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Darboux transformation for soliton solutions of the modified Kadomtsev-Petviashvili-II equation

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Abstract: Soliton solutions as far as hyperbolic cosines to the modified Kadomtsev–Petviashvili II equation are displayed. The behaviour of each line soliton in the far region can be characterized analytically. It is revealed that under certain conditions, there may appear an isolated bump in the interaction centre, which is much higher in peak amplitude than the surrounding line solitons, and the more line solitons interact, the higher isolated bump will form. These results may provide a clue to generation of extreme high-amplitude waves, in a reservoir of small waves, based on nonlinear interactions between the involved waves.

Keywords: Darboux transformation, multisoliton solution, spectral problem, generalized Korteweg-de Vries equations, Boussinesq equation.

1 Introduction

Soliton is some kind of highly nonlinear wave-packet unusual yet ubiquitous in nature; it manifests itself in many physical settings such as the shallow- and deep water waves [1, 2], the light pulses in optical fibres [3, 4], the spatial and spatiotemporal localized structures in nonlinear optical media [5–8], the matter waves in Bose Einstein condensates [9–12], and even the domain walls in supergravity [13]. It exists by a delicate balance between the dispersion (or diffraction) that tends to expand the wave-packet and the nonlinear effect that tends to localize it (in dissipative contexts, it requires also an extra balance between gain and loss [14]). In a strict sense, the soliton concept is a sophisticated mathematical construct associated to the integrability of a class of nonlinear differential equations [15] and typically obtained by means of the inverse scattering transform [16]. The soliton theory of integrable equations is a broad and very active field of mathematical research [17, 18].

The main objective of this paper is to solve the modified Kadomtsev-Petviashvili-II equation using Darboux transformation. This paper is organized as follow:

(1) Section 2 : Historical Background.
(2) Section 3 : Darboux transformation for modified Kadomtsev-Petviashvili-II equation.
(3) Section 4: Comparison.

2 Historical background

This equation describes water waves in (x,y)-plane when the nonlinearity is higher than for the KP equation. It has been introduced in various forms. One of its forms [19, 20] is;

$$4v_t + v_{xxx} - 6v^2 v_x + 6v_x(\partial_x^{-1} v_y) + 3(\partial_x^{-1} v_{yy}) = 0 \tag{1}$$

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where $(\partial_x^{-1} f)(x) = \int_{-\infty}^{x} f(t)dt$ Lax pairs of modified KP equation [20] have the form;

$$\psi_{xx} = \psi_y - 2v\psi_x \tag{2}$$

$$4\psi_t = -4\psi_{xxx} - 12v\psi_{xx} - 6v_x\psi_x - 6(\partial_x^{-1}v_y)\psi_x - 6v^2\psi_x. \tag{3}$$

For simplification assume; $v = u_x$, So eq. (1) rewrites as;

$$4u_{xt} + u_{xxxx} - 6{u_x}^2 u_{xx} + 6u_{xx}\, u_y + 3u_{yy} = 0 \tag{4}$$

and its Lax pair eqs. (2), (3) rewrite as;

$$\psi_{xx} = \psi_y - 2u_x\psi_x \tag{5}$$

$$4\psi_t = -4\psi_{xxx} - 12u_x\psi_{xx} - 6u_{xx}\psi_x - 6u_y\psi_x - 6{u_x}^2\psi_x. \tag{6}$$

3 One soliton solution for the modified Kadomtsev-Petviashvili-II (mKPII) equation

Consider the first DT with the form,

$$\psi[1] = \psi_x - \sigma\psi\ , \sigma = \frac{\psi_1'}{\psi_1},\ \psi_1' = \frac{d\psi_1}{dx} \tag{7}$$

and ψ , $\psi_1$are two solutions of the Lax pair equations. $\psi[1]$ satisfies eq. (5);

$$\psi_{xx}[1] = \psi_y[1] - 2u_x[1]\psi_x[1] \tag{8}$$

Differentiate $\psi[1]$ w.r.t *(x)* and *(y)* once separately;

$$\psi_y[1] = \psi_{xy} - \sigma_y\psi - \sigma\psi_y \tag{9}$$

$$\psi_x[1] = \psi_y - 2u_x\psi_x - \sigma_x\psi - \sigma\psi_x \tag{10}$$

Substituting from eq. (9), eq. (10) in eq. (8) we obtain;

$$\psi_{xx}[1] = \psi_{xy} + (-\sigma_y + 2u_x[1]\sigma_x)\psi + (4u_x + 2\sigma)u_x[1]\psi_x + (-\sigma - 2u_x[1])\psi_y \tag{11}$$

Differentiating eq. (7) w.r.t *(x)* twice;

$$\psi_{xx}[1] = \psi_{xxx} - \sigma_{xx}\psi - 2\sigma_x\psi_x - \sigma\psi_{\ xx} \tag{12}$$

Differentiate first Lax pair eq. (5) w.r.t (x) once;

$$\psi_{xxx} = \psi_{xy} - 2u_{xx}\psi_x - 2u_x\psi_{xx} \tag{13}$$

Substitute from eq. (13) and eq. (5) in (12);

$$\psi_{xx}[1] = \psi_{xy} - 2u_{xx}\psi_x - 2u_x(\psi_y - 2u_x\psi_x) - \sigma_{xx}\psi - 2\sigma_x\psi_x - \sigma(\psi_y - 2u_x\psi_x)$$

Rearranging ψ coefficients yields;

$$\psi_{xx}[1] = \psi_{xy} - \sigma\psi_y + \left(-2u_{xx} + 4{u_x}^2 - 2\sigma_x + 2u_x\sigma\right)\psi_x - \sigma_{xx}\psi \tag{14}$$

Comparing coefficients of ψ_x in eq. (11) and eq. (14) gives;

$$(4u_x + 2\sigma)\,u_x[1] = -2u_{xx} + 4{u_x}^2 - 2\sigma_x + 2u_x\sigma$$

So;

$$u_x[1] = u_x[0] - \frac{u_{xx}[0] + \sigma_x}{\sigma + 2u_x[0]} \tag{15}$$

as $v = u_x$, this equation reduces to;

$$v[1] = v[0] - \frac{v_x[0] + \sigma_x}{\sigma + 2v[0]} \tag{16}$$

These are mKP equation recurrent solutions. They are in the following sections tested for several seeds solutions.

3.1 First initial (seed) solution

Assuming an initial wave;

$$\begin{aligned} u[0] &= 0 \\ v[0] &= 0 \end{aligned} \tag{17}$$

Hence Lax pair equations (5), (6) reduced to;

$$\psi_{xx} = \psi_y \tag{18}$$

$$4\psi_t = -4\psi_{xxx} \tag{19}$$

For $\xi = x + y - t$ the solution of this system of equation is;

$$\psi(x,y,t) = e^{(x+y-t)} + 1 \tag{20}$$

We then evaluate

$$\sigma = \frac{e^{(x+y-t)}}{e^{(x+y-t)} + 1} \tag{21}$$

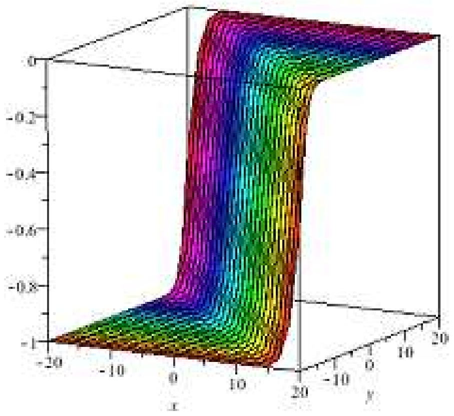
Substitute from eq. (21) in eq. (15) and eq. (16);

$$\mathrm{u}[1] = -\ln(\sigma) = -\ln\left(\frac{e^{(x+y-t)}}{e^{(x+y-t)} + 1}\right) \tag{22}$$

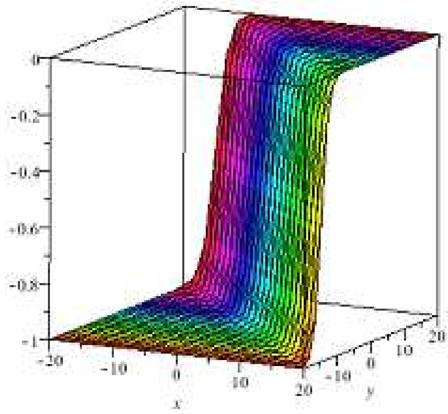
which is a new solution of the equation (4). and;

$$v[1] = \frac{-\sigma_x}{\sigma} = -\frac{1}{e^{(x+y-t)} + 1} \tag{23}$$

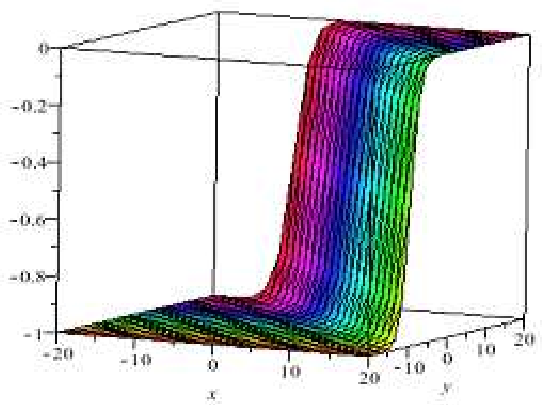
This is a new solution of the mKP-II equation (1) illustrated in Fig.5.1 at different times.



(a). v[1] for v_0=0,u_0=0, t=0.



(b). v[1] for v_0=0,u_0=0, t=5.



(c). v[1] for v_0=0,u_0=0, t=10.

Fig. 1: Solution of the mKP-II equation (1) with initial seed solution v[0]=0, u[0]=0.

3.2 Second seed solution

Assuming an initial wave;

$$\begin{aligned} u[0] &= 0.5x \\ v[0] &= 0.5 \end{aligned} \tag{24}$$

Lax pair equations (5), (6) reduced to;

$$\psi_{xx} = \psi_y - \psi_x \tag{25}$$

$$4\psi_t = -4\psi_{xxx} - 6\psi_{xx} - \frac{3}{2}\psi_x \tag{26}$$

Solve two equations together;

$$\psi(x,y,t) = e^{0.5(\lambda-1)x+0.25\left(\lambda^2-1\right)y-\frac{1}{16}\left(2\lambda^3-3\lambda+1\right)t} + e^{-0.5(\lambda+1)x+0.25\left(\lambda^2-1\right)y+\frac{1}{16}\left(2\lambda^3-3\lambda-1\right)t} \tag{27}$$

So;

$$\sigma = 0.5\,\lambda\tanh\left(0.5\lambda x - \frac{1}{16}\left(2\lambda^3-3\lambda\right)t\right) - 0.5 \tag{28}$$

Substituting from eq. (28) in eq. (15) and eq. (16) we obtain;

$$u[1] = 0.5x - \ln(-0.5\lambda\tanh\left(-0.5\lambda x + \frac{1}{16}\left(2\lambda^3-3\lambda\right)t\right) + 0.5) \tag{29}$$

$$v[1] = 0.5 - 0.25\frac{\lambda^2(1-\tanh(-0.5\lambda x+\frac{1}{16}\left(2\lambda^3-3\lambda\right)t)^2)}{-0.5\lambda\tanh\left(-0.5\lambda x+\frac{1}{16}\left(2\lambda^3-3\lambda\right)t\right)+0.5} \tag{30}$$

This is a Darboux solution of the mKP-II equation (1) depicted in Fig.5.2 at different times. We notice that the wave moves slowly to negative values of x.

3.3 *Third seed solution*

Assuming an initial seed wave;

$$\begin{aligned} u[0] &= x \\ v[0] &= 1 \end{aligned} \tag{31}$$

Lax pair equations (5), (6) reduced to;

$$\psi_{xx} = \psi_y - 2\psi_x \tag{32}$$

$$4\psi_t = -4\psi_{xxx} - 12\psi_{xx} - 6\psi_x \tag{33}$$

Solving these equations together, we get;

$$\psi_1(x,y,t) = e^{(\lambda-1)x+\left(\lambda^2-1\right)y-\left(\lambda^3-\frac{3}{2}\lambda+0.5\right)t} + e^{-(\lambda+1)x+\left(\lambda^2-1\right)y+\left(\lambda^3-\frac{3}{2}\lambda+0.5\right)t} \tag{34}$$

Evaluating $\sigma = \frac{\psi_1'}{\psi_1}$

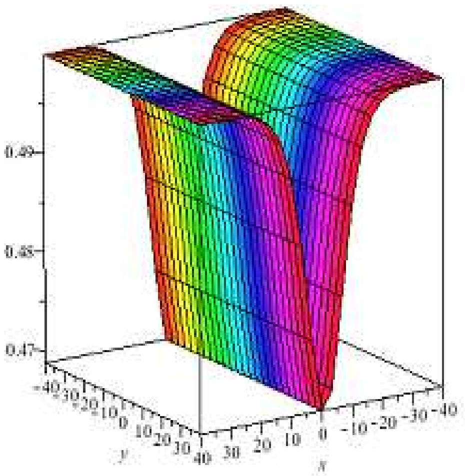
$$\sigma = \lambda\tanh\left(\lambda x - \left(\lambda^3-\frac{3}{2}\lambda\right)t\right) - 1 \tag{35}$$

Substitute from eq. (35) in eq. (15) and eq. (16);

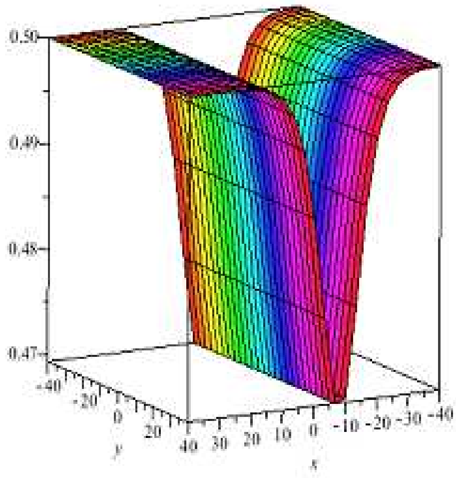
$$u[1] = x - \ln\left(1-\lambda\tanh(-\lambda x+\left(\lambda^3-\frac{3}{2}\lambda\right)t)\right) \tag{36}$$

$$v[1] = 1 - \frac{\lambda^2(1-\tanh(-\lambda x+\left(\lambda^3-\frac{3}{2}\lambda\right)t)^2)}{-\lambda\tanh\left(-\lambda x+\left(\lambda^3-\frac{3}{2}\lambda\right)t\right)+1} \tag{37}$$

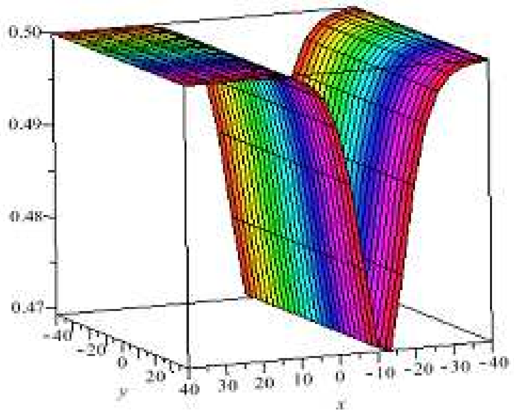
This is a new solution of the mKP-II equation (1) depicted in Fig 5.3 at different times. We notice in this figure that the wave pic moves quicker on the negative x axis that in the previous case with seed wave v[0]=0.5, u[0]=*0.5x*.



(a). v[1] for v_0=0.5, u_0 =0.5x,$\lambda = 0.25$, t=0.



(b). v[1] for v_0=0.5, u_0 =0.5x ,$\lambda = 0.25$, t=20.



(c). v[1] for v_0=0.5, u_0 =0.5x, $\lambda = 0.25$, t=30.

Fig. 2: Solution of the mKP-II equations (1) for v[1] for v_0=0.5, u_0 =0.5x, $\lambda = 0.25$.

3.4 Comparison

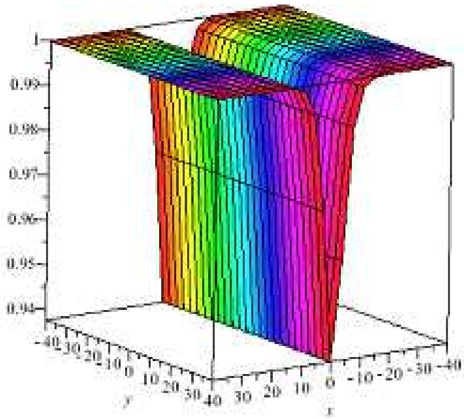
Wazwaz [19] did solve eq. (1) using Hirota-bilinear method. He assumed a solution in the form;

$$v(x,y,t) = \frac{e^{\left(k_1x+k_1^2y-\left(k_1^3+\alpha k_1^2+\beta k_1\right)t\right)}}{1+e^{\left(k_1x+k_1^2y-\left(k_1^3+\alpha k_1^2+\beta k_1\right)t\right)}} \tag{38}$$

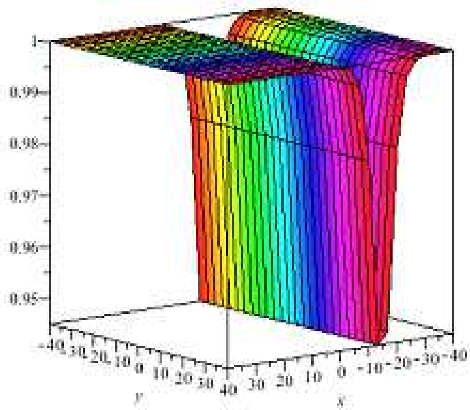
Setting $k_1 = -1, \alpha\ , \beta = 0$ in this equation yields;

$$v(x,y,t) = -\frac{e^{(-x+y+t)}}{1+e^{(-x+y+t)}} = \frac{-1}{1+e^{(x-y-t)}} \tag{39}$$

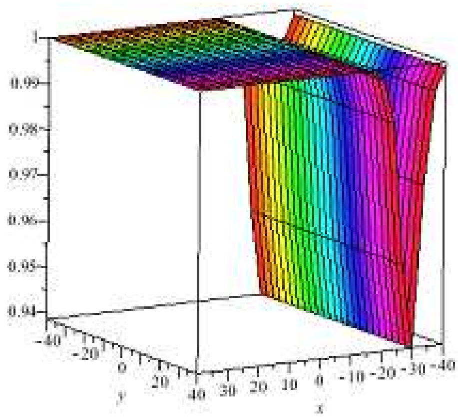
A plot of the solution eq. (38) in Fig. 5.4 together with our solution in eq. (23) shows a similarity of form.



(a). v[1] for v_0=1, u_0=x, $\lambda = 0.25$, t=0.



(b). v[1] for v_0=1, u_0=x, $\lambda = 0.25$, t=10.



(c). v[1] for v_0=1, u_0=x, $\lambda = 0.25$, t=20.

Fig. 3: Solution of the mKP-II equation (1) with initial solution v[0]=1, u[0]= x, $\lambda = 0.25$.

4 Conclusion

In conclusion, we presented soliton solutions to the KP-II equation, using Darboux transformation. We showed that these resonant soliton solutions could exhibit rich intriguing interaction patterns on a finite background, some of which may change drastically with the evolution time, some may not, behaving like a spatiotemporal bullet propagating along a certain direction. Despite all this, both the amplitudes and propagation directions of these resonant line solitons occurring in the interaction centre and in the asymptotic far region can be well characterized analytically. We unveiled further that under certain conditions, there may appear an isolated bump in the interaction centre, which is much higher in peak amplitude than the surrounding line solitons, and the more line solitons interact, the higher isolated bump will form. The results reported in this work may provide a clue to forming extreme high-amplitude waves (e.g., rogue waves), in a reservoir of small waves, based on nonlinear wave interactions.

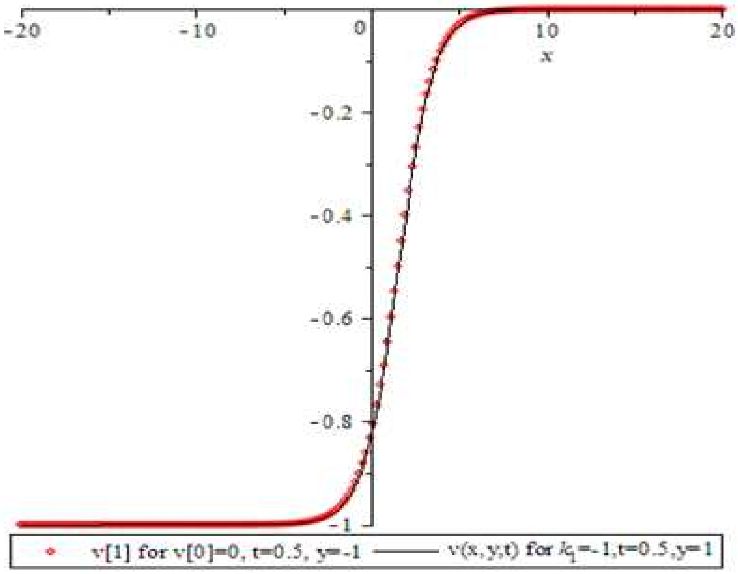


Fig. 4: Comparison of solution of mKP-II equation by using Hirota-bilinear method with our result equation (23) using DT.

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