if we define

$$F(t,\lambda):=\begin{cases}\left|\int_t^b e^{2i\lambda^{1/2}x}q(x)\,dx\right|/\int_t^b|q(x)|\,dx, & \text{if } \int_t^b|q(x)|\,dx\ne 0,\\ 0, & \text{if } \int_t^b|q(x)|\,dx=0,\end{cases} \tag{11}$$

then $0\le F(t,\lambda)\le 1$ and we set $\eta(\lambda):=\sup_{a\le t\le b}F(t,\lambda)$. $\eta(\lambda)$ is well defined by (11) and $\eta(\lambda)\to 0$ as $\lambda\to\infty$ [12].

Our method of approximating a solution of $v'(t,\lambda)=-\lambda+q-v^2$ on $[a,b]$ is similar to [12], so we set

$$v(t,\lambda):=i\lambda^{1/2}+\sum_{n=1}^{\infty}v_n(t,\lambda). \tag{12}$$

When we put this serie into the Riccati equation and solve differential equations, we hold

$$v_1(t,\lambda)=-e^{-2i\lambda^{1/2}t}\int_t^b e^{2i\lambda^{1/2}x}q(x)\,dx,$$

$$v_2(t,\lambda)=e^{-2i\lambda^{1/2}t}\int_t^b e^{2i\lambda^{1/2}x}v_1^2(x,\lambda)\,dx, \tag{13}$$

$$v_n(t,\lambda)=e^{-2i\lambda^{1/2}t}\int_t^b e^{2i\lambda^{1/2}x}\left[\,v_{n-1}^2(x,\lambda)+2v_{n-1}(x,\lambda)\sum_{m=1}^{n-2}v_m(x,\lambda)\,\right]dx,\quad n\ge 3.$$

Also we found $\theta(b,\lambda)-\theta(a,\lambda)=\int_a^b T(x,\lambda)\,dx$, so with (8) and (12) we have

$$\theta(b,\lambda)-\theta(a,\lambda)=\int_a^b\left[\lambda^{1/2}+Im\sum_{n=1}^{\infty}v_n(x,\lambda)\right]dx,$$

then

$$\theta(b,\lambda)-\theta(a,\lambda)=\lambda^{1/2}(b-a)+\sum_{n=1}^{\infty}Im\int_a^b v_n(x,\lambda)\,dx. \tag{14}$$

Theorem 3. *[1] If $v(t,\lambda)$ as in (12), as $\lambda\to\infty$*

$$v(t,\lambda)=i\lambda^{1/2}+v_1(t,\lambda)+O\left(\eta^2(\lambda)\right)$$

where

$$v_1(t,\lambda) = -e^{-2i\lambda^{1/2}t}\int_t^b e^{2i\lambda^{1/2}x}q(x)\,dx = -\left[\cos 2\lambda^{1/2}t - i\sin 2\lambda^{1/2}t\right]\times\int_t^b\left[\cos 2\lambda^{1/2}x + i\sin 2\lambda^{1/2}x\right]q(x)\,dx$$

and $\eta(\lambda)$ *is defined (11).*

After some calculations by using the last theorem, with (4) we gain

$$S(t,\lambda) = -\left(\cos 2\lambda^{1/2}t\right)\int_t^b\left[\cos 2\lambda^{1/2}x\right]q(x)\,dx - \left(\sin 2\lambda^{1/2}t\right)\int_t^b\left[\sin 2\lambda^{1/2}x\right]q(x)\,dx + O\left(\eta^2(\lambda)\right).$$

Let define the following notations:

$$\sin\xi_t := \int_t^b\left(\cos 2\lambda^{1/2}x\right)q(x)\,dx,$$
$$\cos\xi_t := \int_t^b\left(\sin 2\lambda^{1/2}x\right)q(x)\,dx,$$

thus we can write $S(t,\lambda)$ as

$$S(t,\lambda) = -\sin\left(2\lambda^{1/2}t + \xi_t\right) + O\left(\eta^2(\lambda)\right). \tag{15}$$

Similarly, with (5) we find $T(t,\lambda)$ as

$$T(t,\lambda) = \lambda^{1/2} - \cos\left(2\lambda^{1/2}t + \xi_t\right) + O\left(\eta^2(\lambda)\right). \tag{16}$$

Also, by using integration by part to (13), we determine

$$\int_a^b v_1(x,\lambda)\,dx = \frac{i}{2\lambda^{1/2}}\int_a^b e^{2i\lambda^{1/2}(x-a)}q(x)\,dx$$

and again with integration by part

$$\int_a^b v_1(x,\lambda)\,dx = \frac{i}{2}\lambda^{-1/2}[\int_a^b iq(x)\left[\sin 2\lambda^{1/2}x\right]\left[\cos 2\lambda^{1/2}a\right]dx - \int_a^b iq(x)\left[\cos 2\lambda^{1/2}x\right]\left[\sin 2\lambda^{1/2}a\right]dx$$
$$+\frac{i}{2}\lambda^{-1/2}[\int_a^b q(x)\left[\cos 2\lambda^{1/2}x\right]\left[\cos 2\lambda^{1/2}a\right]dx + \int_a^b q(x)\left[\sin 2\lambda^{1/2}x\right]\left[\sin 2\lambda^{1/2}a\right]dx,$$

so

$$Im\int_a^b v_1(x,\lambda)\,dx = \frac{1}{2}\lambda^{-1/2}\left[\cos 2\lambda^{1/2}a\right]\int_a^b q(x)\left[\cos 2\lambda^{1/2}x\right]dx + \left[\sin 2\lambda^{1/2}a\right]\int_a^b q(x)\left[\sin 2\lambda^{1/2}x\right]dx.$$

We also have from equation (13),

$$\int_a^b v_2(x,\lambda)\,dx = \frac{i}{2\lambda^{1/2}}\int_a^b\left[1 - e^{2i\lambda^{1/2}(x-a)}\right]v_1^2(x,\lambda)\,dx$$

and for $n \geq 3$

$$\int_a^b v_n(x,\lambda)\,dx = \frac{i}{2\lambda^{1/2}}\times\int_a^b[\,1 - e^{2i\lambda^{1/2}(x-a)}\,][\,v_{n-1}^2(x,\lambda) + 2v_{n-1}(x,\lambda)\sum_{m=1}^{n-2}v_m(x,\lambda)\,]\,dx.$$

Thus, with the last equations

$$\int_a^b \sum_{n=1}^{\infty} Im\{v_n(x,\lambda)\}\,dx = \sum_{n=1}^{\infty} Im\left\{\int_a^b v_n(x,\lambda)\,dx\right\} = \frac{1}{2}\lambda^{-1/2}\sin\left(2\lambda^{1/2}t+\xi_t\right)+O\left(\lambda^{-1}\eta(\lambda)\right). \tag{17}$$

4 Proof of the main results

Proof of theorem 1.

(i) If $a_2' \neq 0$ and $b_2' \neq 0$, the real solution of $y''(t)+[\lambda-q(t)]y(t)=0$ is $y(t,\lambda)=R(t,\lambda)\cos\theta(t,\lambda)$ from (6). We use this equation for boundary $t=a$, we find

$$R(a,\lambda)\left\{\cos\theta(a,\lambda)\left[\left(-a_2'\lambda+a_2\right)\frac{R'(a,\lambda)}{R(a,\lambda)}+\left(-a_1'\lambda+a_1\right)\right]+\left(a_2'\lambda-a_2\right)\theta'(a,\lambda)\sin\theta(a,\lambda)\right\}=0.$$

If we choose α_1 as

$$\sin\alpha_1 := \left(-a_2'\lambda+a_2\right)\frac{R'(a,\lambda)}{R(a,\lambda)}+\left(-a_1'\lambda+a_1\right),$$
$$\cos\alpha_1 := \left(a_2'\lambda-a_2\right)\theta'(a,\lambda),$$

we have $R(a,\lambda)\sin[\alpha_1+\theta(a,\lambda)]=0$ so $\sin(\alpha_1+\theta(a,\lambda))=0$, or $\theta(a,\lambda)=-\alpha_1$. Using by equations (7) and (8) as $S(t,\lambda)=\frac{R'(t,\lambda)}{R(t,\lambda)}$, $T(t,\lambda)=\theta'(t,\lambda)$ and their asymptotic expansions (15)-(16), we calculate

$$\frac{\sin\alpha_1}{\cos\alpha_1}=\frac{\begin{array}{c}-a_1'\lambda+\lambda a_2'\sin\left(2\lambda^{1/2}a+\xi_a\right)+a_1-a_2\sin\left(2\lambda^{1/2}a+\xi_a\right)\\ +O\left(\eta^2(\lambda)\right)+O\left(\lambda\eta^2(\lambda)\right)\end{array}}{\begin{array}{c}a_2'\lambda^{3/2}-\lambda a_2'\cos\left(2\lambda^{1/2}a+\xi_a\right)-\lambda^{1/2}a_2+a_2\cos\left(2\lambda^{1/2}a+\xi_a\right)\\ +O\left(\eta^2(\lambda)\right)+O\left(\lambda\eta^2(\lambda)\right)\end{array}},$$

hence

$$\frac{\sin\alpha_1}{\cos\alpha_1}=\frac{\begin{array}{c}-a_1'\lambda+\lambda a_2'\sin\left(2\lambda^{1/2}a+\xi_a\right)+a_1-a_2\sin\left(2\lambda^{1/2}a+\xi_a\right)\\ +O\left(\eta^2(\lambda)\right)+O\left(\lambda\eta^2(\lambda)\right)\end{array}}{a_2'\lambda^{3/2}\left[\begin{array}{c}1-\lambda^{-1/2}\cos\left(2\lambda^{1/2}a+\xi_a\right)-\lambda^{-1}\frac{a_2}{a_2'}+\lambda^{-3/2}\frac{a_2}{a_2'}\cos\left(2\lambda^{1/2}a+\xi_a\right)\\ +O\left(\lambda^{-3/2}\eta^2(\lambda)\right)+O\left(\lambda^{-1/2}\eta^2(\lambda)\right)\end{array}\right]}.$$

Then

$$\tan\alpha_1=\left[\begin{array}{c}-\frac{a_1'}{a_2'}\lambda^{-1/2}+\lambda^{-1/2}\sin\left(2\lambda^{1/2}a+\xi_a\right)+\frac{a_1}{a_2'}\lambda^{-3/2}-\lambda^{-3/2}\frac{a_2}{a_2'}\sin\left(2\lambda^{1/2}a+\xi_a\right)\\ +O\left(\lambda^{-1/2}\eta^2(\lambda)\right)\end{array}\right]$$
$$\times\left[\begin{array}{c}1+\lambda^{-1/2}\cos\left(2\lambda^{1/2}a+\xi_a\right)+\lambda^{-1}\frac{a_2}{a_2'}-\lambda^{-3/2}\frac{a_2}{a_2'}\cos\left(2\lambda^{1/2}a+\xi_a\right)\\ +O\left(\lambda^{-1/2}\eta^2(\lambda)\right)\end{array}\right],$$

so

$$\tan\alpha_1=-\frac{a_1'}{a_2'}\lambda^{-1/2}+\lambda^{-1/2}\sin\left(2\lambda^{1/2}a+\xi_a\right)-\frac{a_1'}{a_2'}\lambda^{-1}\cos\left(2\lambda^{1/2}a+\xi_a\right)$$
$$+O\left(\lambda^{-1/2}\eta^2(\lambda)\right).$$

In the last equation, by using Taylor expansion of $\arctan x$ at $x=0$, we obtain

$$\alpha_1 = -\frac{a_1'}{a_2'}\lambda^{-1/2} + \lambda^{-1/2}\sin\left(2\lambda^{1/2}a+\xi_a\right) - \frac{a_1'}{a_2'}\lambda^{-1}\cos\left(2\lambda^{1/2}a+\xi_a\right) + O\left(\lambda^{-1/2}\eta^2(\lambda)\right). \tag{18}$$

Similarly, when we use the form $y(t,\lambda)=R(t,\lambda)\cos\theta(t,\lambda)$ for boundary $t=b$, we find

$$R(b,\lambda)\left\{\cos\theta(b,\lambda)\left[\left(-b_2'\lambda+b_2\right)\frac{R'(b,\lambda)}{R(b,\lambda)}+\left(-b_1'\lambda+b_1\right)\right]-\left(-b_2'\lambda+b_2\right)\theta'(b,\lambda)\sin\theta(b,\lambda)\right\}=0.$$

If we choose α_2 as

$$\sin\alpha_2 := \left(-b_2'\lambda+b_2\right)\frac{R'(b,\lambda)}{R(b,\lambda)}+\left(-b_1'\lambda+b_1\right),$$
$$\cos\alpha_2 := \left(-b_2'\lambda+b_2\right)\theta'(b,\lambda),$$

we have $R(b,\lambda)\sin[\alpha_2-\theta(b,\lambda)]=0$ so $\sin[\alpha_2-\theta(b,\lambda)]=0$, or $\theta(b,\lambda)=\alpha_2+(n+1)\pi$. Using by $S(t,\lambda)=\frac{R'(t,\lambda)}{R(t,\lambda)}$, $T(t,\lambda)=\theta'(t,\lambda)$ and their asymptotic expansions (15)-(16) we can write

$$\frac{\sin\alpha_2}{\cos\alpha_2}=\frac{-\lambda b_1'+b_1+O\left(\eta^2(\lambda)\right)+O\left(\lambda\eta^2(\lambda)\right)}{-\lambda^{3/2}b_2'+\lambda^{1/2}b_2+O\left(\eta^2(\lambda)\right)+O\left(\lambda\eta^2(\lambda)\right)},$$

so

$$\tan\alpha_2=\left[\lambda^{-1/2}\frac{b_1'}{b_2'}-\lambda^{-3/2}\frac{b_1}{b_2'}+O\left(\lambda^{-1/2}\eta^2(\lambda)\right)\right]\times\left[1+\lambda^{-1}\frac{b_2}{b_2'}+O\left(\lambda^{-1/2}\eta^2(\lambda)\right)\right]$$
$$=\lambda^{-1/2}\frac{b_1'}{b_2'}+O\left(\lambda^{-1/2}\eta^2(\lambda)\right).$$

In the last equation, by using Taylor expansion of $\arctan x$ at $x=0$, we obtain

$$\alpha_2=\lambda^{-1/2}\frac{b_1'}{b_2'}+O\left(\lambda^{-1/2}\eta^2(\lambda)\right). \tag{19}$$

Let use these findings (17), (18) and (19) in $\theta(b,\lambda)-\theta(a,\lambda)=\lambda^{1/2}(b-a)+\sum_{n=1}^{\infty} Im\int_a^b v_n(x,\lambda)\,dx$, we see that

$$(n+1)\pi+\lambda^{-1/2}\frac{b_1'}{b_2'}-\frac{a_1'}{a_2'}\lambda^{-1/2}+\lambda^{-1/2}\sin\left(2\lambda^{1/2}a+\xi_a\right)-\lambda^{-1}\frac{a_1'}{a_2'}\cos\left(2\lambda^{1/2}a+\xi_a\right)+O\left(\lambda^{-1/2}\eta^2(\lambda)\right)$$
$$=\lambda^{1/2}(b-a)+\frac{1}{2}\lambda^{-1/2}\sin\left(2\lambda^{1/2}t+\xi_t\right)+O\left(\lambda^{-1}\eta(\lambda)\right).$$

We prove the theorem by using definitions of $\sin\xi_t,\cos\xi_t$ and $\eta(\lambda)$; also series error computation in the last equation.

Proof of theorem 2.

(ii) If $a_2'=0$ and $b_2'=0$, our problem is

$$y''(t)+\{\lambda-q(t)\}y(t)=0,\ \ t\in[a,b],$$
$$a_1y(a)+a_2y'(a)=\lambda a_1'y(a),$$
$$b_1y(b)+b_2y'(b)=\lambda b_1'y(b).$$

The real solution of $y''(t)+[\lambda-q(t)]y(t)=0$ is $y(t,\lambda)=R(t,\lambda)\cos\theta(t,\lambda)$. We use this equation for boundary $t=a$, we find

$$R(a,\lambda)\left\{\cos\theta(a,\lambda)\left[a_2\frac{R'(a,\lambda)}{R(a,\lambda)}-a_1'\lambda+a_1\right]-a_2\theta'(a,\lambda)\sin\theta(a,\lambda)\right\}=0.$$

If we choose α_3 as

$$\sin\alpha_3:=a_2\frac{R'(a,\lambda)}{R(a,\lambda)}-a_1'\lambda+a_1,$$
$$\cos\alpha_3:=-a_2\theta'(a,\lambda),$$

we have $R(a,\lambda)\sin(\alpha_3+\theta(a,\lambda))=0$ so $\sin(\alpha_3+\theta(a,\lambda))=0$, or $\theta(a,\lambda)=-\alpha_3$. Using by $S(t,\lambda)=\frac{R'(t,\lambda)}{R(t,\lambda)}$, $T(t,\lambda)=\theta'(t,\lambda)$ and their asymptotic expansions, one writes

$$\frac{\cos\alpha_3}{\sin\alpha_3}=\frac{-a_2\lambda^{-1/2}+a_2\cos\left(2\lambda^{1/2}a+\xi_a\right)+O\left(\eta^2(\lambda)\right)}{-a_1'\lambda+a_1-a_2\sin\left(2\lambda^{1/2}a+\xi_a\right)+O\left(\eta^2(\lambda)\right)},$$

or

$$\cot\alpha_3=\frac{-a_2\lambda^{-1/2}+a_2\cos\left(2\lambda^{1/2}a+\xi_a\right)+O\left(\eta^2(\lambda)\right)}{-a_1'\lambda\left[1-\frac{a_1}{a_1'}\lambda^{-1}+\frac{a_2}{a_1'}\lambda^{-1}\sin\left(2\lambda^{1/2}a+\xi_a\right)+O\left(\lambda^{-1}\eta^2(\lambda)\right)\right]},$$

so

$$\cot\alpha_3=\frac{a_2}{a_1'}\lambda^{-1/2}-\frac{a_2}{a_1'}\lambda^{-1}\cos\left(2\lambda^{1/2}a+\xi_a\right)+O\left(\lambda^{-1}\eta^2(\lambda)\right)\times\left[1+\frac{a_1}{a_1'}\lambda^{-1}-\frac{a_2}{a_1'}\lambda^{-1}\sin\left(2\lambda^{1/2}a+\xi_a\right)+O\left(\lambda^{-1}\eta^2(\lambda)\right)\right],$$

then

$$\cot\alpha_3=\frac{a_2}{a_1'}\lambda^{-1/2}-\frac{a_2}{a_1'}\lambda^{-1}\cos\left(2\lambda^{1/2}a+\xi_a\right)+\frac{a_1a_2}{\left[a_1'\right]^2}\lambda^{-3/2}-\frac{a_2^2}{\left(a_1'\right)^2}\lambda^{-3/2}\sin\left(2\lambda^{1/2}a+\xi_a\right)+O\left(\lambda^{-1}\eta^2(\lambda)\right).$$

In the last equation, by using Taylor expansion of *arccotx* at $x=0$, we obtain

$$\begin{aligned}-\theta(a,\lambda)=\alpha_3=&\frac{\pi}{2}-\frac{a_2}{a_1'}\lambda^{-1/2}+\frac{a_2}{a_1'}\lambda^{-1}\cos\left(2\lambda^{1/2}a+\xi_a\right)-\frac{a_1a_2}{\left[a_1'\right]^2}\lambda^{-3/2}\\&+\frac{a_2^2}{\left(a_1'\right)^2}\lambda^{-3/2}\sin\left(2\lambda^{1/2}a+\xi_a\right)+\frac{a_2^3}{3\left(a_1'\right)^3}\lambda^{-3/2}+O\left(\lambda^{-1}\eta^2(\lambda)\right).\end{aligned}\tag{20}$$

For boundary $t=b$, by using $y(b,\lambda)=R(b,\lambda)\cos\theta(b,\lambda)$, we find

$$R(b,\lambda)\left\{\cos\theta(b,\lambda)\left[b_2\frac{R'(b,\lambda)}{R(b,\lambda)}-b_1'\lambda+b_1\right]-b_2\theta'(b,\lambda)\sin\theta(b,\lambda)\right\}=0.$$

If we choose α_4 as

$$\sin\alpha_4:=b_2\frac{R'(b,\lambda)}{R(b,\lambda)}-b_1'\lambda+b_1,$$
$$\cos\alpha_4:=b_2\theta'(b,\lambda),$$

we have $R(b,\lambda)\sin[\alpha_4-\theta(b,\lambda)]=0$ so $\sin[\alpha_4-\theta(b,\lambda)]=0$, or $\theta(b,\lambda)=(n+1)\pi+\alpha_4$. Using by $S(t,\lambda)=\frac{R'(t,\lambda)}{R(t,\lambda)}$, $T(t,\lambda)=\theta'(t,\lambda)$ and their asymptotic expansions, one writes

$$\begin{aligned}\cot\alpha_4 &= \frac{\lambda^{-1/2}b_2+O\left(\lambda^{-1}\eta^2(\lambda)\right)}{-b_1'\lambda\left[1-\frac{b_1}{b_1'}\lambda^{-1}+O(\lambda^{-1}\eta^2(\lambda))\right]}\\ &= \left[-\lambda^{-1/2}\frac{b_2}{b_1'}+O\left(\lambda^{-2}\eta^2(\lambda)\right)\right]\times\left[1+\frac{b_1}{b_1'}\lambda^{-1}+O\left(\lambda^{-1}\eta^2(\lambda)\right)\right],\end{aligned}$$

then

$$\cot\alpha_4=-\lambda^{-1/2}\frac{b_2}{b_1'}-\lambda^{-3/2}\frac{b_1b_2}{(b_1')^2}+O\left(\lambda^{-1}\eta^2(\lambda)\right).$$

In the last equation, by using Taylor expansion of *arccotx* at $x=0$, we obtain

$$\alpha_4=\frac{\pi}{2}+\lambda^{-1/2}\frac{b_2}{b_1'}+\lambda^{-3/2}\frac{b_1b_2}{(b_1')^2}-\lambda^{-3/2}\frac{b_2^3}{3(b_1')^3}+O\left(\lambda^{-1}\eta^2(\lambda)\right)$$

so

$$\theta(b,\lambda)=(n+1)\pi+\frac{\pi}{2}+\lambda^{-1/2}\frac{b_2}{b_1'}+\lambda^{-3/2}\frac{b_1b_2}{(b_1')^2}-\lambda^{-3/2}\frac{b_2^3}{3(b_1')^3}+O\left(\lambda^{-1}\eta^2(\lambda)\right). \tag{21}$$

Let use these findings in $\theta(b,\lambda)-\theta(a,\lambda)=\lambda^{1/2}(b-a)+\sum_{n=1}^{\infty}Im\int_a^b v_n(x,\lambda)\,dx$, we estimate that

$$\begin{aligned}&(n+1)\pi+\frac{\pi}{2}+\lambda^{-1/2}\frac{b_2}{b_1'}+\lambda^{-3/2}\frac{b_1b_2}{(b_1')^2}-\lambda^{-3/2}\frac{b_2^3}{3(b_1')^3}\\ &+\frac{\pi}{2}-\frac{a_2}{a_1'}\lambda^{-1/2}+\frac{a_2}{a_1'}\lambda^{-1}\cos\left(2\lambda^{1/2}a+\xi_a\right)-\frac{a_1a_2}{[a_1']^2}\lambda^{-3/2}\\ &+\frac{a_2^2}{[a_1']^2}\lambda^{-3/2}\sin\left(2\lambda^{1/2}a+\xi_a\right)+\frac{(a_2)^3}{3[a_1']^3}\lambda^{-3/2}+O\left(\lambda^{-1}\eta^2(\lambda)\right)\\ &=\lambda^{1/2}(b-a)+\frac{1}{2}\lambda^{-1/2}\sin\left(2\lambda^{1/2}t+\xi_t\right)+O\left(\lambda^{-1}\eta(\lambda)\right).\end{aligned}$$

We prove the theorem by using definitions of $\sin\xi_t,\cos\xi_t$ and $\eta(\lambda)$; also series error computation in the last equation.

Similarly, Theorem 1 (ii) follows from (14), (18) and (21); Theorem 2 (i) follows from (14), (20) and (19).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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