New Trends in Mathematical Sciences

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Fixed point theorem for composite functions

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Abstract: This study is about the part of the composition of functions obtained from the series iterations. In this study, we tried to show what can be said about the fixed point of compositions on every step of the way by using lines and counterparts on the plane geometry when, as series, the compositions of a function which has a fixed point are calculated. Also, we showed the relation between the x-intercepts of series composition of an F(x) function and the fixed point of the composition. Before everything else, with this study, the writer's aim is to stop being an abstract concept of the fixed point theory thanks to the concepts of the plane geometry and so to take interest of the readers to its visiual side.

Keywords: Composite function, plane geometry, fixed point.

1 Introduction

The simplest fixed point theorem known for fixed point theory is was developed by L.E.J. Brouwer in 1909-1913 as " If [a, b] is a closed subrange of R, then there is a fixed point of F: [a, b] $\rightarrow$[a, b] continuous conversion in the [a, b] subrange". The author then expanded this theorem and stipulated that; **"Let C be a unit sphere on R^n . In this case, F : C $\rightarrow$ C continuous (norm functions) transformation has a fixed point"** [1].

In 1922, S.Banach proved the theory under the name of Constriction principle, stipulating that "If (X, d) is a full metric space, there is a single fixed point for each F: X $\rightarrow$ X function" [2].

In 1930s, J.Schauder expanded this theorem by stating that "Let (X, $\|.\|$) be a Banach space (Full norm space). If C$\subset$ X is a compact and convex subset according to a norm, then each **F: C $\rightarrow$ C** continuous function has a fixed point" [6].

1965's have been the golden years of the fixed point theory, and many important theorems that have been proved during these years. In 1965 D. Göhde, F.E.Browder and W.A. Kirk, and in 1968 R. Kanan made important contributions to fixed-point studies [5].

Although the method of approaching the fixed point of functions with successive iterations is quite old, it was initially used by Picard, and Italian mathematician [4].

Fixed point studies have been done in many fields of mathematics such as analysis algebra topology and plane geometry. In this study, we will start from the definition of fixed point and examine the fixed points of linear functions of $F(x) = ax+b$, where a, b $\in$ R, in other words, linear equations in the plane geometry, by taking series compositions of these functions. At the end of our study, we will show that the fixed point of $F(x) = ax+b$ is the same as the fixed point of series compositions, and the points at which the compositions intersect the x-axis are convering on the fixed point.

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Definition 1. *(Fixed Point) Let X be a non-empty set and F : X→ X be a function. If there is a x ∈ X where F(x) = x, then x is the fixed point of function F. Then, x ∈ X point that satisfy this equation are the fixed points of function F [5].*

Example 1. Let us find the fixed points of $F(x) = x^3$ defined under $F : R \to$R.

Solution 1.

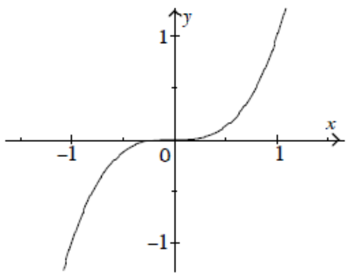

Fig. 1

Due to the definition of fixed point, F(x) = x should be satisfied. The function is given as F(x) = x^3 therefore x^3= x . When we gather the variables on one side, then the equation becomes x^3- x = 0. If we perform factorization on this equation, we obtain x(x-1)(x+1) = 0, and then the solution becomes: x = -1, x = 0, x = 1. These are the fixed points of function F.

Definition 2. *(Fixed points of composite functions) With a,b ∈R in plane geometry, graphs of $ax + by + c = 0$ equations consist of lines. If we rearrange the equation, we can denote these equations as $y = mx + nm, n \in R$. We will denote this as $y = F(x) = mx + n$. These functions are called linear functions. We will then take series compositions of $F(x)$ and draw the graphs of these functions in R^2. Now let us examine two special cases for $m > 0$ and $m < 0$. We will then discuss the theoretical aspects of this topic.*

2 Materials and methods

Let us first analyze the fixed points of series compositions, and the relationship of the points where these composites intersect the x-axis with the fixed points, based on two examples. Then let us propose our theorem and prove it.

Example 2.

$$y_1 = F(x) = 2x - 4.$$

Let us find the series composites of $F(x)$.

$$\begin{aligned} y_2 =& F^2(x) = F(F(x)) = 4x - 12. \\ y_3 =& F^3(x) = F(F(F(x))) = 8x - 28. \\ y_4 =& F^4(x) = F(F(F(F(x)))) = 16x - 60. \end{aligned}$$

Let us draw these equations and the $y = x$ line and see what happens. The lines and the point at which they intersect can be seen below (Figure 2).

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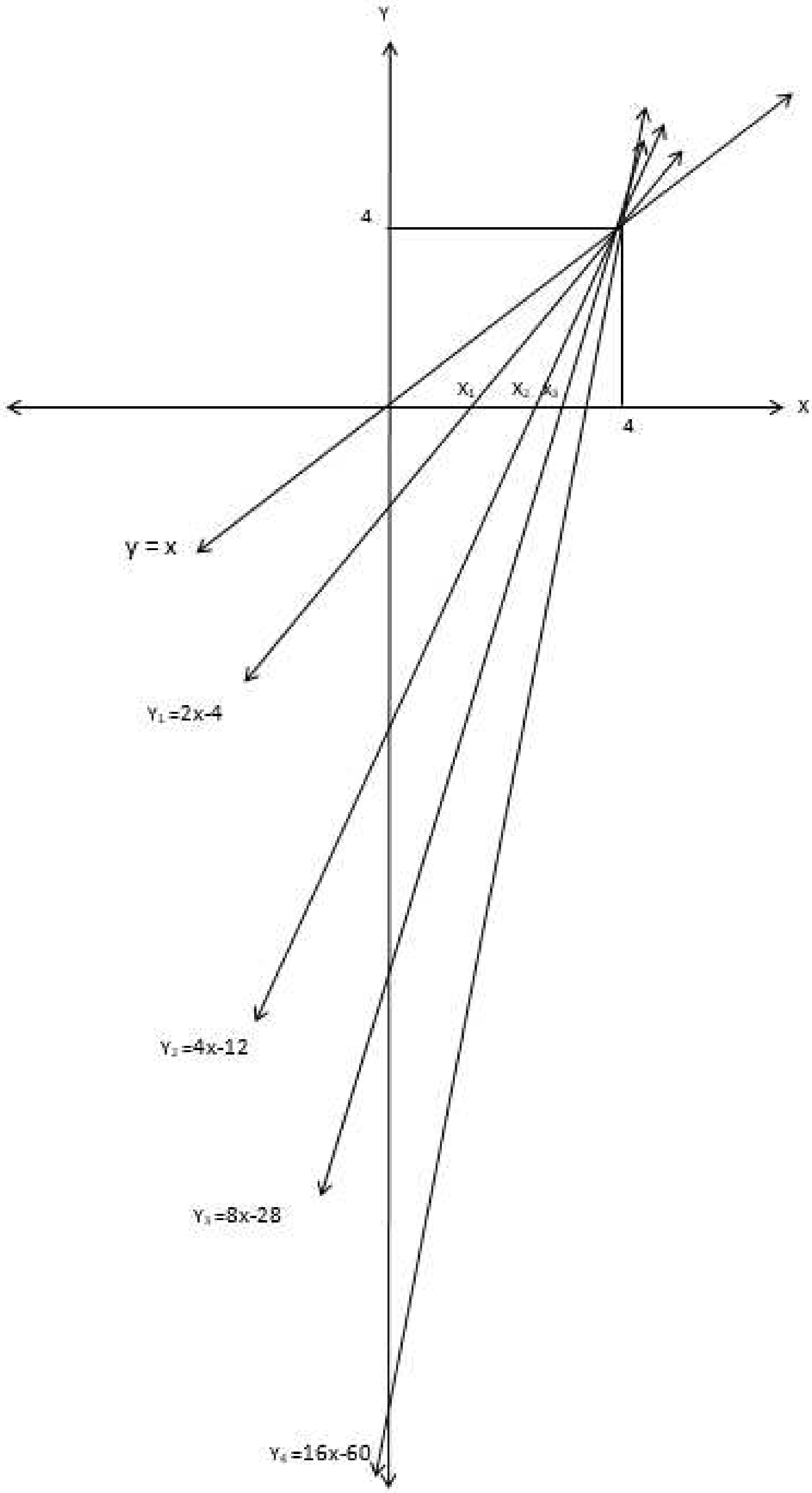

Fig. 2

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As it can be seen, the point $(4,4)$ is where the $F(x) = 2x-4$ line intersects with the $y = x$ line. Since $F(4) = 4, x = 4$is a fixed point of $F(x)$. We can easily see that x =4 is the only solution point of $y = F(x) = F(F(x)) = F(F(F(x))) = F(F(F(F(x)))) = x$ functions. Therefore $x = 4$ is also the fixed point of composites.

Furthermore, if we denote the points where the elements of the $\{F^n(x) : n \in N\}$ composite set intersect the x-axis as $\{X_n : n \in N\}$ (Figure 2), then $lim X_n = 4$ when $n \to \infty$.

Now let us examine our second example.

Example 3. Let

$$y_1 = F(x) = -2x+6$$

Let us find the series composites of $F(x)$.

$$y_2 = F^2(x) = F(F(x)) = 4x-6. y_3 = \quad F^3(x) = F(F(F(x))) = -8x+18. y_4 = F^4(x) = F(F(F(F(x)))) = 16x-30.$$

Let us draw these equations and the y = x line and see what happens. The lines and the point at which they intersect can be seen below (Figure 3).

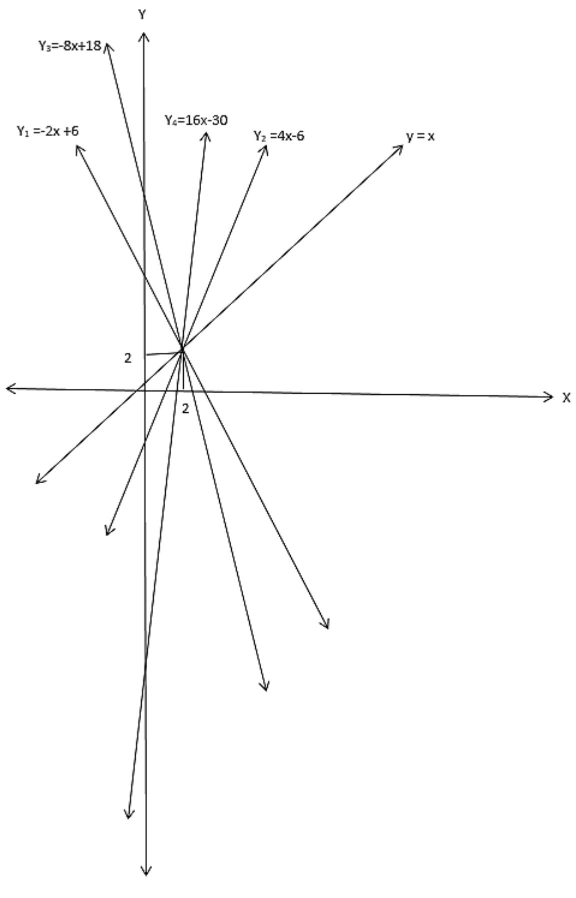
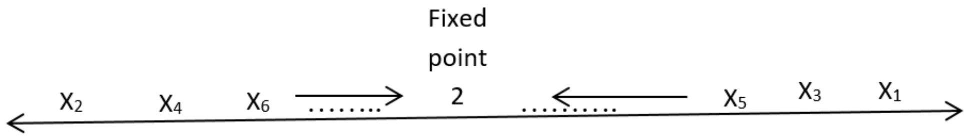


Fig. 3

As it can be seen, $(2,2)$ is the point where $F(x) = -2x+6$ intersects with the line $y = x$. Since $F(2) = 2, x = 2$ is the fixed point of $F(x)$. It can be easily seen that $x = 2$ is the only solution point of $y = F(x) = F(F(x)) = F(F(F(x))) = F(F(F(F(x)))) = x$ equations. Then, this point is also the fixed point of composites.

Furthermore, for this type of function, the points where the composites intersect the x-axis can be visualized as follows (Figure 3). Let us denot the points where the elements of the $\{F^n(x) : n \in N\}$ composite set intersect the x-axis as $\{X_n : n \in N\}$.



As it can be seen, the points where the composites intersect the x-axis also converge to the fixed point for this type of functions, in other words, $lim X_n = 2$ when $n \to \infty$. Now let us discuss the theoretical aspects of these situations we have noticed in these two examples.

Definition 3. *Let $y = F(x) = ax+b$. Then*

$$
\begin{aligned}
&F^2(x) = F(F(x)) = a^2x+ab+b. \\
&F^3(x) = F(F(F(x))) = a^3x+a^2b+ab+b. \\
&F^4(x) = F(F(F(F(x)))) = a^4x+a^3b+a^2b+ab+b. \\
&\quad\vdots \\
&F^n(x) = a^nx+a^{n-1}b+a^{n-2}b+\ldots+ab+b.
\end{aligned}
$$

Anyone who knows mathematics at high school level will easily understand the results below.

(1) With $F(x) = ax+b = x$, fixed point of F(x) is then found as $x = \frac{b}{1-a}$.
(2) If we rearrage $F^2(x) = F(F(x)) = a^2x+ab+b = x$, we obtain $b(a+1) = x(1-a^2)$, we then we write this as $b(a+1) = x(1-a)(1+a)$ and simplify, and find the fixed point of $F^2(x)$ also as $x = \frac{b}{1-a}$.
(3) If we rearrange the equation $F^n(x) = a^nx+a^{n-1}b+a^{n-2}b+\cdots+ab+b = x$ in its most general form, we obtain $b(a^{n-1}+a^{n-2}+\cdots+a+1) = x(1-a^n)$. When we perform the necessary operations, we then get $b(\frac{a^n-1}{a-1}) = x(1-a^n)$, and when we simplify this equation, we find the fixed point of $F^n(x) as x = \frac{b}{1-a}$.
(4) For $y = F(x) = ax+b$, the tangent of the angle that the line makes with respect to x-axis in the positive direction is defined as the slope of the line. slope $[F(x)] = a$ is a known information. Now, let us obtain the slopes of the composites and see the relationship between them.

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We obtain

$$
\begin{aligned}
slope[F^2(x)] &= a^2.\\
slope[F^3(x)] &= a^3.\\
slope[F^4(x)] &= a^4.\\
&\vdots\\
slope[F^n(x)] &= a^n.
\end{aligned}
$$

When we examine the slopes of composite functions with respect to the status of a in slope[F(x)] = a , how does the angle between the composites and the line y = x change? Investigation of this question is left to the reader. By status of a, we mean the investigation of conditions such as (-∞, -1], (-1, 0) (0, 1), (1, ∞).

Now let us examine the relationship between the fixed point and the points where the composite functions intersect the x-axis.

3 Findings and discussion

Theorem 1. *For a function defined as* $y = F(x) = ax + b$,

(1) *The fixed point of the function is* $x = \frac{b}{1-a}$.
(2) *If we denote the points where the elements of the composite set* $\{F^n(x) : n \in N\}$ *intersect with the x-axis as* $\{X_n : n \in N\}$, *then* $lim X_n = \frac{b}{1-a}$ *when* $n \to \infty$.

Proof. (1) When $y = F(x) = ax + b = x$, we obtain $x = \frac{b}{1-a}$.
(2) Now let us obtain the points of composite functions intersecting the x-axis, and write $y_i = 0$ for $i \in N$ in the composites. We obtain
  (i) If $F(x) = ax + b = 0$, then $x_1 = -\frac{b}{a}$.
  (ii) If $F^2(x) = F(F(x)) = a^2x + ab + b = 0$, then $x_2 = -\frac{b}{a} - \frac{b}{a^2}$.
  (iii) If $F^3(x) = F(F(F(x))) = a^3x + a^2b + ab + b = 0$, then $x_3 = -\frac{b}{a} - \frac{b}{a^2} - \frac{b}{a^3}$.
  (iv) If $F^4(x) = F(F(F(F(x)))) = a^4x + a^3b + a^2b + ab + b = 0$, then $x_4 = -\frac{b}{a} - \frac{b}{a^2} - \frac{b}{a^3} - \frac{b}{a^4}$ and in the most general form.
  (v) If $F^n(x) = a^nx + a^{n-1}b + a^{n-2}b + \cdots + ab + b = 0$, then we get $x_n = -\frac{b}{a} - \frac{b}{a^2} - \frac{b}{a^3} - \frac{b}{a^4} - \ldots. - \frac{b}{a^n}$ and when this statement is rearranged after the denominators are equalized, then we obtain $x_n = \frac{b}{1-a}(1 - \frac{1}{a^n})$.

For anyone with mathematics knowledge at high school level; We obtain $lim X_n = lim \frac{b}{1-a}(1 - \frac{1}{a^n}) = \frac{b}{1-a}$ when $n \to \infty$.

The proof is complete, and we show that the points where the composites intersect the x-axis converge to the fixed point, and the limit value of the series consisting of the fixed points is equal to the fixed point.

4 Conclusion

In this study, we showed that the fixed point of series composites of linear functions such as y= F(x)= ax+b is equal to the fixed point of y= F(x), and the points where the composites intersect the x-axis converge to the fixed point. Furthermore, this situation we have demonstrated for linear functions is also valid for any function with a fixed point.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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