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A note on $I-$ convergence of filters

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Abstract: In this paper, we make further investigation on $I-$ convergence of filters and introduce $I-$ closed sets in a topological space. We have used $I-$ closed sets to determine topology on an arbitrary set.

Keywords: $I-$ convergence, filter, ideal, closure, closed set.

1 Historical background and introduction

Convergence of sequences has always been an area of interest for researchers. Several new types of convergence of sequences were introduced and studied by the researchers and named it as usual convergence, uniform convergence, strong convergence, weak convergence etc. Later on, the idea of statistical convergence was introduced by Zygmund [21] (published in Warsaw in 1935)where he called it "'almost convergence"'. Statistical convergence is the generalization of usual convergence,which is based on the natural density of subsets of natural numbers $\mathbb{N}$. Formally, the concept of statistical convergence was introduced by Fast[1] in 1951. Alot of investigations have been done on this convergence and its topological consequences after the initial works by Fridy[2] and Š*alá*t [16].

$I-$convergence is the generalization of the statistical convergence, which is based on the notion of ideal of a set X. The notion of $I-$convergence of sequences was introduced and studied by Kostyrko, Š*alá*t and Wilczy*ń*ski [5] in the year 2000. The idea of $I-$convergence was extended from real number space to metric space [5], normed linear space [17] and to a topological space [8]. The concept of $I-$ convergence of nets was introduced by Lahiri and Dass [9] and they established the basic topological nature of these convergence.

In general topology, the notion of nets and filters are the two basic tools to describe convergence.Therefore, it was obvious to extend the idea of $I-$convergence to filters.The concept of $I-$convergence was extended to filters in [3] [4].In an ordinary space, there are three basic notions from which all others can be derived. These notions are convergence, closure and neighborhood. This paper is the extension of the work done in [3] [4]. Our motivation mainly arises from the results obtained for $I-$ convergence of nets. In this paper, we have used the idea of $I-$convergence of filters to introduce $I-$closed sets and to derive certain conditions of $I-$closed sets, which are sufficient to determine a topology on any non empty set X.

2 $I-$ convergence of filters and $I-$limit points

We first recall the following definitions.

Definition 1. [20] *Let X be a non-empty set. Then a family $\mathscr{F} \subset 2^X$ is called a Filter on X if,*

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(i) $\phi \notin \mathscr{F}$
(ii) $A,B \in \mathscr{F}$ *implies* $A \cap B \in \mathscr{F}$ *and*
(iii) $]\, A \in \mathscr{F}, B \supset A$ *implies* $B \in \mathscr{F}$

Definition 2. [20] *Let X be a non-empty set. Then a family* $\mathscr{I} \subset 2^X$ *is called a Ideal on X if,*

(i) $\phi \in \mathscr{I}$
(ii) $A,B \in \mathscr{F}$ *implies* $A \cup B \in \mathscr{I}$ *and*
(iii) $A \in \mathscr{F}, B \subset A$ *implies* $B \in \mathscr{I}$

Definition 3. [3] *Let X be a topological space and* $\mathscr{F}$ *be a filter on X. We say that* $\mathscr{F}$ *I- converges to a point* $x_0 \in X$*, if for each nbd. U of* $x_0, y \in X : y \notin U \in I$ *Here* x_0 *is called the I-limit of* $\mathscr{F}$ *and we write I- lim F=* x_0.

Theorem 1. *The set of* $I-$*limits of Filter on a topological space X is a closed set.*

Proof. Let S be the set of all I-limit points of a filter $\mathscr{F}$ on a topological space X. We shall show that S is a closed set. For this, let $y \in \overline{S}$. Then for any neighborhood U of y, $U \cap S \neq \phi$. Let $z \in U \cap Z$. Then $z \in S$. Take a neighborhood V of z such that $V \subseteq U$. Clearly, $\{y \in X : y \notin V\} \in \mathscr{I}$. Also, $\{y \in X : y \notin U\} \subset \{y \in X : y \notin V\} \subset \mathscr{I}$. Since $\mathscr{I}$ is closed under subsets, it follows that $\{y \in X : y \notin U\} \in \mathscr{I}$. Thus, $y \in S$. This proves that S is a closed set.

Theorem 2. [4] *Let* $E \subset X$. *Then* $x_0 \in \overline{E}$ *if and only if there is a filter* $\mathscr{F}$ *on X such that* $E \in \mathscr{F}$ *and I-lim* $\mathscr{F}$*=*x_0.

In the next result, we show that the condition $E \in \mathscr{F}$ can be dropped.

Theorem 3. *Let* $E \subset X$. *Then* $x_0 \in \overline{E}$ *if and only if there is a filter* $\mathscr{F}$ *on E and I-lim* $\mathscr{F} = x_0$.

Proof. Let $x_0 \in \overline{E}$. Then each nbd. U of x_0, $U \cap E \neq \phi$.. Let $B = \left\{U \cap E : U \in \mathscr{U}_{x_0}\right\}$.
Clearly, B is a filter base for some filter $\mathscr{F}$ on E. For any nbd. U of x_0, $U \cap E \subset U$, implies $U \in \mathscr{F}$. Also $E \in \mathscr{F}$. Let $V \subset X$ such that $U \cap V = \phi$. Now, $U \in \mathscr{F}$ and $U \cap V = \phi$, implies $V \in \mathscr{I}$.
Thus, $\{V \subset X : U \cap V = \phi\} \in \mathscr{I}$. Therefore, I-lim $\mathscr{F} = x_0$.
Conversely, suppose there is a filter $\mathscr{F}$ on E. and $I-$lim $\mathscr{F} = x_0$. We shall show that $x_0 \in \overline{E}$. For this, let U be a nbd. of x_0. Since $I-$lim $\mathscr{F} = x_0, \{V \subset X : U \cap V = \phi\} \in \mathscr{I}$.. Since $U \cap (X \setminus U) = \phi$, implies $X \setminus U \in \mathscr{I}$. Thus, $U \in \mathscr{F}$. Now E, U $\in \mathscr{F}$ and $\mathscr{F}$ is closed under finite intersection,implies $U \cap E \in \mathscr{F}$ and so $U \cap E \neq \phi$. Thus, $x_0 \in \overline{E}$.

Theorem 4. *If a filter* $\mathscr{F}$ $I-$ *converges to a point* $x \in X$*, then each filter base of* $\mathscr{F}$ *converges to x.*

Proof. Let $\mathscr{F}$ be a filter on X such that $\mathscr{F}$ I- converges to a point $x \in X$. Then for each nbd. U of x,

$$\{y \in X : y \notin U\} \in I(\mathscr{F})$$

implies

$$\{y \in X : y \in U^c\} \in I(\mathscr{F})$$

Let $\mathscr{F}'$ be any filter base of $\mathscr{F}$. Since for each nbd U of x,$\{y \in X : y \in U^c \in I(\mathscr{F})\}$ implies $\{y \in X : y \in U\} \in \mathscr{F}$. Thus, $U \in \mathscr{F}$. Now, $U \in \mathscr{F}$ and $\mathscr{F}'$ is a filter base so there is some $B \in \mathscr{F}'$ such that $B \subset U$. Therefore, for each nbd. U of x, there is some $B \in \mathscr{F}'$ such that $B \subset U$. This proves that $\mathscr{F}'$ converges to x.

3 $I-$ closure and $I-$ closed sets

In this section,we shall study the topology generated by $I-$ convergence of filters. This new type of convergence allows us to define closed sets mirroring what one knows from regular convergence of filters. Throughout this section X stands for a Hausdroff space.

Definition 4. *Let X be a topological space and $A \subseteq X$. We define $I-$ closure of A in X as the collection*

$$\{x \in X : \text{there is a filter} \mathscr{F} \text{ on A and } \mathscr{F} \text{ I-converges to x.}\}$$

We denote $I-$ closure of A by $I-\overline{A}$.

Clearly, $\overline{A} \subseteq I-\overline{A}$, for any set A in a topological space X. For, if $x \in \overline{A}$, then there is a filter $\mathscr{F}$ on A such that $\mathscr{F}$ converges to x. Since every convergent filter is $I-$ convergent, it follows that x$\in I-\overline{A}$.

Definition 5. *Let X be a topological space and $A \subseteq X$. We say that A is $I-$ closed if for each filter $\mathscr{F}$ on A converges to a point in A.*

Let τ_f be the collection of $I-$ closed sets on a topological space X. We investigate whether T_f defines a topology.Clearly, ϕ, X $\in \tau_f$ for any topological space X.

Theorem 5. *Arbitrary intersection of $I-$closed sets is $I-$closed.*

Proof. Let $\{A_\alpha : \alpha \in \Delta\}$ be a family $I-$ closed sets. We have to show that $\bigcap_{\alpha\in\Delta} A_\alpha$ is a $I-$ closed set. For this, let $\mathscr{F}$ be a filter on $\bigcap_{\alpha\in\Delta} A_\alpha$ such that I-lim $\mathscr{F} = x$. We shall show that $x \in \bigcap_{\alpha\in\Delta} A_\alpha$. Since $\mathscr{F}$ is a filter on $\bigcap_{\alpha\in\Delta} A_\alpha$, implies $\mathscr{F}$ is a filter on $A_\alpha, \forall\ \alpha \in \Delta$. Also, each $A_\alpha, \alpha \in \Delta$ is $I-$ closed and $I-$ limit of a filter is unique, it follows that $x \in A_\alpha, \forall\ \alpha \in \Delta$. This implies, $x \in \bigcap_{\alpha\in\Delta} A_\alpha$. Thus, $\bigcap_{\alpha\in\Delta} A_\alpha$ is a $I-$ closed set. Hence the proof.

Next we shall show that finite union of $I-$closed sets is $I-$closed.

Theorem 6. *The family τ_f is closed under finite union.*

Proof. Let A,B $\in \tau_f$. We shall show that $A \cup B \in \tau_f$. For this, let $\mathscr{F}$ be a filter on $A \cup B$, such that $\mathscr{F}$ I- converges to x. We shall show that $x \in A \cup B$. Since $\mathscr{F}$ $I-$ converges to x, for each nbd. U of x

$$\{y \in A \cup B : y \notin U\} \in \mathscr{I}(\mathscr{F})$$

implies

$$\{y \in A \cup B : y \in U^c\} \in \mathscr{I}(\mathscr{F})$$

implies $\{y \in A : y \in U^c\} \cup \{y \in B : y \in U^c\} \in \mathscr{I}(\mathscr{F})$ implies $\{y \in A : y \in U^c\} \in \mathscr{I}(\mathscr{F})$. Now, $\mathscr{F}$ is a filter on $A \cup B$. WLOG, assume that the collection $\mathscr{F}' = \{A \cap F : F \in \mathscr{F}\}$ is a filter base for some filter on A. Let $\mathscr{G}$ be the filter generated by $\mathscr{F}'$ on A. Clearly, $\mathscr{G}$ converges to x and so $I-$converges to x. Since A is $I-$ closed, it follows that $x \in A$. Therefore, $x \in A \cup B$. Hence the proof.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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