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Lucas collocation method to determination spherical curves in euclidean 3-space

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Abstract: In this study, we give a necassary and sufficient condition for an arbitrary-speed regular space curve to lie on a sphere centered at origin. Also, we obtain the position vector of any regular arbitrary-speed space curve lying on a sphere centered at origin satisfies a third-order linear differential equation whose coefficients is related to speed function, curvature and torsion. Then, a collocation method based on Lucas polynomials is developed for the approximate solutions of this differential equation. Moreover, by means of the Lucas collacation method, we approximately obtain the parametric equation of the spherical curve by using this differential equation. Furthermore, an example is given to demonstrate the efficiency of the method and the results are compared with figures and tables.

Keywords: Spherical curves, Frenet frame, Lucas polynomial and series, collocation points, differential equation.

1 Introduction

In books on differential geometry, for a curve necessary condition to lie in a sphere has been given and the spherical curves have been investigated many mathematicians. For example, in [1], Wong gave a necessary and sufficient condition for a curve to lie in a sphere without having to assume that its torsion is nowhere zero. Then, Breuer and Gottlieb showed that the differential equation characterizing a spherical curve could be solved explicitly to express the radius of curvature of the curve in terms of its torsion [2]. After then, Wong proved that the explicit characterization of spherical curves obtained by Breuer and Gottlieb is, without any precondition on the curvature and torsion, a necessary and sufficient condition for a curve to be a spherical curve [3]. Özdamar and Hacısalihoğlu, gave the characterizations for the regular curves each of lies on the S^{n-1} of n-dimensional Euclidean Space E^n and they expressed these characterizations in the higher curvatures of the curves [4]. Mehlum and Wimp gaved that the position vector of any 3-space curve lying on a sphere satisfies a third-order linear differential equation whose coefficients involve a single arbitrary function $A(s)$ [5].

For the solution of linear integro-differential equations, a Taylor collocation method was given by Karamete and Sezer [6]. Then, Taylor collocation method was given by Sezer et al. to find the approximate solutions of high-order systems of linear differential equations with variable coefficients [7]. Also, Yüzbaşı and Sezer, presented an exponential matrix method for the solutions of systems of high-order linear differential equations with variable coefficients [8]. Furthermore, in [9] Çetin et al. developed an approximation method based on Lucas polynomials for the solution of the system of high-order linear differential equations with variable coefficients under the mixed conditions. Also in [10] Çetin et al. a collocation method based on Lucas polynomials for solving high-order linear differential equations with variable coefficients under the boundary conditions is presented by transforming the problem into a system of linear algebraic equations with Lucas coefficients.

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In this paper, we developed a Lucas collocation method to find the approximate solutions of third-order linear differential equations with variable coefficients. Then, we obtain that the position vector of any arbitrary-speed regular space curve lying on a sphere centered at **0** in Euclidean 3-Space satisfies a third-order linear differential equation. Furthermore, by means of this method we obtain a Lucas polynomial solution of differential equations characterizing the position vector of any regular arbitrary-speed space curve lying on a sphere centered at **0** according to Frenet frame in Euclidean 3-space E^3. Then we give an example and compare the results to demonstrate the efficiency of the method.

2 Preliminaries

In this section, we give some basic concepts on differential geometry of space curves and spherical curves in Euclidean 3-Space.

A parametrized differentiable curve is a differentiable map $\alpha : I \to \mathrm{R}^3$ of an open interval $I = (a,b)$ of the real line R into R^3. A parametrized differentiable curve $\alpha : I \to \mathrm{R}^3$ is said to be *regular* if $\alpha'(t) \neq 0$ for all $t \in I$ [11].

The velocity vector of a regular curve $\alpha(t)$ at $t = t_0$ is the derivative $\frac{d\alpha}{dt}$ evaluated at $t = t_0$. The velocity vector field is the vector valued function $\frac{d\alpha}{dt}$. The speed of $\alpha(t)$ at $t = t_0$ is the lenght of the velocity vector at $t = t_0$, $|\alpha'(t_0)|$ [12].

The unit *tangent* vector, the unit *binormal* vector and the unit *principal normal* vector of the regular curve α are given by

$$T(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|}, B(t) = \frac{\alpha'(t) \wedge \alpha''(t)}{\|\alpha'(t) \wedge \alpha''(t)\|}, N(t) = B(t) \wedge T(t)$$

respectively [13] .

If α is a regular curve in R^3 with $\kappa > 0$, then the Frenet formulaes

$$\begin{aligned} T' &= v\kappa N \\ N' &= -v\kappa T + v\tau B \\ B' &= -v\tau N \end{aligned}$$

where κ is the curvature of the curve α, τ is the torsion of the curve α and $v = \|\alpha'\|$ is the speed function of the curve α, respectively [13].

For a general parameter t of a space curve α, the curvature and the torsion of the curve are given by

$$\kappa(t) = \frac{\|\alpha'(t) \wedge \alpha''(t)\|}{\|\alpha'(t)\|^3}, \tau(t) = \frac{\det(\alpha'(t), \alpha''(t), \alpha'''(t))}{\|\alpha'(t) \wedge \alpha''(t)\|^2}.$$

where $\det(\alpha'(t), \alpha''(t), \alpha'''(t)) = \langle \alpha'(t) \wedge \alpha''(t), \alpha'''(t) \rangle$ [13].

A *Clelies* is a curve lying on a sphere centered at origin parametrized by

$$\alpha(s) = (r\sin(mt)\cos(t), r\sin(mt)\sin(t), r\cos(mt))$$

where r is radius of the sphere [14].

3 Differential equation characterizing spherical curves

In this section, we give a necassary and sufficient condition for an arbitrary-speed regular space curve to lie on a sphere centered at origin. Then, we obtain that position vector of any arbitrary-speed regular space curve lying on a sphere satisfies a third-order linear differential equation with variable coefficients.

Theorem 1. *Let α be a Frenet frame curve of class C^4 in R^3 with $\tau \neq 0$ everywhere. Then α lies on a sphere if and only if the following equation holds.*

$$\frac{\tau}{\kappa} = \left(\frac{\kappa'}{\tau\kappa^2}\right)'$$

[15].

Now, we obtain the necessary and sufficient condition for an arbitrary-speed regular space curve α to lie on a sphere centered at **0** in Euclidean 3-Space E^3, parametrized by s. Denote by $\{T, N, B\}, \kappa, \tau$ and v the moving Frenet frame, curvature, torsion and speed function along the curve α, respectively. Then, we can write

$$\alpha(s) = \lambda_1(s)T(s) + \lambda_2(s)N(s) + \lambda_3(s)B(s) \tag{1}$$

Differentiating (1) with respect to s and by using the Serret-Frenet formulas for non-unit speed regular curve, we get

$$vT = \left(\lambda_1' - v\kappa\lambda_2\right)T + \left(v\kappa\lambda_1 + \lambda_2' - v\tau\lambda_3\right)N + \left(v\tau\lambda_2 + \lambda_3'\right)B \tag{2}$$

From (2) we can write

$$v = \lambda_1' - v\kappa\lambda_2 \tag{3}$$

$$0 = v\kappa\lambda_1 + \lambda_2' - v\tau\lambda_3 \tag{4}$$

$$0 = v\tau\lambda_2 + \lambda_3' \tag{5}$$

The condition that α lie on a sphere centered at the origin is

$$\langle \alpha, \alpha' \rangle = 0. \tag{6}$$

Substituting (1) in (6), we obtain $\lambda_1 = 0$. and hence, from (3) and (4), we obtain

$$\lambda_2 = \frac{-1}{\kappa} \quad \textit{and} \quad \lambda_3 = \frac{\kappa'}{v\kappa^2\tau} \tag{7}$$

Substituting (7) into the (5), we obtain the necessary and sufficient condition for any arbitrary-speed regular space curve α to lie on a sphere centered at **0** in Euclidean 3-Space E^3, as follows

$$\frac{v\tau}{\kappa} - \left(\frac{\kappa'}{v\tau\kappa^2}\right)' = 0.$$

We can express T, N and Bin terms of $\alpha', \alpha'', \alpha'''$ and κ, τ, v via the Serret-Frenet formulae as follows.

$$T = \frac{1}{v}\alpha' \tag{8}$$

$$N = \frac{1}{v^2\kappa}\alpha'' - \frac{v'}{v^3\kappa}\alpha' \tag{9}$$

$$B = \frac{1}{v^3\kappa\tau}\left\{\alpha''' - \left(\frac{3v'}{v} + \frac{\kappa'}{\kappa}\right)\alpha'' + \left(\frac{3(v')^2}{v^2} + \frac{v'\kappa'}{v\kappa} + v^2\kappa^2 - \frac{v''}{v}\right)\alpha'\right\}. \tag{10}$$

Substituting (7), (8), (9) and (10) in (1), we get

$$p_3(s)\alpha'''(s) + p_2(s)\alpha''(s) + p_1(s)\alpha'(s) + p_0(s)\alpha(s) = 0 \tag{11}$$

where

$$p_3(s) = v^2\kappa\kappa', p_2(s) = -v^4\kappa^2\tau^2 - 3vv'\kappa\kappa' - v^2(\kappa')^2$$

$$p_1(s) = v^3v'\kappa^2\tau^2 - vv''\kappa\kappa' + v^4\kappa^3\kappa' + 3(v')^2\kappa\kappa' + vv'(\kappa')^2, p_0(s) = -v^6\kappa^4\tau^2.$$

This differential equation is third-order with respect to position vector of α characterizing spherical curves according to Frenet frame in Euclidean 3-Space E^3.

4 Lucas collocation method for third-order linear differential equation with variable coefficients

Sezer et al. gave a Taylor collocation method to find the approximate solutions of high-order systems of linear differential equations with variable coefficients in [7]. Then Çetin et al. presented an approximation method based on Lucas polynomials for the solution of the system of high-order linear differential equations with variable coefficients under the mixed conditions [9]. Moreover, Çetin et al. gave a collocation method based on Lucas polynomials for solving high-order linear differential equations with variable coefficients under the boundary conditions [10].

In this section we have developed Lucas collocation method to solve the third-order linear differential equations with variable coefficients in the form

$$L[y(x)] = \sum_{k=0}^{3} p_k(x)y^{(k)}(x) = g(x), \quad 0 \le a \le x \le b \tag{12}$$

under the mixed conditions

$$\sum_{k=0}^{2}\left(a_{jk}y^{(k)}(a) + b_{jk}y^{(k)}(b)\right) = c_j, \quad j = 0,1,2 \tag{13}$$

where $y^{(0)}(x) = y(x)$ is an unknown function, $p_k(x)$ and $g(x)$ are known continuous functions defined on interval $[a,b]$, and coefficients a_{jk}, b_{jk} and c_j are real constants.

In addition, by improving the present method with the help of residual error function used in [16-19], we obtain the corrected approximate solution of Eq.(12) expressed in the truncated Lucas series form

$$y_{N,M}(x) = y_N(x) + e_{N,M}(x) \tag{14}$$

where

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_nL_n(x) \tag{15}$$

is the Lucas polynomial solution and

$$e_{N,M}(x)=\sum_{n=0}^{M}a_n^*L_n(x),\quad M>N \tag{16}$$

is the solution of the error problem obtained with help of the residual error function. Here a_n and a_n^*, $(n=0,1,2,\dots,N)$ are unknown Lucas coefficients, and $L_n(x)$, $(n=0,1,2,\dots,N)$ are Lucas polynomials defined by

$$L_0(x)=2;L_n(x)=\sum_{k=0}^{[\![n/2]\!]}\frac{n}{n-k}\binom{n-k}{k}x^{n-2k},\ (n\geq 1)\ [[n/2]]=\begin{cases} n/2 & ,\ n \text{ even}\\ (n-1)/2 & ,\ n \text{ odd}\end{cases}$$

[20, 21].

In order to find solution of Eq.(12), with mixed conditions (13), we can use collocation points defined by

$$x_i=a+\frac{b-a}{N}i,\quad i=0,1,\dots,N,\quad 0\leq a\leq x\leq b. \tag{17}$$

The Lucas polynomials $L_n(x)$ can be written in the matrix form as

$$\mathrm{L}(x)=\mathrm{X}(x)\mathrm{D}^T \tag{18}$$

where

$$\mathrm{L}(x)=\left[L_0(x)\ L_1(x)\ L_2(x)\ \cdots\ L_N(x)\right],\mathrm{X}(x)=\left[1\ x\ x^2\ \cdots\ x^N\right]$$

and if N is odd,

$$\mathrm{D}=\begin{bmatrix}
2 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & \frac{1}{1}\binom{1}{0} & 0 & 0 & 0 & \cdots & 0\\
\frac{2}{1}\binom{1}{1} & 0 & \frac{2}{2}\binom{2}{0} & 0 & 0 & \cdots & 0\\
0 & \frac{3}{2}\binom{2}{1} & 0 & \frac{3}{3}\binom{3}{0} & 0 & \cdots & 0\\
\frac{4}{2}\binom{2}{2} & 0 & \frac{4}{3}\binom{3}{1} & 0 & \frac{4}{4}\binom{4}{0} & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\frac{n-1}{(n-1)/2}\binom{(n-1)/2}{(n-1)/2} & 0 & \frac{n-1}{(n+1)/2}\binom{(n+1)/2}{(n-3)/2} & 0 & \frac{n-1}{(n+3)/2}\binom{(n+3)/2}{(n-5)/2} & & 0\\
0 & \frac{n}{(n+1)/2}\binom{(n+1)/2}{(n-1)/2} & 0 & \frac{n}{(n+3)/2}\binom{(n+3)/2}{(n-3)/2} & 0 & \cdots & \frac{n}{n}\binom{n}{0}
\end{bmatrix}.$$

If N is even,

$$
\mathrm{D}=\begin{bmatrix}
2 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & \frac{1}{1}\binom{1}{0} & 0 & 0 & 0 & \cdots & 0\\
\frac{2}{1}\binom{1}{1} & 0 & \frac{2}{2}\binom{2}{0} & 0 & 0 & \cdots & 0\\
0 & \frac{3}{2}\binom{2}{1} & 0 & \frac{3}{3}\binom{3}{0} & 0 & \cdots & 0\\
\frac{4}{2}\binom{2}{2} & 0 & \frac{4}{3}\binom{3}{1} & 0 & \frac{4}{4}\binom{4}{0} & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & \frac{n-1}{n/2}\binom{n/2}{(n-2)/2} & 0 & \frac{n-1}{(n+2)/2}\binom{(n+2)/2}{(n-4/2} & 0 & \cdots & 0\\
\frac{n}{n/2}\binom{n/2}{n/2} & 0 & \frac{n}{(n+2)/2}\binom{(n+2)/2}{(n-2)/2} & 0 & \frac{n}{(n+4)/2}\binom{(n+4)/2}{(n-4)/2} & \cdots & \frac{n}{n}\binom{n}{0}
\end{bmatrix}.
$$

We can write the approximate solution $y_N(x)$ given by (15) in the matrix form

$$
y_N(x)=\mathrm{L}(x)\mathrm{A} \tag{19}
$$

where

$$
\mathrm{A}=\begin{bmatrix} a_0 \; a_1 \; a_2 \; \cdots \; a_N \end{bmatrix}^T .
$$

From (18) and (19), we obtain the matrix relation

$$
y_N(x)=\mathrm{X}(x)\,\mathrm{D}^T\mathrm{A}. \tag{20}
$$

Also, the relation between the matrix $\mathrm{X}(x)$ and its derivatives $\mathrm{X}^{(k)}(x)$ is

$$
\mathrm{X}^{(k)}(x)=\mathrm{X}(x)\,\mathrm{B}^k \tag{21}
$$

where

$$
\mathrm{B}=\begin{bmatrix}
0 & 1 & 0 & \cdots & 0\\
0 & 0 & 2 & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & \cdots & N\\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}.
$$

By using relations (20) and (21), we obtain the following relation.

$$
y_N^{(k)}(x)=\mathrm{X}(x)\,\mathrm{B}^k\mathrm{D}^T\mathrm{A}. \tag{22}
$$

By substituting (18) and (22) into Eq.(12), we obtain the matrix equation

$$
\sum_{k=0}^{3} p_k(x)\mathrm{X}(x)\,\mathrm{B}^k\mathrm{D}^T\mathrm{A}=g(x) \tag{23}
$$

and by using collocation points (17) into Eq.(23), the system of matrix equations can be obtained as

$$\sum_{k=0}^{3} p_k(x_i)\mathrm{X}(x_i)\mathrm{B}^k\mathrm{D}^T\mathrm{A} = g(x_i)$$

or the compact form

$$\left\{\sum_{k=0}^{3} \mathrm{P}_k\mathrm{X}\mathrm{B}^k\mathrm{D}^T\right\}\mathrm{A} = \mathrm{G} \tag{24}$$

where

$$\mathrm{P}_k = \begin{bmatrix} p_k(x_0) & 0 & \cdots & 0 \\ 0 & p_k(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_k(x_N) \end{bmatrix}, \mathrm{X} = \begin{bmatrix} \mathrm{X}(x_0) \\ \mathrm{X}(x_1) \\ \vdots \\ \mathrm{X}(x_N) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^N \\ 1 & x_1 & x_1^2 & \cdots & x_1^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^N \end{bmatrix}, \mathrm{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}.$$

Thus, the fundamental matrix equation (24) corresponding to Eq.(12) can be written in the form

$$\mathrm{WA} = \mathrm{G} \;\; or \;\; [\mathrm{W};\mathrm{G}], \mathrm{W} = \sum_{k=0}^{3} \mathrm{P}_k\mathrm{X}\mathrm{B}^k\mathrm{D}^T \tag{25}$$

Eq.(25) indicates a system of $(N+1)$ linear algebraic equations with unknown Lucas coefficients a_n $(n = 0,1,...,N)$. Now, by means of Eq.(22), we obtain the matrix forms for the conditions (13) as follows

$$\sum_{k=0}^{2} \left[a_{jk}\mathrm{X}(a) + b_{jk}\mathrm{X}(b)\right]\mathrm{B}^k\mathrm{D}^T\mathrm{A} = [c_j], \quad (j = 0,1,2)$$

or briefly,

$$\mathrm{U}_j\mathrm{A} = [c_j] \;\; or \;\; [\mathrm{U}_j; c_j], (j = 0,1,2) \tag{26}$$

where

$$\mathrm{U}_j = \sum_{k=0}^{2} \left[a_{jk}\mathrm{X}(a) + b_{jk}\mathrm{X}(b)\right]\mathrm{B}^k\mathrm{D}^T = \left[u_{j0} \; u_{j1} \; u_{j2} \; \cdots \; u_{jN}\right], \quad (j = 0,1,2).$$

Consequently, by replacing the row matrices (26) by last rows of the matrix (25), we have

$$\left[\widetilde{\mathrm{W}};\widetilde{\mathrm{G}}\right] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & g(x_0) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & g(x_1) \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ w_{(N-3)0} & w_{(N-3)1} & \cdots & w_{(N-3)N} & ; & g(x_{N-3}) \\ u_{00} & u_{00} & \cdots & u_{0N} & ; & c_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & c_1 \\ u_{20} & u_{21} & \cdots & u_{2N} & ; & c_2 \end{bmatrix},$$

which is a linear algebraic system. If rank$\widetilde{\mathrm{W}}$ =rank$\left[\widetilde{\mathrm{W}};\widetilde{\mathrm{G}}\right] = N+1$, then we can write$\mathrm{A} = \left(\widetilde{\mathrm{W}}\right)^{-1}\widetilde{\mathrm{G}}$. Hence, the unknown Lucas coefficients matrix $\mathrm{A} = \left[a_0 \; a_1 \; a_2 \; \cdots \; a_N\right]^T$ is determined and by substituting the coefficients $a_0, a_1, a_2, ..., a_n$ into Eq.(15), the Lucas polynomial solution of the differential equation is obtained.

Now, we give an error estimation for the Lucas polynomial solution (15) with the residual error function [16-19]. Moreover, we improve the solution (15) by means of the residual error function. Firstly, we can define the residual function of the method as

$$R_N(x) = L[y_N(x)] - g(x). \tag{27}$$

Here, $y_N(x)$ is the Lucas polynomial solution given by (15) of the problem (12) and (13). Hence, $y_N(x)$ satisfies the problem

$$\begin{cases} L[y_N(x)] = \sum_{k=0}^{3} p_k(x) y_N^{(k)}(x) = g(x) + R_N(x) \\ \sum_{k=0}^{2} \left(a_{jk} y_N^{(k)}(a) + b_{jk} y_N^{(k)}(b) \right) = c_j, \quad j = 0,1,2. \end{cases}$$

Also, the error function $e_N(x)$ can be defined as

$$e_N(x) = y(x) - y_N(x) \tag{28}$$

where $y(x)$ is the exact solution of the problem (12) and (13). From Eqs.(12), (13), (27) and (28), we gain the error differential equation

$$L[e_N(x)] = L[y(x)] - L[y_N(x)] = -R_N(x)$$

with the homogeneous mixed conditions

$$\sum_{k=0}^{2} \left(a_{jk} e_N^{(k)}(a) + b_{jk} e_N^{(k)}(b) \right) = 0, \ j = 0,1,2$$

or openly, the error problem

$$\begin{cases} \sum_{k=0}^{3} p_k(x) e_N^{(k)}(x) = -R_N(x) \\ \sum_{k=0}^{2} \left(a_{jk} e_N^{(k)}(a) + b_{jk} e_N^{(k)}(b) \right) = 0, \quad j = 0,1,2. \end{cases} \tag{29}$$

Here, note that the nonhomegeneous mixed conditions

$$\sum_{k=0}^{2} \left(a_{jk} y^{(k)}(a) + b_{jk} y^{(k)}(b) \right) = c_j, \ j = 0,1,2$$

and

$$\sum_{k=0}^{2} \left(a_{jk} y_N^{(k)}(a) + b_{jk} y_N^{(k)}(b) \right) = c_j, \ j = 0,1,2$$

are reduced to homogeneous mixed conditions

$$\sum_{k=0}^{2} \left(a_{jk} e_N^{(k)}(a) + b_{jk} e_N^{(k)}(b) \right) = 0, \ j = 0,1,2.$$

The error problem (29) can be solved by using the prosedure given above. Thus, we obtain the approximation $e_{N,M}(x)$ to $e_N(x)$ as follows

$$e_{N,M}(x) = \sum_{n=0}^{M} a_n^* L_n(x), \ M > N.$$

Consequently, the corrected Lucas polynomial solution $y_{N,M}(x) = y_N(x) + e_{N,M}(x)$ is obtained by means of the polynomials $y_N(x)$ and $e_{N,M}(x)$. Also, we construct the error function $e_N(x) = y(x) - y_N(x)$, the estimated error function $e_{N,M}(x)$ and the corrected error function $E_{N,M}(x) = e_N(x) - e_{N,M}(x) = y(x) - y_{N,M}(x)$.

5 Illustration

In this section, we give an example and we compare the results. Here, $\alpha_i(s)$ and $\alpha_{i,N}(s)$ represent the coordinate functions of position vector of the curve $\alpha(s)$ and approximate solutions of the differential equations, respectively. All numerical computations are calculated by using a computer programme written in *Maple*.

Example 1. We consider the *Clelies curve* in E^3 $\alpha : [0,2\pi] \to E^3$ parametrized by

$$\alpha(t) = \left(\sin\left(\frac{t}{2}\right)\cos(s), \sin\left(\frac{t}{2}\right)\sin(s), \cos\left(\frac{t}{2}\right)\right).$$

This curve is regular but non-unit speed curve which lie on unit-sphere. Speed function, curvature and torsion of the curve α are as follows.

$$v(t) = \left\|\alpha'(t)\right\| = \frac{1}{2}\sqrt{-4\cos^2\left(\frac{t}{2}\right)+5} \tag{30}$$

$$\kappa(t) = \frac{\sqrt{48\cos^4\left(\frac{t}{2}\right) - 156\cos^2\left(\frac{t}{2}\right)+125}}{\left(-4\cos^2\left(\frac{t}{2}\right)+5\right)^{\frac{3}{2}}} \tag{31}$$

$$\tau(t) = \frac{12\left(2\cos^2\left(\frac{t}{2}\right)-5\right)\sin\left(\frac{t}{2}\right)}{48\cos^4\left(\frac{t}{2}\right) - 156\cos^2\left(\frac{t}{2}\right)+125}. \tag{32}$$

By substituting (30), (31) and (32) in (11), we obtain differential equation charecterizing the spherical Clelies curve α is as follow.

$$p_3(t)\alpha'''(t) + p_2(t)\alpha''(t) + p_1(t)\alpha'(t) + p_0(t)\alpha(t) = 0 \tag{33}$$

where

$$p_3(t) = \frac{6\cos\left(\frac{t}{2}\right)\sin\left(\frac{t}{2}\right)\left(4\cos^4\left(\frac{t}{2}\right) - 16\cos^2\left(\frac{t}{2}\right)+15\right)}{64\cos^6\left(\frac{t}{2}\right) - 240\cos^4\left(\frac{t}{2}\right) + 300\cos^2\left(\frac{t}{2}\right) - 125}$$

$$p_2(t) = \frac{9\sin^2\left(\frac{t}{2}\right)\left(4\cos^4\left(\frac{t}{2}\right) - 12\cos^2\left(\frac{t}{2}\right)+5\right)}{64\cos^6\left(\frac{t}{2}\right) - 240\cos^4\left(\frac{t}{2}\right) + 300\cos^2\left(\frac{t}{2}\right) - 125}$$

$$p_1(t) = \frac{3}{2}\frac{\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)\left(4\cos^4\left(\frac{t}{2}\right) - 40\cos^2\left(\frac{t}{2}\right)+75\right)}{64\cos^6\left(\frac{t}{2}\right) - 240\cos^4\left(\frac{t}{2}\right) + 300\cos^2\left(\frac{t}{2}\right) - 125}$$

$$p_0(t) = -\frac{9}{4}\frac{4\cos^6\left(\frac{t}{2}\right) - 24\cos^4\left(\frac{t}{2}\right) + 45\cos^2\left(\frac{t}{2}\right) - 25}{64\cos^6\left(\frac{t}{2}\right) - 240\cos^4\left(\frac{t}{2}\right) + 300\cos^2\left(\frac{t}{2}\right) - 125}.$$

We consider $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$, then Eq.(33) satisfies for $\alpha_1(t)$, $\alpha_2(t)$ and $\alpha_3(t)$ as follows

$$p_3(t)\alpha_1'''(t) + p_2(t)\alpha_1''(t) + p_1(t)\alpha_1'(t) + p_0(t)\alpha_1(t) = 0 \tag{34}$$

$$p_3(t)\alpha_2'''(t) + p_2(t)\alpha_2''(t) + p_1(t)\alpha_2'(t) + p_0(t)\alpha_2(t) = 0 \tag{35}$$

$$p_3(t)\alpha_3'''(t) + p_2(t)\alpha_3''(t) + p_1(t)\alpha_3'(t) + p_0(t)\alpha_3(t) = 0. \tag{36}$$

We suppose that the parametric equation of the curve α is unknown. We have only speed function, curvature and torsion of the curve α. Then we can find the parametric equation of the curve α approximately by means of Lucas collacation method and considering the Eq.(33).

We consider the initial conditions for $\alpha_1(t)$, $\alpha_2(t)$ and $\alpha_3(t)$as follows

$$\alpha_1(0) = 0,\ \alpha_1^{'}(0) = 0,5\ \alpha_1^{''}(0) = 0$$

$$\alpha_2(0) = 0,\ \alpha_2^{'}(0) = 0,\ \alpha_2^{''}(0) = 1$$

$$\alpha_3(0) = 1,\ \alpha_3^{'}(0) = 0,\ \alpha_3^{''}(0) = -0,25.$$

The approximate solutions $\alpha_{1,4}(t)$, $\alpha_{2,4}(t)$ and $\alpha_{3,4}(t)$ by the truncated Lucas series for $N = 4$ is

$$\alpha_{i,4}(t) = \sum_{n=0}^{4} a_{i,n} L_n(t)\ ,\quad (i = 1,2,3).$$

Now, let us compute the coefficients $a_{i,n}$, $(n = 0,1,2,3,4)$ of the approximate solutions. The set of collocation points for $a = 0$, $b = 2\pi$ and $N = 4$ is calculated as

$$\left\{t_0 = 0,\ t_1 = \frac{\pi}{2},\ t_2 = \pi,\ t_3 = \frac{3\pi}{2},\ t_4 = 2\pi\right\}.$$

From Eq.(24), the fundamental matrix equation for each Eq.(34), Eq.(35) and Eq.(36) is

$$\left\{\sum_{k=0}^{3} \mathrm{P}_k \mathrm{T}\,\mathrm{B}^k \mathrm{D}^T\right\} \mathrm{A} = \mathrm{G}.$$

By using the prosedure given above, we obtain the Lucas poynomial solutions for $N = 4$ as

$$\alpha_{1,4}(t) = 0.4999999996t - 0.1033282131t^3 + (0.1145284738e - 1)t^4$$

$$\alpha_{2,4}(t) = -(0.26e - 9) + 0.5t^2 - 0.2389089492t^3 + (0.2231474722e - 1)t^4$$

$$\alpha_{3,4}(t) = 0.9999999995 + (0.2e - 10)t - 0.125t^2 + (0.402468024e - 2)t^3 + (0.1160878758e - 2)t^4$$

In order to compute the Lucas polynomial solution, let us consider the error problem

$$\begin{aligned} p_3(t)e'''_{1,4}(t) + p_2(t)e''_{1,4}(t) + p_1(t)e'_{1,4}(t) + p_0(t)e_{1,4}(t) &= -R_{1,4} \\ p_3(t)e'''_{2,4}(t) + p_2(t)e''_{2,4}(t) + p_1(t)e'_{2,4}(t) + p_0(t)e_{2,4}(t) &= -R_{2,4} \\ p_3(t)e'''_{3,4}(t) + p_2(t)e''_{3,4}(t) + p_1(t)e'_{3,4}(t) + p_0(t)e_{3,4}(t) &= -R_{3,4} \end{aligned}$$

where

$$\left.\begin{aligned} R_{1,4}(t) &= p_3(t)\alpha'''_{1,4}(t) + p_2(t)\alpha''_{1,4}(t) + p_1(t)\alpha'_{1,4}(t) + p_0(t)\alpha_{1,4}(t) \\ R_{2,4}(t) &= p_3(t)\alpha'''_{2,4}(t) + p_2(t)\alpha''_{2,4}(t) + p_1(t)\alpha'_{2,4}(t) + p_0(t)\alpha_{2,4}(t) \\ R_{3,4}(t) &= p_3(t)\alpha'''_{3,4}(t) + p_2(t)\alpha''_{3,4}(t) + p_1(t)\alpha'_{3,4}(t) + p_0(t)\alpha_{3,4}(t) \end{aligned}\right\} \tag{37}$$

By solving the error problem (37) for $M = 5$ with the method, the estimated Lucas error function approximation $e_{1,4,5}(t)$, $e_{2,4,5}(t)$ and $e_{3,4,5}(t)$ to $e_{1,4}(t)$, $e_{2,4}(t)$ and $e_{3,4}(t)$ is obtained as

$$\begin{aligned} e_{1,4,5}(t) = {} & (0.222044604925031e - 15)t - 0.335907477291688t^3 + 0.147313123124575t^4 \\ & -(0.145247893504105e - 1)t^5 \end{aligned}$$

$$e_{2,4,5}(t) = -(0.138777878078145e-15)+(0.111022302462516e-15)t \\ -(0.555111512312578e-16)t^2 - 0.238494735315633t^3 \\ +0.107194173804204t^4 - (0.106643253034255e-1)t^5$$

$$e_{3,4,5}(t) = -(0.173472347597681e-17)+(0.346944695195361e-17)t \\ -(0.534226555720575e-2)t^3 + (0.236392416157558e-2)t^4 \\ -(0.227723431226885e-3)t^5$$

Thus, we can calculate the corrected Lucas polynomial solutions

$$\alpha_{1,4,5}(t) = 0.499999999600000t - 0.439235690391688t^3 + 0.158765970504575t^4 \\ -(0.145247893504105e-1)t^5$$

$$\alpha_{2,4,5}(t) = -(0.260000138777878e-9)+0.5t^2 - 0.477403684515633t^3 \\ +0.129508921024204t^4 + (0.111022302462516e-15)t \\ -(0.106643253034255e-1)t^5$$

$$\alpha_{3,4,5}(t) = 0.999999999500000 + (0.200000034694470e-10)t - 0.125t^2 \\ +(0.352480291957558e-2)t^4 - (0.227723431226885e-3)t^5 \\ -(0.131758531720575e-2)t^3$$

Similarly, we can calculate the corrected Lucas polynomial solutions for $\alpha_i(s)$, $(i = 1,2,3)$ by using present method for different values of M.

For $N = 4$ and $M = 8$

$$\alpha_{1,4,8}(t) = 0.499999999599978t + 0.306532559723370t^3 - 0.444149870356995t^4 \\ -(0.186620060767598e-13) - (0.532907051820075e-14)t^2 \\ +(0.174641358983243)t^5 - (0.286040945209451e-1)t^6 \\ +(0.194021418478947e-2)t^7 - (0.356421367872731e-4)t^8$$

$$\alpha_{2,4,8}(t) = -(0.259996639461156e-9) + 0.499999999999996t^2 + 0.224416858744376t^3 \\ -0.248981167081697t^4 + (0.128196064874686e-13)t \\ +(0.213309695850870e-1)t^5 + (0.147212321823962e-1)t^6 \\ -(0.335863274832651e-2)t^7 + (0.204842806713442e-3)t^8$$

$$\alpha_{3,4,8}(t) = 0.999999999500000 + (0.199999702454266e-10)t - 0.125t^2 \\ +(0.787720390942864e-4)t^3 + (0.254392355708362e-2)t^4 \\ +(0.192991422948481e-4)t^5 - (0.256915769161762e-4)t^6 \\ +(0.629632428667440e-6)t^7 + (0.407675668361376e-7)t^8$$

For $N = 4$ and $M = 13$

$$\alpha_{1,4,13}(t) = 0.499999999599173t - 0.260125216685152t^3 - (0.161578050696862e-1)t^4 \\ -(0.375578231641654e-11) - (0.454118933077138e-11)t^2 \\ +(0.464218823180868e-1)t^5 - (0.997436281466191e-2)t^6 \\ +(0.303085602484618e-2)t^7 - (0.156638931981066e-2)t^8 \\ +(0.411485747537591e-3)t^9 - (0.539918403345653e-4)t^{10} \\ +(0.352490067494012e-5)t^{11} - (0.911934775912629e-7)t^{12} \\ -(0.101566509306906e-9)t^{13}$$

$$\begin{aligned}
\alpha_{2,4,13}(t) = &-(0.257271019489540e-9)+0.500000000001660t^2-(0.646796711174480e-2)t^3 \\
&-(0.940898075448491e-1)t^4-(0.803453640963370e-12)t \\
&+(0.534514547759334e-6)t^{12}-(0.132234193418015e-7)t^{13} \\
&-(0.978240920331576e-2)t^5+(0.148807268548122e-1)t^6 \\
&-(0.359982011372225e-2)t^7+(0.102181351910415e-2)t^8 \\
&-(0.360440584949504e-3)t^9+(0.769153381880423e-4)t^{10} \\
&-(0.890477068009215e-5)t^{11}
\end{aligned}$$

$$\begin{aligned}
\alpha_{3,4,13}(t) = &\,0.999999999499971+(0.201201043977291e-10)t-0.125000000000007t^2 \\
&+(0.412274200171665e-5)t^3+(0.259806624082214e-2)t^4 \\
&+(0.553872304938226e-5)t^5-(0.253558352879775e-4)t^6 \\
&+(0.170714609813919e-5)t^7-(0.462427797626050e-6)t^8 \\
&+(0.128169082819624e-6)t^9-(0.204317393646569e-7)t^{10} \\
&+(0.207431984062146e-8)t^{11}-(0.125009847463864e-9)t^{12} \\
&+(0.337713469364518e-11)t^{13}
\end{aligned}$$

When the values of $\alpha_{i,N,M}(t)$, $(i=1,2,3)$ for $N=4$, $M=5,8,13$ are written in

$$\alpha_{N,M}(t) = (\alpha_{1,N,M}(t), \alpha_{2,N,M}(t), \alpha_{3,N,M}(t))$$

the parametric equation of the curve $\alpha_{N,M}(t) \cong \alpha(t)$ is found. Now we compare the corrected absolute error functions in Tables 1-3.

Table 1: Corrected absolute error functions ($|E_{1,N,M}(t)|$) for $N=4$ and $M=5,8,13$.

t_i	Corrected absolute error functions $\lvert E_{1,N,M}(t)\rvert = \lvert\alpha_1(t)-\alpha_{1,N,M}(t)\rvert$		
	$\lvert E_{1,4,5}(t_i)\rvert$	$\lvert E_{1,4,8}(t_i)\rvert$	$\lvert E_{1,4,13}(t_i)\rvert$
0	0	011866e-13	0.3756e-11
$\pi/3$	0.058173632	0.276326840	0.003215525
$2\pi/3$	0.085536668	0.703709963	0.006995609
π	0.027889603	0.276845034	0.001717034
$4\pi/3$	0.392355438	0.457343549	0.005477075
$5\pi/3$	1.485985927	0.382910418	0.003910408
2π	0.602818793	0.673770187	0.000128008

Table 2: Corrected absolute error functions ($|E_{2,N,M}(t)|$) for $N=4$ and $M=5,8,13$.

t_i	Corrected absolute error functions $\lvert E_{2,N,M}(t)\rvert = \lvert\alpha_2(t)-\alpha_{2,N,M}(t)\rvert$		
	$\lvert E_{2,4,5}(t_i)\rvert$	$\lvert E_{2,4,8}(t_i)\rvert$	$\lvert E_{2,4,13}(t_i)\rvert$
0	0.2600e-9	0.2600e-9	0.2573e-9
$\pi/3$	0.290627483	0.115529399	0.001906142
$2\pi/3$	0.880523887	0.298494761	0.004139286
π	0.515855720	0.120235950	0.001009328
$4\pi/3$	0.553938273	0.187144292	0.003254205
$5\pi/3$	0.982494585	0.141416172	0.002353825
2π	1.267127566	1.162168851	0.000950739

Table 3: Corrected absolute error functions ($|E_{3,N,M}(t)|$) for $N = 4$ and $M = 5, 8, 13$.

t_i	Corrected absolute error functions $\lvert E_{3,N,M}(t)\rvert = \lvert\alpha_3(t) - \alpha_{3,N,M}(t)\rvert$		
	$\lvert E_{3,4,5}(t_i)\rvert$	$\lvert E_{3,4,8}(t_i)\rvert$	$\lvert E_{3,4,13}(t_i)\rvert$
0	0.5000e-9	0.5000e-9	0.5000e-9
$\pi/3$	0.000664249	0.000037843	0.000001268
$2\pi/3$	0.001771263	0.000096440	0.000002810
π	0.000893956	0.000037970	0.000000754
$4\pi/3$	0.001401566	0.000062277	0.000002133
$5\pi/3$	0.003040652	0.000050771	0.000001526
2π	0.001924771	0.000213768	0.000002288

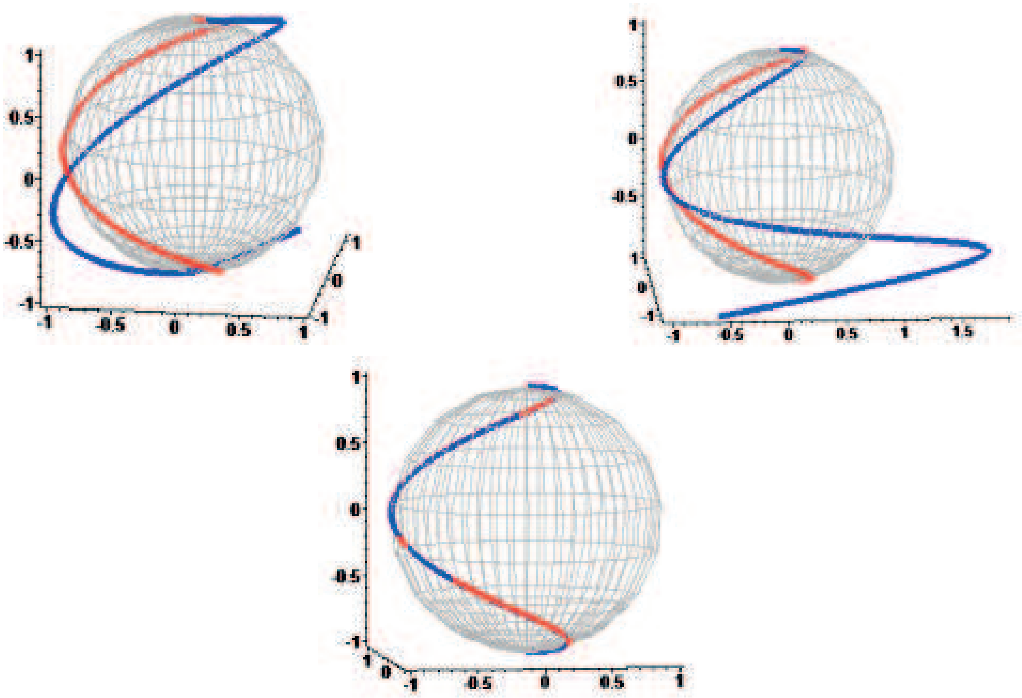

Fig. 1: Graphics of curve α and curves $\alpha_{4,5}$, $\alpha_{4,8}$ ve $\alpha_{4,13}$.

6 Conclusions

In this study, we have developed a Lucas collocation method to find the approximate solutions of third-order linear differential equations with variable coefficients. Then, we have gave a necassary and sufficient condition for an arbitrary-speed regular space curve to lie on a sphere centered at origin. Then, we have obtained that position vector of any arbitrary-speed regular space curve lying on a sphere satisfies a third-order linear differential equation with variable coefficients in E^3. And then, by means of this method we have gave approximate solutions of these differential equation characterizing spherical curves. We have gave an example to show efficiency of this method.

In Tables 1-3, we obtained corrected absolute error functions for various values of N and M. It is seen from these comparisions that the corrected absolute errors are very close to zero when the values of N and M is selected big. Also, we have seen from the Figure 1 that the Lucas collocation method used for approximate solutions is very effective.

This method can be developed for the differential equations which is obtained in differential geometry another special curves.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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