New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2018.311

Some characterizations of the slant helices according to N-Bishop frame in Euclidean 3-space

Hatice Kusak Samanci[1], *Huseyin Kocayigit*[2], *Ayhan Yildiz*[1]

[1]Bitlis Eren University, Dept. of Mathematics, Bitlis, Turkey
[2]Celal Bayar University, Dept. of Mathematics, Manisa, Turkey

Received: 13 August 2018, Accepted: 12 October 2018
Published online: 24 October 2018.

Abstract: A slant helix is a kind of helix whose normal line makes a constant angle with a fixed direction. Due to the diversity of the application areas of the slant helices, a lot of work has been done in recent years. In this paper, some characterizations of the slant helices according to N-Bishop Frame in Euclidean 3- space are given.

Keywords: Helix, slant helix, N-Bishop frame, Euclidean space.

1 Introduction

Helix is a special kind of curve used widely in many sciences. It plays a particularly important role in the study of DNA structure. In general, a curve is called a general helix if its tangent vector makes a constant angle with a fixed certain direction in space. The most prominent feature of the overall helix is the constant rate of curvature and torsion. There are various types of helices. For example, if both curvature and torsion are fixed functions that are different form zero, this kind of the helix is called as a circular helix. If the curvature is zero, then it indicates a straight line. If the torsion is zero, it indicates a circle, and these form the special cases of the helices. On the other hand, these helices types are called slant helices if the principal normal vector makes a constant angle with a fixed certain direction. A lot of work has been done on helices and slant helices until now. Kula and his colleagues conducted some studies on the characterizations of the slant helices and spherical indicatrix in 2005 and 2010, [Kula, 2005; Kula 2010]. Izumiya and his colleagues first defined slant helices in 2004, [Izumiya, 2004]. With this study, the concept of slant helices was introduced into the literature. Recently, slant helices have begun to be examined with the help of the Bishop frame, which is a different alternative frame. This alternative frame appeared in 1975 by L. Bishop. The Bishop frame which can be formed without the need for a second derivative, is a frame parallel to the tangent vector field T and having more advantageous than the Frenet frame, [Bishop, 1975]. Bükcü and his colleagues examined slant helices in 2009 according to the Bishop frame, [Bükcü, 2009]. In addition, Bükcü and Karacan reviewed the characterization of slant helices according to Bishop frame in Minkowski space in 2008 [Bükcü, 2008; Karacan 2008]. Kocayiğit and his colleagues also examined timelike curves according to the Bishop darboux vector and gave some characterizations, [Kocayiğit, 2013]. A new type of Bishop frame named with Type-2 Bishop frame in 2010 was created for the first time by S. Yılmaz and his colleagues, [Yılmaz, 2010]. Later on, these authors examined the new Bishop frame in Minkowski space, [Yilmaz, 2015]. Using the Type-2 Bishop frame, different researchers have presented many studies until now. S. Kızıltuğ and his colleagues examined the slant helices according to the type-2 Bishop frame in 2013, [Kızıltuğ, 2013]. On the other hand, Uzunoğlu and his friends have defined a new frame which include the vectors $\left\{N, C = \frac{N'}{\|N'\|}, W = \frac{\tau T + \kappa B}{\|\kappa^2+\tau^2\|}\right\}$ in 2016, [Uzunoğlu, 2016]. Keskin and his colleagues obtained a

* Corresponding author e-mail: hkusak@beu.edu.tr

different Bishop frame named N-Bishop frame by using this new frame $\{N, C, W\}$ produced by Uzunoğlu [Keskin, 2017]. In our study, we obtained some characterizations of the slant helices according to the N-Bishop frame were obtained. Thanks to this study, we have introduced a different approach to slant helices using N-Bishop frame. Some properties of the slant helices have been obtained according to the derivatives of the N-Bishop frame elements. Then, covariant derivatives of N-Bishop elements and base characterizations were obtained.

2 Preliminaries

Frenet frame formulas describe the kinematic properties of a particle moving along a continuous and differentiable curve in Euclidean 3-space. Serret-Frenet frames derivative formulas can be obtained by

$$T'(s) = \kappa N(s), N'(s) = -\kappa T(s) + \tau B(s), B'(s) = -\tau N(s).$$

Bishop has created a new alternative frame which is called Bishop frame in 1975. The alternative Bishop frame is relatively parallel the unit tangent field T, and the derivative matrix of the Bishop frame is calculated by

$$T'(s) = k_1 N_1(s) + k_2 N_2(s), N_1'(s) = -k_1 T(s), N_2'(s) = -k_2 T(s)$$

where k_1 and k_2 are curvatures of the Bishop frame, [Bishop, 1975]. In addition, Bukcu et al. studied on the Bishop frames in Minkowski space, [Bükcü, 2008-2010, Karacan 2008]. On the other hand, Yılmaz et al. defined a new Bishop frame named with Type-2 Bishop frame which is parallel to binormal vector field in 2010, and its derivatives are identified by

$$N_1'(s) = -k_1 B(s), N_2'(s) = -k_2 B(s), B'(s) = k_1 N_1(s) + k_2 N_2(s)$$

where k_1 and k_2 are curvatures of the Type-2 Bishop frame, [Yılmaz, 2010]. Uzunoğlu and his colleagues gave a formula of a different alternative moving frame along a curve in Euclidean 3-space denoted by

$$\left\{N, C = \frac{N'}{\|N'\|}, W = \frac{\tau T + \kappa B}{\|\kappa^2 + \tau^2\|}\right\}.$$

The derivatives of the alternative moving frame is calculated by

$$N'(s) = f(s)C(s), C'(s) = -f(s)N(s) + g(s)W(s), W'(s) = -g(s)C(s)$$

where $f = \sqrt{\kappa^2 + \tau^2}$ and are

$$g = \frac{\kappa^2 \left(\frac{\tau}{\kappa}\right)'}{\kappa^2 + \tau^2} = \sigma f$$

the differentiable functions, [Uzunoğlu, 2016]. Keskin and Yaylı were first introduced N-Bishop frame for a normal direction curve which is defined as an integral curve of the principal normal of a curve. The derivative matrix of this new N-Bishop frame is given by

$$N'(s) = k_1 N_1(s) + k_2 N_2(s), N'_1(s) = -k_1 N(s), N'_2(s) = -k_2 N(s),$$

[Keskin-Yaylı, 2017].

3 Main results

In this section we will give some characterizations of the slant helices according to N-Bishop frame.

Definition 1. *Let* $\xi : \mathrm{I} \to \mathrm{E}^3$*be a regular curve. The curve* ξ *is called a slant helix according to N-Bishop frame* $\{\mathbf{N}, \mathbf{N}_1, \mathbf{N}_2\}$*, whether the unit vector* $\mathbf{N}_1$ *makes a constant angle* φ *with a fixed certain direction u in Euclidean 3-space; that is* $\langle \mathbf{N}_1(s), \mathbf{v}\rangle = \cos\varphi$ *for all* $s \in I$*.*

Theorem 1. *Assume that the curve* $\xi : \mathrm{I} \to \mathrm{E}^3$ *is a unit speed curve with nonzero N-Bishop frame curvatures denoted by* k_1 *and* k_2*. Thus the curve* ξ *is a slant helix if and only if the curvatures satisfy the condition* $\frac{k_1}{k_2} = const.$ *for* $\forall s \in \mathrm{I}$.

Proof. $(\Rightarrow)$: Let ξ be a slant helix in E^3. From the definition of the slant helix, the second frame vector $\mathbf{N}_1$ of the N-Bishop frame $\{\mathbf{N}, \mathbf{N}_1, \mathbf{N}_2\}$holds the condition $\langle \mathbf{N}_1, \mathbf{v}\rangle = \cos\varphi$ (const), where the angle φ is constant for all points of the curve. Taking the differentiation of the inner product $\langle \mathbf{N}_1, \mathbf{v}\rangle = \cos\varphi$, we get $\langle \mathbf{N}, \mathbf{v}\rangle = 0$. Taking the differential of $\langle \mathbf{N}, \mathbf{v}\rangle = 0$, we get also the equation $\langle \mathbf{N}', \mathbf{v}\rangle = 0$. By using the derivative formulas of N-Bishop in this equation, we obtain the inner product as $\langle k_1\mathbf{N}_1 + k_2\mathbf{N}_2, \mathbf{v}\rangle = 0$ and

$$\begin{aligned} k_1\langle \mathbf{N}_1, \mathbf{v}\rangle + k_2\langle \mathbf{N}_2, \mathbf{v}\rangle &= 0, \\ k_1(\cos\varphi) + k_2(\sin\varphi) &= 0, \\ \frac{k_1}{k_2} &= -\tan\varphi \ (const) \end{aligned}$$

If the inner product is $\langle \mathbf{N}, \mathbf{v}\rangle = 0$, then the vector $\mathbf{v} \in s_p\{\mathbf{N}_1, \mathbf{N}_2\}$is written with linear combination of $\mathbf{N}_1$ and $\mathbf{N}_2$ as

$$\mathbf{v} = (\cos\varphi)\mathbf{N}_1 + (\sin\varphi)\mathbf{N}_2 \tag{1}$$

By differentiation of the Equation (1), we get

$$\begin{aligned} \mathbf{v}' &= (\cos\varphi)\mathbf{N}_1{}' + (\sin\varphi)\mathbf{N}_2{}' \\ &= (\cos\varphi)(-k_1\mathbf{N}) + (\sin\varphi)(-k_2\mathbf{N}) \\ &= -\left[k_1(\cos\varphi) + k_2(\sin\varphi)\right]\mathbf{N} = 0 \end{aligned} \tag{2}$$

where the derivative is found zero and the vector $\mathbf{v}$ is a constant vector.

$(\Leftarrow)$: Suppose that the ratio $\frac{k_1}{k_2}$ is constant. Therefore the ratio can be written as $\frac{k_1}{k_2} = \lambda$ where the coefficient λ can be chosen as $\lambda = -\tan\varphi$. From this equation, we have $\frac{k_1}{k_2} = -\tan\varphi$ and then $k_1(\cos\varphi) + k_2(\sin\varphi) = 0$. By differentiating the equation (1), we obtain

$$\begin{aligned} \mathbf{v}' &= (\cos\varphi)\mathbf{N}_1' + (\sin\varphi)\mathbf{N}_2' \\ &= (\cos\varphi)(-k_1\mathbf{N}) + (\sin\varphi)(-k_2\mathbf{N}) \\ &= -\left[k_1(\cos\varphi) + k_2(\sin\varphi)\right]\mathbf{N} = 0. \end{aligned}$$

Then it is clear that the vector $\mathbf{v}$ is constant. Now we will Show the inner product $\langle \mathbf{N}_1, \mathbf{v}\rangle$ is constant. By substitute the components of the vector $\mathbf{v}$ in the Eq.(1), the inner product

$$\begin{aligned} \langle \mathbf{N}_1, \mathbf{v}\rangle &= \langle \mathbf{N}_1, (\cos\varphi)\mathbf{N}_1 + (\sin\varphi)\mathbf{N}_2\rangle \\ &= (\cos\varphi)\langle \mathbf{N}_1, \mathbf{N}_1\rangle + (\sin\varphi)\langle \mathbf{N}_1, \mathbf{N}_2\rangle \\ &= (\cos\varphi)\underbrace{\langle \mathbf{N}_1, \mathbf{N}_1\rangle}_{1} + (\sin\varphi)\underbrace{\langle \mathbf{N}_1, \mathbf{N}_2\rangle}_{0} \\ &= \cos\varphi \ (const). \end{aligned}$$

is satisfied. Then it is seen that the curve ξ is a slant helix. Thus, the proof of the theorem is completed.

Theorem 2. *Suppose that the curve* $\xi : \mathrm{I} \to \mathrm{E}^3$ *is a unit speed curve with nonzero N-Bishop frame curvatures denoted by* k_1 *and* k_2. *Thus the curve* ξ *is a slant helix if and only if* $\det(\mathbf{N}_1', \mathbf{N}_1'', \mathbf{N}_1''') = 0$.

Proof. $(\Rightarrow)$: Let the curve ξ be a slant helix. Thus the ratio $\frac{k_1}{k_2}$ is constant. By differentiating the equation $\mathbf{N}_1' = -k_1\mathbf{N}$ and using the derivation formula, the second derivation of $\mathbf{N}_1$ is

$$\begin{aligned} \mathbf{N}_1'' &= -k_1'\mathbf{N} - k_1\mathbf{N}' \\ &= -k_1'\mathbf{N} - k_1(k_1\mathbf{N}_1 + k_2\mathbf{N}_2) \\ &= -k_1'\mathbf{N} - {k_1}^2\mathbf{N}_1 - k_1k_2\mathbf{N}_2. \end{aligned}$$

The third derivative of $\mathbf{N}_1$ is obtained by

$$\begin{aligned} \mathbf{N}_1''' &= -2k_1k_1'\mathbf{N}_1 - {k_1}^2\mathbf{N}_1' - {k_1}''\mathbf{N} - k_1'\mathbf{N}' - k_1'k_2\mathbf{N}_2 - k_2'k_1\mathbf{N}_2 - k_1k_2\mathbf{N}_2' \\ &= -2k_1k_1'\mathbf{N}_1 - {k_1}^2(-k_1\mathbf{N}) - {k_1}''\mathbf{N} - k_1'(k_1\mathbf{N}_1 + k_2\mathbf{N}_2) - k_1'k_2\mathbf{N}_2 - k_2'k_1\mathbf{N}_2 - k_1k_2(-k_2\mathbf{N}) \\ &= ({k_1}^3 - {k_1}'' + k_1{k_2}^2)\mathbf{N} + (-3k_1k_1')\mathbf{N}_1 + (-2{k_1}'k_2 - {k_2}'k_1)\mathbf{N}_2. \end{aligned}$$

Now, if we put the above equations in $\det(\mathbf{N}_1', \mathbf{N}_1'', \mathbf{N}_1''')$, the determination will be

$$\begin{aligned} \det\left(\mathbf{N}_1', \mathbf{N}_1'', \mathbf{N}_1'''\right) &= \begin{vmatrix} -k_1 & 0 & 0 \\ -{k_1}' & -{k_1}^2 & -k_1k_2 \\ {k_1}^3 - {k_1}'' + k_1{k_2}^2 & -3k_1k_1' & -2{k_1}'k_2 - {k_2}'k_1 \end{vmatrix} \\ &= -k_1\left[2{k_1}^2k_1'k_2 + {k_2}'{k_1}^3 - 3{k_1}^2k_2{k_1}'\right] \\ &= k_1\left[{k_1}^2k_2{k_1}' - {k_2}'{k_1}^3\right] \\ &= {k_1}^3\left[k_2{k_1}' - k_1{k_2}'\right] \\ &= {k_1}^3{k_2}^2\left[\frac{k_1}{k_2}\right]'. \end{aligned}$$

Since the curve ξ is slant helix, the ratio $\frac{k_1}{k_2}$ is constant. Then the derivative of the ratio is $\left[\frac{k_1}{k_2}\right]' = 0$. As a result the determination is $\det(\mathbf{N}_1', \mathbf{N}_1'', \mathbf{N}_1''') = 0$.

$(\Leftarrow)$: Otherwise, suppose that $\det(\mathbf{N}_1', \mathbf{N}_1'', \mathbf{N}_1''') = 0$. Hence the determination can be written as $\det(\mathbf{N}_1', \mathbf{N}_1'', \mathbf{N}_1''') = {k_1}^3{k_2}^2\left[\frac{k_1}{k_2}\right]' = 0$ and because of $k_2 \neq 0$ and $k_1 \neq 0$, after the simplification this equation we obtain $\left[\frac{k_1}{k_2}\right]' = 0$ and $\frac{k_1}{k_2} = \text{const}$. Thus the curve ξ is a slant helix. The proof is done.

Theorem 3. *Let the curve* $\xi : \mathrm{I} \to \mathrm{E}^3$ *be a unit speed curve with nonzero N-Bishop frame curvatures denoted by* k_1 *and* k_2. *Thus the curve* ξ *is a slant helix if and only if* $\det(\mathbf{N}_2', \mathbf{N}_2'', \mathbf{N}_2''') = 0$.

Proof. $(\Rightarrow)$: If the curve ξ is a slant helix, then the fraction $\frac{k_1}{k_2}$ is constant. By differentiating the equation $\mathbf{N}_2' = -k_2\mathbf{N}$, the second derivative of $\mathbf{N}_2$ is obtained as

$$\begin{aligned}\mathbf{N}_2'' &= -{k_2}'\mathbf{N} - k_2\mathbf{N}' \\ &= -{k_2}'\mathbf{N} - k_2\left(k_1\mathbf{N}_1 + k_2\mathbf{N}_2\right) \\ &= -{k_2}'\mathbf{N} - k_1k_2\mathbf{N}_1 - {k_2}^2\mathbf{N}_2.\end{aligned}$$

Using the derivative formulas of N-Bishop frame, the third derivative of $\mathbf{N}_2$ can be calculated by

$$\begin{aligned}\mathbf{N}_2''' &= -{k_1}'k_2\mathbf{N}_1 - k_1{k_2}'\mathbf{N}_1 - k_1k_2\mathbf{N}_1' - {k_2}''\mathbf{N} - {k_2}'\mathbf{N}' - 2k_2{k_2}'\mathbf{N}_2 - {k_2}^2{\mathbf{N}_2}' \\ &= -{k_1}'k_2\mathbf{N}_1 - k_1{k_2}'\mathbf{N}_1 - k_1k_2(-k_1\mathbf{N}) - {k_2}''\mathbf{N} - {k_2}'\left(k_1\mathbf{N}_1 + k_2\mathbf{N}_2\right) - 2k_2{k_2}'\mathbf{N}_2 - {k_2}^2(-k_2\mathbf{N}) \\ &= ({k_2}^3 - {k_2}'' + k_2{k_1}^2)\mathbf{N} + (-2k_1{k_2}' - k_2{k_1}')\mathbf{N}_1 + (-3k_2{k_2}')\mathbf{N}_2.\end{aligned}$$

Now substituiting above results in the determination $\det\left(\mathbf{N}_2', \mathbf{N}_2'', \mathbf{N}_2'''\right)$, we get

$$\begin{aligned}\det\left(\mathbf{N}_2', \mathbf{N}_2'', \mathbf{N}_2'''\right) &= \begin{vmatrix} -k_2 & 0 & 0 \\ -{k_2}' & -k_1k_2 & -{k_2}^2 \\ {k_2}^3 - {k_2}'' + k_2{k_1}^2 & -2k_1{k_2}' - k_2{k_1}' & -3k_2{k_2}' \end{vmatrix} \\ &= -k_2(3k_1{k_2}^2k_2' - 2k_1{k_2}^2k_2' - {k_2}^3k_1') \\ &= {k_2}^3(k_2k_1' - k_1k_2') \\ &= {k_2}^5\left[\frac{k_1}{k_2}\right]'.\end{aligned}$$

where the curve ξ is a slant helix and $\frac{k_1}{k_2}$ is constant. Thus the derivative of the fraction is $\left[\frac{k_1}{k_2}\right]' = 0$. Consequently, it is easy to see that $\det\left(\mathbf{N}_2', \mathbf{N}_2'', \mathbf{N}_2'''\right) = 0$. $(\Leftarrow)$: Otherwise, assume that $\det\left(\mathbf{N}_2', \mathbf{N}_2'', \mathbf{N}_2'''\right) = 0$. From the following equation

$$\det\left(\mathbf{N}_2', \mathbf{N}_2'', \mathbf{N}_2'''\right) = {k_2}^5\left[\frac{k_1}{k_2}\right]' = 0,$$

we compute $\left[\frac{k_1}{k_2}\right]' = 0$ and the ratio $\frac{k_1}{k_2}$ is constant for $k_2 \neq 0$. Hence, the curve ξ will be a slant helix. The proof is completed.

Theorem 4. *Assume that the curve* $\xi : \mathrm{I} \to \mathrm{E}^3$ *is a unit speed curve with nonzero N-Bishop frame curvatures denoted by* k_1 *and* k_2. *The curve* ξ *is a slant helix if and only if* $\det\left(\mathbf{N}', \mathbf{N}'', \mathbf{N}'''\right) = 0$.

Proof. $(\Rightarrow)$:If ξ be a slant helix, then $\frac{k_1}{k_2}$ is constant. The derivatives of N can be compute as

$$\begin{aligned}\mathbf{N}' &= k_1\mathbf{N}_1 + k_2\mathbf{N}_2 \\ \mathbf{N}'' &= {k_1}'\mathbf{N}_1 + k_1{\mathbf{N}_1}' + {k_2}'\mathbf{N}_2 + k_2{\mathbf{N}_2}' \\ &= {k_1}'\mathbf{N}_1 + k_1(-k_1\mathbf{N}) + {k_2}'\mathbf{N}_2 + k_2(-k_2\mathbf{N}) \\ &= (-{k_1}^2 - {k_2}^2)\mathbf{N} + {k_1}'\mathbf{N}_1 + {k_2}'\mathbf{N}_2.\end{aligned}$$

Using the derivative formulas of N-Bishop frame, the third derivative is obtained by

$$\begin{aligned}\mathbf{N}''' &= (-2k_1k_1{}' - 2k_2k_2{}')\mathbf{N} + (-k_1{}^2 - k_2{}^2)\mathbf{N}' + k_1{}''\mathbf{N}_1 + k_1{}'\mathbf{N}_1{}' + k_2{}''\mathbf{N}_2 + k_2{}'\mathbf{N}_2{}' \\ &= (-2k_1k_1{}' - 2k_2k_2{}')\mathbf{N} + (-k_1{}^2 - k_2{}^2)(k_1\mathbf{N}_1 + k_2\mathbf{N}_2) + k_1{}''\mathbf{N}_1 + k_1{}'(-k_1\mathbf{N}) + k_2{}''\mathbf{N}_2 + k_2{}'(-k_2\mathbf{N}) \\ &= (-3k_1k_1{}' - 3k_2k_2{}')\mathbf{N} + (k_1{}'' - k_1{}^3 - k_1k_2{}^2)\mathbf{N}_1 + (k_2{}'' - k_1{}^2k_2 - k_2{}^3)\mathbf{N}_2.\end{aligned}$$

and the determination can be calculated by

$$\begin{aligned}\det(\mathbf{N}',\mathbf{N}'',\mathbf{N}''') &= \begin{vmatrix} 0 & k_1 & k_2 \\ (-k_1{}^2 - k_2{}^2) & k_1{}' & k_2{}' \\ (-3k_1k_1{}' - 3k_2k_2{}') & (k_1{}'' - k_1{}^3 - k_1k_2{}^2) & (k_2{}'' - k_1{}^2k_2 - k_2{}^3) \end{vmatrix} \\ &= -k_1\left[(-k_1{}^2 - k_2{}^2)(k_2{}'' - k_1{}^2k_2 - k_2{}^3) + 3k_1k'_1k_2{}' + 3k_2k_2'^2\right] \\ &+ k_2\left[(-k_1{}^2 - k_2{}^2)(k_1{}'' - k_1{}^3 - k_1k_2{}^2) + 3k_1k'^2{}_1 + 3k_2k_2'k_1{}'\right] \\ &= \left(3k_1k_2{}^2k_1{}' + 3k_2{}^3k_2{}'\right)\left(\frac{k_1}{k_2}\right)' - (k_1{}^2 + k_2{}^2)\left[k_2k_1{}'' - k_1k_2{}''\right] \\ &= \left(3k_1k_2{}^2k_1{}' + 3k_2{}^3k_2{}'\right)\left(\frac{k_1}{k_2}\right)' - \left[\left(\frac{k_1}{k_2}\right)^2 + 1\right]\left[\left(\frac{k_1}{k_2}\right)''k_2{}^4 + 2k_2{}^3k_2{}'\left(\frac{k_1}{k_2}\right)'\right].\end{aligned} \tag{3}$$

Since the curve ξ is a slant helix, the ratio $\frac{k_1}{k_2}$ is constant. Hence $\left[\frac{k_1}{k_2}\right]' = 0$ and from the Eq. (3) the determination is zero, i.e. $\det(\mathbf{N}',\mathbf{N}'',\mathbf{N}''') = 0$. $(\Leftarrow)$: Assume that $\det(\mathbf{N}',\mathbf{N}'',\mathbf{N}''') = 0$. According to Eq.(3), it is possible to say $\left[\frac{k_1}{k_2}\right]' = 0$ and $\frac{k_1}{k_2} =$ const. Therefore the curve ξ is a slant helix. Then the theorem is proven. Let the curve ξ be slant helix on an Euclidean manifold. The covariant derivative of N-Bishop frame according to the vector $\alpha'(s) = T$ can be defined by

$$\begin{aligned} D_T\mathbf{N} &= k_1\mathbf{N}_1 + k_2\mathbf{N}_2 \\ D_T\mathbf{N}_1 &= -k_1\mathbf{N} \\ D_T\mathbf{N}_2 &= -k_2\mathbf{N}. \end{aligned} \tag{4}$$

For each $s \in \mathrm{I}$, $\mathbf{N}_1(s)$ and $\mathbf{N}_2(s)$ are two N-Bishop vector field where the curvatures k_1 and k_2 are functions with s parameters.

Theorem 5. *Suppose that the curve $\xi : \mathrm{I} \to \mathrm{E}^3$ is a unit speed curve with nonzero N-Bishop frame curvatures denoted by k_1 and k_2. The curve ξ is a slant helix if and only if*

$$D_T(D_TD_T\mathbf{N}_1) = D_T\mathbf{N}_1\left(\frac{k_1''}{k_1} - k_2^2 - k_1^2\right) - 3k_1'D_T\mathbf{N}. \tag{5}$$

Proof. $(\Rightarrow)$: Note that since the curve ξ is a slant helix, taking the derivative of the Eq.(5) and using the Eq. (4) the covariant derivative of $D_T\mathbf{N}_1 = -k_1\mathbf{N}$ is obtained by

$$\begin{aligned} D_T(D_T\mathbf{N}_1) &= D_T(-k_1\mathbf{N}) \\ &= -k_1{}'\mathbf{N} - k_1D_T\mathbf{N} \\ &= -k_1{}'\mathbf{N} - k_1(k_1\mathbf{N}_1 + k_2\mathbf{N}_2) \\ &= -k_1{}'\mathbf{N} - k_1{}^2\mathbf{N}_1 - k_1k_2\mathbf{N}_2 \end{aligned} \tag{6}$$

and the covariant derivative of the Eq.(6) is calculated by

$$\begin{aligned} D_T(D_TD_T\mathbf{N}_1) &= D_T(-k_1{}'\mathbf{N}-k_1{}^2\mathbf{N}_1-k_1k_2\mathbf{N}_2) \\ &= -k_1{}''\mathbf{N}-k_1{}'D_T\mathbf{N}-2k_1k_1{}'\mathbf{N}_1-k_1{}^2D_T\mathbf{N}_1-(k_1{}'k_2+k_1k_2{}')\mathbf{N}_2-k_1k_2D_T\mathbf{N}_2 \\ &= -k_1{}''\mathbf{N}-k_1{}'D_T\mathbf{N}-2k_1k_1{}'\mathbf{N}_1-(k_1{}'k_2+k_1k_2{}')\mathbf{N}_2-k_1k_2(-k_2\mathbf{N})-k_1{}^2D_T\mathbf{N}_1. \end{aligned} \tag{7}$$

From above equations it is easy to say that $\frac{k_1}{k_2}$ is constant. Then we have $\left(\frac{k_1}{k_2}\right)'=0$. If we take this derivative, we get the following properties

$$\begin{aligned} \frac{k_1'k_2-k_1k_2'}{k_2^2} &= 0 \\ k_1'k_2-k_1k_2' &= 0 \\ k_1'k_2 &= k_1k_2'. \end{aligned} \tag{8}$$

Furthermore, by using the Eq. (4), we can write the equation

$$N=\frac{-1}{k_1}D_T\mathbf{N}_1 \tag{9}$$

Now, by substituting (8) and (9) in the Eq.(7) and the required arrangements are made, then the third covariant derivative of $\mathbf{N}_1$ is obtained by

$$\begin{aligned} D_T(D_TD_T\mathbf{N}_1) &= -k_1{}''\mathbf{N}+k_1k_2^2\mathbf{N}-k_1{}'D_T\mathbf{N}-2k_1'\left(\underbrace{k_1\mathbf{N}_1+k_2\mathbf{N}_2}_{D_TN}\right)-k_1{}^2D_T\mathbf{N}_1 \\ &= (-k_1{}''+k_1k_2^2)\mathbf{N}-k_1{}^2D_T\mathbf{N}_1-3k_1{}'D_T\mathbf{N} \\ &= (-k_1{}''+k_1k_2^2)(-\frac{1}{k_1}D_T\mathbf{N}_1)-k_1{}^2D_T\mathbf{N}_1-3k_1{}'D_T\mathbf{N} \\ &= D_T\mathbf{N}_1(\frac{k_1{}''}{k_1}-k_2^2-k_1{}^2)-3k_1{}'D_T\mathbf{N}. \end{aligned}$$

($\Leftarrow$): Conversely, Let's assume that the equality is correct in (5). In this case, let's try to show that this curve ξ is a slant helix. If we take the covariant derivative of (9), we have

$$\begin{aligned} D_T\mathbf{N} &= D_T\left(\frac{-1}{k_1}D_T\mathbf{N}_1\right) \\ &= \frac{k_1'}{k_1^2}D_T\mathbf{N}_1-\frac{1}{k_1}D_TD_T\mathbf{N}_1. \end{aligned}$$

In this equation, necessary arrangements are made by taking the covariant derivative again, i.e.

$$\begin{aligned} D_TD_T\mathbf{N} &= D_T\left(\frac{k_1'}{k_1^2}D_T\mathbf{N}_1-\frac{1}{k_1}D_TD_T\mathbf{N}_1\right) \\ &= \left(\frac{k_1'}{k_1^2}\right)'D_T\mathbf{N}_1+\frac{k_1'}{k_1^2}D_TD_T\mathbf{N}_1+\frac{k_1'}{k_1^2}D_TD_T\mathbf{N}_1-\frac{1}{k_1}D_TD_TD_T\mathbf{N}_1. \end{aligned} \tag{10}$$

By substituting (5) in (10), the covariant derivative can also be written as

$$\begin{aligned} D_T D_T \mathbf{N} &= \left(\frac{k_1'}{k_1^2}\right)' D_T \mathbf{N}_1 + \frac{k_1'}{k_1^2} D_T D_T \mathbf{N}_1 + \frac{k_1'}{k_1^2} D_T D_T \mathbf{N}_1 - \frac{1}{k_1}\left(D_T \mathbf{N}_1 \left(\frac{k_1''}{k_1} - k_2^2 - k_1^2\right) - 3k_1' D_T \mathbf{N}\right) \\ &= \left(\frac{k_1'}{k_1^2}\right)' D_T \mathbf{N}_1 + \frac{k_1'}{k_1^2} D_T D_T \mathbf{N}_1 + \frac{k_1'}{k_1^2} D_T D_T \mathbf{N}_1 + D_T \mathbf{N}_1 \left(-\frac{k_1''}{k_1^2} + \frac{k_2^2}{k_1} + k_1\right) + \frac{3k_1'}{k_1} D_T \mathbf{N} \\ &= \left(\left(\frac{k_1'}{k_1^2}\right)' - \frac{k_1''}{k_1^2} + \frac{k_2^2}{k_1} + k_1\right) D_T \mathbf{N}_1 + \frac{2k_1'}{k_1^2} D_T D_T \mathbf{N}_1 + \frac{3k_1'}{k_1} D_T \mathbf{N}. \end{aligned}$$

If we replace the Eq.(6) and Eq.(4) and make the necessary arrangements, then we have

$$\begin{aligned} D_T D_T \mathbf{N} &= \left(\left(\frac{k_1'}{k_1^2}\right)' - \frac{k_1''}{k_1^2} + \frac{k_2^2}{k_1} + k_1\right) D_T \mathbf{N}_1 - \frac{2(k_1')^2}{k_1^2}\mathbf{N} - 2k_1'\mathbf{N}_1 - \frac{2k_1' k_2}{k_1}\mathbf{N}_2 + \frac{3k_1'}{k_1}(k_1 \mathbf{N}_1 + k_2 \mathbf{N}_2) \\ &= \left(\left(\frac{k_1'}{k_1^2}\right)' - \frac{k_1''}{k_1^2} + \frac{k_2^2}{k_1} + k_1\right) D_T \mathbf{N}_1 + k_1'\mathbf{N}_1 + \frac{k_1' k_2}{k_1}\mathbf{N}_2 - \frac{2(k_1')^2}{k_1^2}\mathbf{N}. \end{aligned} \tag{11}$$

On the other hand, using the Eq.(4), the second covariant derivative $D_T D_T \mathbf{N}$ can also be found by

$$\begin{aligned} D_T D_T \mathbf{N} &= D_T (k_1 \mathbf{N}_1 + k_2 \mathbf{N}_2) \\ &= k_1' \mathbf{N}_1 + k_1 D_T \mathbf{N}_1 + k_2' \mathbf{N}_2 + k_2 D_T \mathbf{N}_2 \\ &= k_1' \mathbf{N}_1 + k_1 D_T \mathbf{N}_1 + k_2' \mathbf{N}_2 - k_2^2 \mathbf{N}. \end{aligned} \tag{12}$$

Now, by equating (11)and (12), we get $\frac{k_1' k_2}{k_1} = k_2'$. Then making some arrangements, the equality is formed by $\frac{k_1'}{k_1} = \frac{k_2'}{k_2}$ and $k_2 k_1' - k_1 k_2' = 0$ then $\left(\frac{k_1}{k_2}\right)' = 0$. This result requires $\frac{k_1}{k_2} = const$. Hence, ξ is a slant helix.

Theorem 6. *Assume that the curve $\xi : \mathrm{I} \to \mathrm{E}^3$ is a unit speed curve on an Euclidean manifold. The curve ξ is a slant helix if and only if*

$$D_T (D_T D_T \mathbf{N}_2) = D_T \mathbf{N}_2 \left(\frac{k_2''}{k_2} - k_2^2 - k_1^2\right) - 3k_2' D_T \mathbf{N} \tag{13}$$

Proof. Let ξ be a slant helix. The second covariant derivative of $\mathbf{N}_2$ is

$$\begin{aligned} D_T (D_T \mathbf{N}_2) &= D_T (-k_2 \mathbf{N}) \\ &= -{k_2}' \mathbf{N} - k_2 D_T \mathbf{N} \\ &= -{k_2}' \mathbf{N} - k_1 k_2 \mathbf{N}_1 - k_2^2 \mathbf{N}_2 \end{aligned}$$

Furthermore, the third covariant derivative is obtained by

$$\begin{aligned} D_T (D_T D_T \mathbf{N}_2) &= D_T (-{k_2}' \mathbf{N} - k_1 k_2 \mathbf{N}_1 - k_2^2 \mathbf{N}_2) \\ &= -k_2'' \mathbf{N} - k_2' D_T \mathbf{N} - 2k_2 k_2' \mathbf{N}_2 - k_2^2 D_T \mathbf{N}_2 - (k_1' k_2 + k_1 k_2')\mathbf{N}_1 - k_1 k_2 D_T \mathbf{N}_1 \\ &= -k_2'' \mathbf{N} - k_2' D_T \mathbf{N} - 2k_2 k_2' \mathbf{N}_2 - (k_1' k_2 + k_1 k_2')\mathbf{N}_1 - k_1 k_2 (-k_1 \mathbf{N}) - k_2^2 D_T \mathbf{N}_2 \\ &= -{k_2}'' \mathbf{N} + k_1^2 k_2 \mathbf{N} - {k_2}' D_T \mathbf{N} - 2k_2' \underbrace{(k_1 \mathbf{N}_1 + k_2 \mathbf{N}_2)}_{D_T \mathbf{N}} - k_2^2 D_T \mathbf{N}_2 \\ &= (-k_2'' + k_1^2 k_2)\mathbf{N} - k_2^2 D_T \mathbf{N}_2 - 3k_2' D_T \mathbf{N}. \end{aligned}$$

On the other hand, since we can write $N = \frac{-1}{k_2} D_T \mathbf{N}_2$ from the derivative formulas of N-Bishop frame, the above equations can be arranged as

$$\begin{aligned} D_T(D_T D_T \mathbf{N}_2) &= (-k_2'' + k_1^2 k_2)(\frac{-1}{k_2} D_T \mathbf{N}_2) - k_2^2 D_T \mathbf{N}_2 - 3k_2' D_T \mathbf{N} \\ &= D_T \mathbf{N}_2 (\frac{{k_2}''}{k_2} - k_2^2 - {k_1}^2) - 3k_2' D_T \mathbf{N}. \end{aligned}$$

Thus the Eq.(13) is proven. Conversely, the other hand of the theorem can be proven similarly.

Theorem 7. *Let the curve $\xi : \mathrm{I} \to \mathrm{E}^3$ be a unit speed curve The curve ξ is a slant helix if and only if the following equalities*

(1) $D_T(D_T D_T \mathbf{N}) = D_T \mathbf{N} \left(\frac{{k_1}''}{k_1} - k_1^2 - {k_2}^2 \right) + 3k_1' D_T \mathbf{N}_1 + 3k_2' D_T \mathbf{N}_2$

(2) $D_T(D_T D_T \mathbf{N}) = D_T \mathbf{N} \left(\frac{{k_2}''}{k_2} - k_1^2 - {k_2}^2 \right) + 3k_1' D_T \mathbf{N}_1 + 3k_2' D_T \mathbf{N}_2$

(3) $D_T(D_T D_T \mathbf{N}) = D_T \mathbf{N} \left(\frac{{k_1}''}{2k_1} + \frac{{k_2}''}{2k_2} - k_1^2 - {k_2}^2 \right) + 3k_1' D_T \mathbf{N}_1 + 3k_2' D_T \mathbf{N}_2$

are satisfied when each condition is provided separately.

Proof. Consider the curve ξ is a slant helix. Taking the covariant derivative of $D_T \mathbf{N} = k_1 \mathbf{N}_1 + k_2 \mathbf{N}_2$ and replace the derivative formulas of N-Bishop frame, we obtain the equality

$$\begin{aligned} D_T(D_T \mathbf{N}) &= D_T(k_1 \mathbf{N}_1 + k_2 \mathbf{N}_2) \\ &= k_1' \mathbf{N}_1 + k_1 D_T \mathbf{N}_1 + k_2' \mathbf{N}_2 + k_2 D_T \mathbf{N}_2 \\ &= -k_1^2 \mathbf{N} - k_2^2 \mathbf{N} + k_1' \mathbf{N}_1 + k_2' \mathbf{N}_2. \end{aligned}$$

The third covariant derivative of the vector $\mathbf{N}$ can be compute as

$$\begin{aligned} D_T(D_T D_T \mathbf{N}) &= D_T(-k_1^2 \mathbf{N} - k_2^2 \mathbf{N} + k_1' \mathbf{N}_1 + k_2' \mathbf{N}_2) \\ &= -2k_1 k_1' \mathbf{N} - k_1^2 D_T \mathbf{N} - 2k_2 k_2' \mathbf{N} - k_2^2 D_T \mathbf{N} + k_1'' \mathbf{N}_1 + k_1' D_T \mathbf{N}_1 + k_2'' \mathbf{N}_2 + k_2' D_T \mathbf{N}_2 \qquad (14) \\ &= 2k_1' D_T \mathbf{N}_1 + 2k_2' D_T \mathbf{N}_2 - k_1^2 D_T \mathbf{N} - k_2^2 D_T \mathbf{N} + k_1'' \mathbf{N}_1 + k_2'' \mathbf{N}_2 + k_1' D_T \mathbf{N}_1 + k_2' D_T \mathbf{N}_2 \\ &= 3k_1' D_T \mathbf{N}_1 + 3k_2' D_T \mathbf{N}_2 - D_T \mathbf{N}(k_1^2 + k_2^2) + k_1'' \mathbf{N}_1 + k_2'' \mathbf{N}_2. \end{aligned}$$

Now we will proof the property $k_1'' \mathbf{N}_1 + k_2'' \mathbf{N}_2$. Since ξ is slant helix, then the fraction $\frac{k_1}{k_2}$ is constant. Thus $\left(\frac{k_1}{k_2} \right)' = 0$. If we arrange the derivative ratio, we have

$$\begin{aligned} \frac{k_1' k_2 - k_1 k_2'}{k_2^2} &= 0, \\ k_1' k_2 - k_1 k_2' &= 0, \\ \frac{k_1'}{k_2'} &= \frac{k_1}{k_2}. \end{aligned}$$

Because of $\frac{k_1}{k_2} = const$, then the ratio $\frac{k_1'}{k_2'}$ is constant, too. If we apply derivative of the fraction, we get

$$\begin{aligned}\frac{k''_1 k_2' - k_1' k''_2}{k_2'^2} &= 0, \\ k''_1 k_2' - k_1' k''_2 &= 0, \\ \frac{k''_1}{k''_2} &= \frac{k_1'}{k_2'}.\end{aligned}$$

As a result, there is a relation as $\frac{k''_1}{k''_2} = \frac{k_1'}{k_2'} = \frac{k_1}{k_2}$ between the curvatures. If the equality is taken to equal a constant coefficient as $\frac{k_1''}{k_1} = \frac{k''_2}{k_2} = a$, the the sum phrase $k''_1 \mathbf{N}_1 + k''_2 \mathbf{N}_2$ can be denoted by

$$\begin{aligned}k_1'' \mathbf{N}_1 + k_2'' \mathbf{N}_2 &= a k_1 \mathbf{N}_1 + a k_2 \mathbf{N}_2 \\ &= a(k_1 \mathbf{N}_1 + k_2 \mathbf{N}_2) \\ &= a D_T \mathbf{N}.\end{aligned}$$

Consequently, the results are found as

$$k_1'' \mathbf{N}_1 + k_2'' \mathbf{N}_2 = \frac{k_1''}{k_1} D_T \mathbf{N} \;\; or \;\; k_1'' \mathbf{N}_1 + k_2'' \mathbf{N}_2 = \frac{k_2''}{k_2} D_T \mathbf{N}.$$

Using the above equations, a characterization of the third covariant of $\mathbf{N}$ can be computed by

$$\begin{aligned}D_T(D_T D_T \mathbf{N}) &= -D_T \mathbf{N}(k_1^2 + k_2^2) + \frac{k_1''}{k_1} D_T \mathbf{N} + 3k_1' D_T \mathbf{N}_1 + 3k_2' D_T \mathbf{N}_2 \\ &= -D_T \mathbf{N}(k_1^2 + k_2^2) + \frac{k_2''}{k_2} D_T \mathbf{N} + 3k_1' D_T \mathbf{N}_1 + 3k_2' D_T \mathbf{N}_2 \\ &= D_T \mathbf{N}\left(\frac{k_2''}{k_2} - k_1^2 - k_2^2\right) + 3k_1' D_T \mathbf{N}_1 + 3k_2' D_T \mathbf{N}_2.\end{aligned} \tag{15}$$

Furthermore, another characterization of $D_T \mathbf{N}\left(\frac{k_1''}{k_1} + \frac{k_2''}{k_2}\right)$ is

$$\begin{aligned}D_T \mathbf{N}\left(\frac{k_1''}{k_1} + \frac{k_2''}{k_2}\right) &= (k_1 \mathbf{N}_1 + k_2 \mathbf{N}_2)\left(\frac{k_1''}{k_1} + \frac{k_2''}{k_2}\right) \\ &= k_1'' \mathbf{N}_1 + k_2'' \mathbf{N}_2 + \frac{k_1'' k_2}{k_2} \mathbf{N}_1 + \frac{k_1 k_2''}{k_1} \mathbf{N}_2 \\ &= k_1'' \mathbf{N}_1 + k_2'' \mathbf{N}_2 + k_1'' \mathbf{N}_1 + k_2'' \mathbf{N}_2 \\ &= 2\left(k_1'' \mathbf{N}_1 + k_2'' \mathbf{N}_2\right).\end{aligned}$$

If we simply both sides of above equality, then

$$k_1'' \mathbf{N}_1 + k_2'' \mathbf{N}_2 = D_T \mathbf{N}\left(\frac{k_1''}{2k_1} + \frac{k_2''}{2k_2}\right)$$

is obtained. When we substitute above results in the Eq.(15) and make some arrangements, we have the result characterization

$$D_T(D_T D_T \mathbf{N}) = -D_T\mathbf{N}(k_1^2+k_2^2) + D_T\mathbf{N}\left(\frac{k_1''}{2k_1}+\frac{k_2''}{2k_2}\right) + 3k_1' D_T\mathbf{N}_1 + 3k_2' D_T\mathbf{N}_2$$
$$= D_T\mathbf{N}\left(\frac{k_1''}{2k_1}+\frac{k_2''}{2k_2}-k_1^2-k_2^2\right) + 3k_1' D_T\mathbf{N}_1 + 3k_2' D_T\mathbf{N}_2.$$

Otherwise of the theorem can be proven similarly.

4 Conclusion

The concept of helices and slant helices have very important application areas in geometry. Therefore, many researchers have been working on these subjects until now. In our study, we obtained some characterizations of the slant helices according to N-Bishop frame. We think that our work is a useful resource for future studies by researchers.

Acknowledgments

The paper has been presented in ICAAMM 2018 conference, Istanbul and only the abstract of the article has been published in conference proceeding book. The authors would like to thank the referee for the careful review and the valuable comments which really improved the paper.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] Bishop, R.L., *There is more than one way to frame a curve*, The American Mathematical Monthly 82, 246-251, 1975.
[2] Bukcu, B.; Karacan, M.K., *Bishop frame of the spacelike curve with a spacelike principal normal in Minkowski 3-space*, Commun.Fac. Sci. Univ. Ank. Ser. A1 Math. Stat 57, 13-22, 2008.
[3] Bukcu, B.; Karacan, M.K., *The slant helices according to Bishop frame*, International Journal of Computational and Mathematical Sciences 3, 67-70, 2009.
[4] Izumiya, S.; Takeuchi, N., *New special curves and developable surfaces*, Turkish Journal of Mathematics 28, 153-164, 2004.
[5] Karacan, M.K., *Bishop frame of the timelike curve in Minkowski 3-space*, SDU Fen Edebiyat Fakultesi Fen Dergisi 3, 2008.
[6] Keskin, O.; Yayli, Y., *An application of N-Bishop frame to spherical images for direction curves*, International Journal of Geometric Methods in Modern Physics 14, 1750162, 2017.
[7] Kiziltug, S.; Kaya, S.; Tarakci O., *Slant helices according to type-2 Bishop frame in Euclidean 3-space*, International Journal of Pure and Applied Mathematics 85, 211-222, 2013.
[8] Kocayigit, H.;Ozdemir, A.; Cetin M.; Asartepe S.O., Characterizations of Timelike Curves According to the Bishop Darboux Vector in Minkowski 3-Space 3, International Mathematical Forum 19, 903-911, 2013.

[9] Kula, L.; Ekmekci, N.; Yayli Y.; Ilarslan K., *Characterizations of slant helices in Euclidean 3-space*, Turkish Journal of Mathematics34, 261-274, 2010.

[10] Kula, L., Yayli, Y., *On slant helix and its spherical indicatrix* , Applied Mathematics and Computation 169.1, 600-607, 2005.

[11] Lopez, R. *Differential geometry of curves and surfaces in Lorentz-Minkowski space*, Int. Elec. Journ. Geom. 3, 67-101, 2010.

[12] Uzunoglu; B., Gok, I.; Yayli, Y.*A new approach on curves of constant precession*, Applied Mathematics and Computation 275, 317-323, 2016.

[13] Weinstein, T., *An introduction to Lorentz surfaces* Walter de Gruyter, 1996.

[14] Yilmaz, S.; Ozyilmaz, E.; Turgut, M., *New spherical indicatrices and their characterizations,* An. St. Univ. Ovidius Constanta 18, 337-354,2010.

[15] Yilmaz, S.; Unluturk, Y. *A note on spacelike curves according to type-2 Bishop frame in Minkowski 3-space,* Int J Pure Appl Math 103, 321-332, 2015.