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An efficient method for solving a class of linear and nonlinear fractional boundary value problems

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Abstract: In this paper, we investigate the sinc collocation method to obtain the approximate solution of fractional boundary value problems based on the conformable fractional derivative. For this purpose a theorem is proved to represent the terms having fractional derivatives in terms of sinc basis functions. Some problems are solved to illustrate the accuracy and efficiency of the presented method. The obtained solutions are compared with the exact solutions of the problems.*

Keywords: Fractional differential equations, boundary value problems, sinc-collocation method, conformable derivative.

1 Introduction

Fractional calculus is a field of calculus that involves noninteger order differential and integral operators. The history of fractional calculus dates back to the end of the 17th century. In 1695, half-order derivative was mentioned in a letter from L'Hopital to Leibniz [1]. Since then, fractional calculus developed mainly as a pure theoretical field for mathematicians. However, in the last few decades fractional calculus has attracted the interest of many researchers in several areas [2],[3], [4], [5], [6], [7], [8], [9]. Many mathematicians contributed to the development of fractional calculus, therefore many definitions for the fractional derivative are available. The most popular definitions are Riemann-Liouville and Caputo definition of fractional derivatives. Riemann-Liouville and Caputo definitions of α order a^{th} derivative of function f is given as

$$D_a^{\alpha}(f)(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t\frac{f(x)}{(t-x)^{\alpha-n+1}}dx$$

and

$$D_{*,a}^{\alpha}(f)(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t\frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}dx$$

respectively, where $\alpha\in[n-1,n)$, $n=1,2,\ldots$. Recently, Khalil et al.[10] introduced a new definition of fractional derivative called the conformable fractional derivative. In [11], Abdeljawad developed the definition of conformable fractional derivative and set basic concepts of this new fractional calculus.For a detailed overview of the conformable fractional derivative and applications, we refer the reader to [12], [13], [14], [15] and references therein.
Particularly, in this paper, sinc-collocation method is presented to obtain the approximate solution of fractional order boundary value problem with variable coefficients in the following form

$$\mu_2(x)y''(x)+\mu_\alpha(x)y^{(\alpha)}(x)+\mu_1(x)y'(x)+\mu_\beta(x)y^{(\beta)}(x)+\mu_0(x)y(x)+\mu_N(x)N(y(x))=f(x) \tag{1}$$
$$y(a)=0,y(b)=0$$

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The paper organized as follows. In section 2, we have given some definition and theorems for fractional calculus and sinc-collocation method. In section 3, we use sinc-collocation method to obtain an approximate solution of a general fractional differential equation and obtained results are stated as a new theorems. In section 4, by using tables and graphs some test problems are given to show the abilities of present method. Finally, in section 5, we have completed the paper with a conclusion.

2 Preliminaries

In this section, some preliminaries and notations related to fractional calculus and sinc basis functions are given. For more details see [16], [17], [18], [19], [20], [21]

Definition 1. *Let $\alpha \in (n,n+1]$, and f be an $n-$ differentiable function at t, where $t>0$. Then the conformable fractional derivative of f of order α is defined as*

$$T_\alpha(f)(t)=\lim_{\varepsilon\to 0}\frac{f^{(\lceil\alpha\rceil-1)}(t+\varepsilon t^{(\lceil\alpha\rceil-\alpha)})-f^{(\lceil\alpha\rceil-1)}(t)}{\varepsilon} \tag{2}$$

where $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α.

Remark. As a consequence of Definition1, one can easily show that

$$T_\alpha\left(f\right)\left(t\right)=t^{(\lceil\alpha\rceil-\alpha)}f^{\lceil\alpha\rceil}\left(t\right)$$

where $\alpha\in(n,n+1]$, and f is $(n+1)$ differentiable at $t>0$.

Theorem 1. *Let $\alpha\in(n,n+1]$ and $f;g$ be $\alpha-$ differentiable at a point $t>0$. Then*

1.$T_\alpha\left(af+bg\right)=aT_\alpha\left(f\right)+bT_\alpha\left(g\right)$, for all $a,b\in\mathbb{R}$.
2.$T_\alpha\left(t^p\right)=pt^{p-\alpha}$, for all $p\in\mathbb{R}$
3.$T_\alpha\left(\lambda\right)=0$, for all constant functions $f\left(t\right)=\lambda$.
4.$T_\alpha\left(fg\right)=fT_\alpha\left(g\right)+gT_\alpha\left(f\right)$.
5.$T_\alpha\left(\frac{f}{g}\right)=\frac{gT_\alpha(f)+fT_\alpha(g)}{g^2}$.

Definition 2. *The Sinc function is defined on the whole real line $-\infty<x<\infty$ by*

$$sinc(x)=\begin{cases}\frac{\sin(\pi x)}{\pi x} & x\neq 0\\ 1 & x=0.\end{cases}$$

Definition 3. *For $h>0$ and $k=0,\pm1,\pm2,...$ the translated sinc function with space node are given by*

$$S(k,h)(x)=sinc\Big(\frac{x-kh}{h}\Big)=\begin{cases}\frac{\sin\left(\pi\frac{x-kh}{h}\right)}{\pi\frac{x-kh}{h}} & x\neq kh\\ 1 & x=kh.\end{cases}$$

To construct approximation on the interval (a,b) the conformal map

$$\phi(z)=\ln\left(\frac{z-a}{b-z}\right)$$

is employed. The basis functions on the interval (a,b) are derived from the composite translated sinc functions

$$S_k(z)=S(k,h)(z)\circ\phi(z)=sinc\Big(\frac{\phi(z)-kh}{h}\Big).$$

The inverse map of $w=\phi(z)$ is

$$z=\phi^{-1}(w)=\frac{a+be^w}{1+e^w}.$$

The sinc grid points $z_k \in (a,b)$ will be denoted by x_k because they are real. For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_k = \phi^{-1}(kh) = \frac{a+be^{kh}}{1+e^{kh}}, \qquad k = 0, \pm1, \pm2, \ldots$$

3 The sinc-collocation method

Let us assume an approximate solution for $y(x)$ in (1) by finite expansion of sinc basis functions for as follows

$$y_n(x) = \sum_{k=-M}^{N} c_k S_k(x), \qquad n = M+N+1 \tag{3}$$

where $S_k(x)$ is the function $S(k,h) \circ \phi(x)$. Here, the unknown coefficients c_k in (3) are determined by sinc-collocation method via the following theorems.

Theorem 2. *The first and second derivatives of $y_n(x)$ are given by*

$$\frac{d}{dx} y_n(x) = \sum_{k=-M}^{N} c_k \phi'(x) \frac{d}{d\phi} S_k(x) \tag{4}$$

$$\frac{d^2}{dx^2} y_n(x) = \sum_{k=-M}^{N} c_k \left(\phi''(x) \frac{d}{d\phi} S_k(x) + (\phi'(x))^2 \frac{d^2}{d\phi^2} S_k(x) \right) \tag{5}$$

respectively.

Theorem 3. *The conformable fractional derivatives of order β and α of $y_n(x)$ for $1 < \alpha \leq 2$ and $0 < \beta \leq 1$ are given by*

$$y_n^{(\beta)}(x) = \sum_{k=-M}^{N} c_k x^{1-\beta} \phi'(x) \frac{d}{d\phi} S_k(x) \tag{6}$$

$$y_n^{(\alpha)}(x) = \sum_{k=-M}^{N} c_k x^{2-\alpha} \left(\phi''(x) \frac{d}{d\phi} S_k(x) + (\phi'(x))^2 \frac{d^2}{d\phi^2} S_k(x) \right) \tag{7}$$

respectively.

Proof. The conformable fractional derivative of order β of $y_n(x)$ in (3) is written as

$$y_n^{(\beta)}(x) = \sum_{k=-M}^{N} c_k S_k^{(\beta)}(x).$$

Here, according to Remark, we can write

$$S_k^{(\beta)}(x) = x^{1-\beta} S_k'(x).$$

Now, if we use (4), we obtain

$$y_n^{(\beta)}(x) = \sum_{k=-M}^{N} c_k x^{1-\beta} \phi'(x) \frac{d}{d\phi} S_k(x)$$

Similarly, we may write the conformable fractional derivative of order α of $y_n(x)$ in (3) as

$$y_n^{(\alpha)}(x) = \sum_{k=-M}^{N} c_k S_k^{(\alpha)}(x).$$

By using Remark, we have

$$S_k^{(\alpha)}(x) = x^{2-\alpha} S_k''(x).$$

Then by (5), we get the desired result

$$y_n^{(\alpha)}(x) = \sum_{k=-M}^{N} c_k x^{2-\alpha}\left(\phi''(x)\frac{d}{d\phi}S_k(x) + (\phi'(x))^2\frac{d^2}{d\phi^2}S_k(x)\right)$$

Replacing each term of (1) with the approximation given in (3), (7) multiplying the resulting equation by $\{(1/\phi')^2\}$, we obtain the following system

$$\sum_{k=-M}^{N}\left[c_k\left\{\sum_{i=0}^{2} g_i(x)\frac{d^i}{d\phi^i}S_k\right\}\right] + g_N(x)N(c_k)S_k = \left(f(x)\left(\frac{1}{\phi'(x)}\right)^2\right)$$

where

$$g_0(x) = \mu_0(x)\left(\frac{1}{\phi'(x)}\right)^2$$
$$g_1(x) = \left[(\mu_1(x)+\mu_\beta(x)x^{1-\beta})\left(\frac{1}{\phi'(x)} - (\mu_2(x)+\mu_\alpha(x)x^{2-\alpha})\left(\frac{1}{\phi'(x)}\right)'\right)\right]$$
$$g_2(x) = \mu_2(x)+\mu_\alpha(x)x^{2-\alpha}$$
$$g_N(x) = \mu_N(x)\left(\frac{1}{\phi'(x)}\right)^2$$

We know from [20] that

$$\delta_{jk}^{(0)} = \delta_{kj}^{(0)}, \qquad \delta_{jk}^{(1)} = -\delta_{kj}^{(1)}, \qquad \delta_{jk}^{(2)} = \delta_{kj}^{(2)}$$

then setting $x = x_j$, we obtain the following theorem.

Theorem 4. *If the assumed approximate solution of boundary value problem (1) is (3), then the discrete sinc-collocation system for the determination of the unknown coefficients* $\{c_k\}_{k=-M}^{N}$ *is given by*

$$\sum_{k=-M}^{N}\left[c_k\left\{\sum_{i=0}^{2}\frac{g_i(x_j)(-1)^i}{h^i}\delta_{jk}^{(i)}\right\}\right] + g_3(x_j)N(c_k)\delta_{jk}^{(0)} = \left(f(x_j)\left(\frac{1}{\phi'(x_j)}\right)^2\right) \qquad j=-M,\dots,N \tag{8}$$

We now introduce some notations to rewrite in the matrix form for system (8). Let $\mathbf{D}(y)$ denotes a diagonal matrix whose diagonal elements are $y(x_{-M}), y(x_{-M+1}), , y(x_N)$ and non-diagonal elements are zero, let denote a matrix and also let $\mathbf{I}^{(i)}$ denote the matrices

$$\mathbf{I}^{(i)} = [\delta_{jk}^{(i)}], \qquad i = 0,1,2$$

where $\mathbf{D}, \mathbf{I}^{(0)}, \mathbf{I}^{(1)}$ and $\mathbf{I}^{(2)}$ are square matrices of order $n \times n$. In order to calculate unknown coefficients c_k in nonlinear system (8), we rewrite this system by using the above notations in matrix form as

$$\mathbf{A_1 c} + \mathbf{A_2 N(c)} = \mathbf{B} \tag{9}$$

where

$$\mathbf{A_1} = \sum_{i=0}^{2}\frac{1}{h^i}\mathbf{D}(g_i)\mathbf{I}^{(i)}$$
$$\mathbf{A_2} = \mathbf{D}(g_3)\mathbf{I}^{(0)}$$
$$\mathbf{B} = \mathbf{D}\left(\frac{f}{\phi'}\right)\mathbf{1}$$
$$\mathbf{c} = (c_{-M}, c_{-M+1}, \dots, c_N)^T$$

Now we have nonlinear system of n equations in the n unknown coefficients given by (9). We can find the unknown coefficients c_k by solving this system.

4 Computational examples

In this section, some examples are given to illustrate the accuracy of the presented methods by `MATHEMATICA 10`. In all examples, $h = \pi/\sqrt{N}, N = M$ are taken into account.

Example 1. [23] Firstly, consider linear fractional boundary value problem

$$y''(x) + \cos(x)y'(x) - xy^{(0.7)}(x) = f(x)$$

subject to the homogeneous boundary conditions

$$y(0) = 0, \qquad y(1) = 0$$

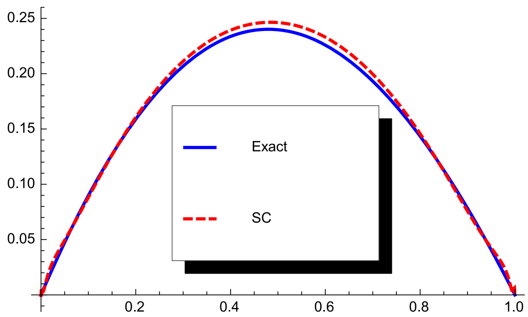
where $f(x) = -\cos^2(x)(x-1) + \sin(x)(x^{1.3} + x - 1) - \cos(x)(\sin(x) - x^{2.3} + x^{1.3} + 2)$. The exact solution of this problem is $y(x) = \sin(x)(1-x)$. The absolute errors which are obtained by using the present method for this problem are presented in Table 1. In Table 2, for the comparison, the absolute errors which are obtained by using the sinc-Galerkin method are reported. Additionally, the graphics of the exact and approximate solutions for different values of N are given in Figure 1.

Table 1: Absolute errors for *Example 1* by using sinc collocation method for different values of N

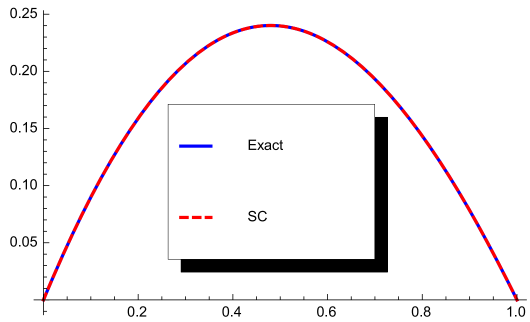
x	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
0.1	1.651×10^{-3}	1.898×10^{-4}	5.158×10^{-6}	1.444×10^{-8}	1.031×10^{-10}
0.2	1.722×10^{-3}	1.699×10^{-4}	5.876×10^{-7}	6.073×10^{-8}	5.149×10^{-11}
0.3	4.494×10^{-3}	8.171×10^{-5}	4.330×10^{-6}	1.957×10^{-9}	2.902×10^{-11}
0.4	5.638×10^{-3}	2.388×10^{-4}	6.934×10^{-7}	5.271×10^{-8}	2.793×10^{-11}
0.5	6.474×10^{-3}	3.914×10^{-4}	8.436×10^{-6}	4.926×10^{-8}	4.329×10^{-11}
0.6	7.144×10^{-3}	5.051×10^{-4}	1.412×10^{-5}	7.960×10^{-8}	5.526×10^{-11}
0.7	6.375×10^{-3}	2.645×10^{-4}	7.600×10^{-6}	1.641×10^{-7}	1.499×10^{-10}
0.8	2.307×10^{-3}	3.267×10^{-4}	8.255×10^{-6}	2.045×10^{-7}	2.693×10^{-10}
0.9	2.629×10^{-3}	1.901×10^{-4}	1.054×10^{-5}	4.069×10^{-9}	2.623×10^{-10}

Table 2: Absolute errors for *Example 1* by using sinc-Galerkin method for different values of N

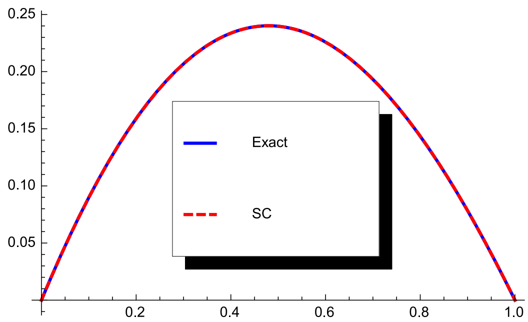
x	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
0.1	4.292×10^{-3}	6.072×10^{-4}	1.513×10^{-5}	1.321×10^{-7}	8.661×10^{-10}
0.2	3.582×10^{-3}	3.610×10^{-4}	2.868×10^{-5}	3.814×10^{-7}	8.919×10^{-10}
0.3	4.518×10^{-3}	3.692×10^{-4}	1.479×10^{-5}	4.035×10^{-7}	8.881×10^{-10}
0.4	2.644×10^{-3}	3.849×10^{-4}	1.962×10^{-5}	3.031×10^{-8}	8.497×10^{-10}
0.5	1.367×10^{-3}	1.823×10^{-4}	1.690×10^{-5}	3.697×10^{-7}	9.330×10^{-10}
0.6	1.701×10^{-3}	1.032×10^{-4}	3.800×10^{-6}	3.052×10^{-7}	9.113×10^{-10}
0.7	2.620×10^{-3}	1.803×10^{-6}	1.908×10^{-5}	2.509×10^{-8}	9.424×10^{-10}
0.8	1.632×10^{-3}	2.900×10^{-4}	6.602×10^{-6}	2.087×10^{-7}	6.082×10^{-11}
0.9	2.410×10^{-3}	8.289×10^{-5}	4.529×10^{-6}	3.354×10^{-7}	5.932×10^{-11}



(a) $N = 4$



(b) $N = 16$



(c) $N = 64$

Fig. 1: Graphs of exact and approximate solutions for Example 1

Example 2. Consider nonlinear fractional boundary value problem

$$y^{(1.3)}(x) + \frac{1}{x} y^{(0.3)}(x) + x\sin(y(x)) = f(x)$$

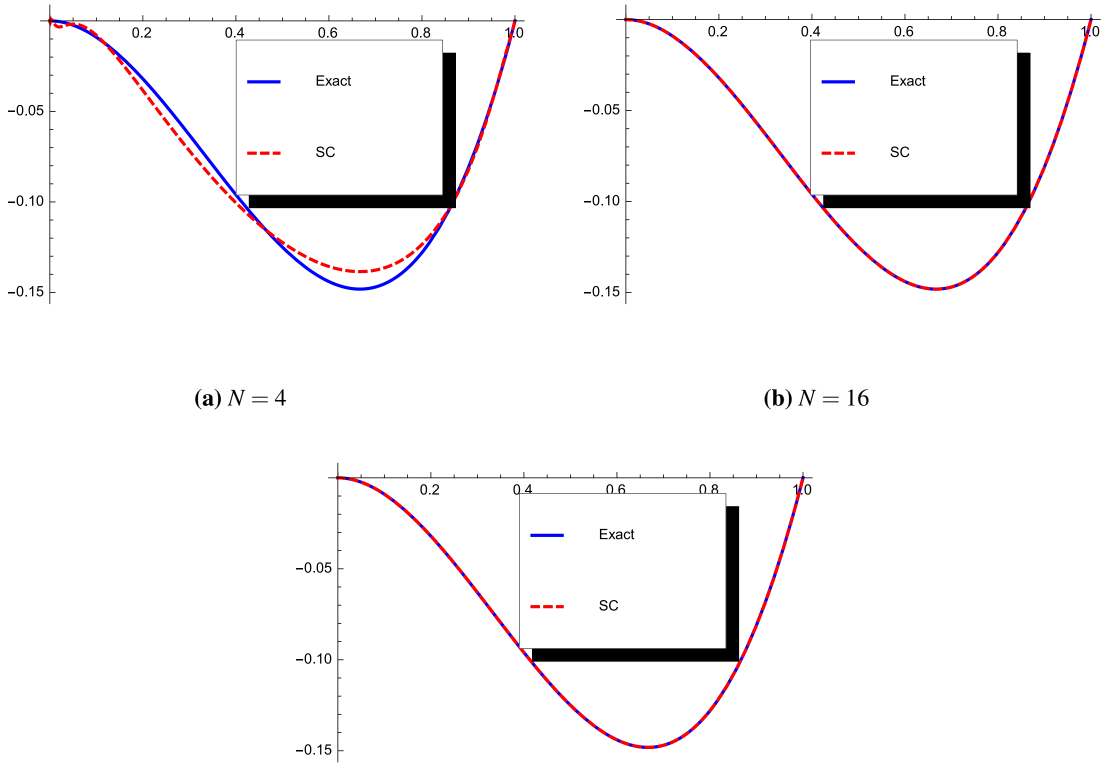
subject to the homogeneous boundary conditions

$$y(0) = 0, \qquad y(1) = 0$$

where $f(x) = -4x^{0.7} + 9x^{1.7} + x\sin[(x-1)x^2]$. The exact solution of this problem is $y(x) = x^2(x-1)$. The absolute errors which are obtained by using the present method for this problem are presented in Table 3. Additionally, the graphics of the exact and approximate solutions for different values of N are given in Figure 2.

Table 3: Absolute errors for *Example 2* for different values of N

x	$N=4$	$N=8$	$N=16$	$N=32$	$N=64$
0.1	1.066×10^{-3}	2.153×10^{-4}	1.890×10^{-5}	9.235×10^{-8}	3.607×10^{-10}
0.2	6.415×10^{-3}	3.374×10^{-4}	3.181×10^{-5}	3.614×10^{-7}	4.699×10^{-10}
0.3	8.735×10^{-3}	8.575×10^{-4}	4.085×10^{-6}	4.482×10^{-7}	1.982×10^{-10}
0.4	4.511×10^{-3}	8.466×10^{-4}	4.521×10^{-5}	3.655×10^{-7}	2.725×10^{-11}
0.5	2.602×10^{-3}	2.398×10^{-4}	7.278×10^{-6}	5.117×10^{-8}	4.255×10^{-11}
0.6	8.345×10^{-3}	1.108×10^{-3}	4.798×10^{-5}	3.388×10^{-7}	6.924×10^{-11}
0.7	9.326×10^{-3}	7.506×10^{-4}	8.622×10^{-6}	3.997×10^{-7}	2.631×10^{-10}
0.8	4.541×10^{-3}	3.689×10^{-4}	2.074×10^{-5}	3.639×10^{-7}	4.223×10^{-10}
0.9	1.340×10^{-3}	1.514×10^{-5}	1.562×10^{-5}	1.904×10^{-8}	3.013×10^{-10}



(a) $N=4$

(b) $N=16$

(c) $N=64$

Fig. 2: Graphs of exact and approximate solutions for Example 2

5 Conclusion

In this study, sinc-collocation method is used to obtain the approximate solutions of a class of fractional order boundary value problems. The presented method is applied to some examples in order to illustrate the accuracy of the method. Obtained numerical solutions are compared with exact solutions and the results are presented in tables and graphical forms. Regarding the results reported in tables and graphical forms, it can be concluded that sinc-collocation method is a very effective method for obtaining the approximate solution of FBVPs.

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