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Approximate solution of the problem of delayed variable boundary value by the CAS wavelet method

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Abstract: In this paper, by considering the boundary value problem

$$x^{''}(t)+a(t)x(t-\tau(t))=f(t), \quad (0\le t\le T)$$

$$x(t)=\varphi(t)(\lambda_0\le t\le 0), \quad x(T)=x(c)(0<c<T)$$

such that $\tau(t)\ge 0$ is an arbitrary continous function on $0\le t\le T$, Fredholm-Volterra Integral Equation, which is equivalent to this problem, was written considering the boundary value problem. Under certain conditions, the Fredholm-Volterra integral equation was transformed into the Fredholm integral equation. The solution of this integral equation is approximated by the "CAS Wavelet" method. Thus, an approximate solution to the given boundary value problem was found.

Keywords: Fredholm-Volterra integral equations, differential equation with retarded argument, CAS wavelet method.

1 Introduction

Many problems is known from the theory of differential equations, in which solutions are sought in response to ordinary differential equations. The rate of change of the process in these problems is only related to the situation at the time of change. Therefore, the unknown function itself and its derivatives in the differential equation are dependent on the value of the independent variable [5-7]. But there are such physical problems that the rate of change of a process is related to its past or future state, not the current state of the process. Such operations are called (or variational) differential equations that deviate from the corresponding differential equations. It is called the variable differential equation, which is delayed by such differential equations, if the unknown function itself is related to the variable $t-\tau(t)$, $(\tau(t)\ge 0)$.

$$\left.\begin{array}{r} x^{''}(t)+a(t)x(t-\tau(t))=f(t), \quad (0\le t\le T) \\ x(t)=\varphi(t)(\lambda_0\le t\le 0), \quad x(T)=x(c)(0<c<T) \end{array}\right\} \tag{1}$$

The differential equation given in (1) above is the delayed variable differential equation. Approximate solution of (1) problem is found by CAS wavelet method.

Wavelet theory is a relatively new emerging field. The wavelet allows the various functions and operators to be displayed correctly [8-9]. In addition, wavelets link between numerical algorithms. In this method, the problem studied is transformed into a corresponding Fredholm-Volterra integral equation, then this integral equalization CAS wavelet method is applied [2-4]. Here, the Fredholm operator belonging to the integral equation makes use of the feature that the core is distorted.

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2 An equivalent integral equation

In the problem (1), if we take $\lambda(t) = t - \tau(t)$ then $t_0 \in [0,t]$ is a point located at the left side of T such that conditions $\lambda(t_0) = 0$ and $\lambda(t) \leq 0 (0 \leq t \leq t_0)$ are satisfied, where, $\lambda_0 = \min_{0 \leq t \leq t_0} \lambda(t)$. We assume that $\lambda(t)$ is a nondecreasing function in the interval $[t, t_0]$ and the equation $\lambda(t) = \sigma$ has a differentiable continuous solution $t = \gamma(\sigma)$ for arbitrary $\sigma \in [0, \lambda(t)]$. It can be seen that if $x^*(t)$ is a solution of the boundary value problem (1) then $x^*(t)$ is also the solution of the equation

$$x(t) = \widetilde{h}(t) + \frac{t}{T}\int_0^T (T-s)a(s)x(s-\tau(s))ds - \int_0^t (t-s)a(s)x(s-\tau(s))ds \tag{2}$$

Here,

$$\widetilde{h}(t) = \varphi(0) - (x_T - \varphi(0))\frac{t}{T} - \frac{t}{T}\int_0^T (T-s)f(s)ds + \int_0^t (t-s)f(s)ds$$

Let $\sigma = s - \tau(s)$ Therefore Eq. (2) can be written as follows:

$$x(t) = h(t) + \frac{t}{T}\int_0^{\lambda(t)} T - \gamma(\sigma)a(\gamma(\sigma))\gamma'(\sigma)x(\sigma)d\sigma$$

$$-\int_0^{\lambda(t)} (t-\gamma(\sigma))a(\gamma(\sigma))\gamma'(\sigma)x(\sigma)d\sigma \tag{3}$$

where,

$$h(t) = \widetilde{h}(t) + \frac{t}{T}\int_{\lambda_0}^0 (T-\gamma(\sigma))a(\gamma(\sigma))\varphi(\sigma)\gamma'(\sigma)d\sigma$$

$$-\int_{\lambda_0}^0 (t-\gamma(\sigma))a(\gamma(\sigma))\varphi(\sigma)\gamma'(\sigma)d\sigma \tag{4}$$

Let

$$K_1(\sigma) = (T-\gamma(\sigma))a(\gamma(\sigma))\gamma'(\sigma)$$

and

$$K(t,\sigma) = -(t-\gamma(\sigma))a(\gamma(\sigma))\gamma'(\sigma)$$

therefore we write

$$x(t) = h(t) + \frac{t}{T}\int_0^{\lambda(T)} K_1(\sigma)x(\sigma)d\sigma - \int_0^{\lambda(t)} K(t,\sigma)x(\sigma)d\sigma \tag{5}$$

or

$$x(t) = h(t) + \frac{t}{T}F_\lambda^T x + V_\lambda x \tag{6}$$

where

$$F_\lambda^T x = \int_0^{\lambda(T)} K_1(\sigma)x(\sigma)d\sigma$$

is the Fredholm operator,

$$V_\lambda x = -\int_0^{\lambda(\mathbf{t})} K(\mathbf{t},\sigma)x(\sigma)d\sigma$$

is the Volterra operator. Eq. (6) is a Fredholm–Volterra integral equation and it is equivalent to the problem (1)

2.1 Numerical Example: Special case of (1) problem

$$\left.\begin{array}{c} x''(t) + tx\left(t - \frac{1}{2}\sqrt{t}\right) = -2t^3 + 2t^{\frac{5}{2}} + \frac{5}{2}t^2 - \frac{3}{2}t^{\frac{3}{2}} - 4, \quad (0 \leq t \leq 1) \\ x(t) = 0, \quad \left(-\frac{1}{16} \leq t \leq 0\right), \quad x(1) = x(\frac{1}{2}) \end{array}\right\} \tag{7}$$

Consider the boundary value problem. here $a(t)=t$, $f(t)=-2t^3+2t^{\frac{5}{2}}+\frac{5}{2}t^2-\frac{3}{2}t^{\frac{3}{2}}-4$, $\tau(t)=\frac{1}{2}\sqrt{t}\geq 0$, $(0\leq t\leq 1)$ and $\ddot{o}(t)\equiv 0$, $t\in[-\frac{1}{16},0]$ are continuous functions. also $t_0=\frac{1}{4}$ in point $\lambda(t)=t-\frac{1}{2}\sqrt{t}$ function takes the value of zero. so $\lambda(t_0)=0$ and $\lambda(t)\leq 0$, $(0\leq t\leq\frac{1}{4})$ and $\min\lambda(t)=-\frac{1}{16}$. now,

$$\lambda(t)=\sigma,\quad \sigma\in[0,\frac{1}{2}]$$

in the equation $t=\gamma(\sigma)$ find the solution. So,

$$t-\frac{1}{2}\sqrt{t}-\sigma=0$$

root of the equation $[0,1]$

$$t=\frac{1}{16}(1+\sqrt{1+16\sigma})^2=\gamma(\sigma)$$

$\gamma'(\sigma)$ function $[0,\frac{1}{2}]$ interval and can be differentiated.. Accordingly, the problem (7) is the special case of the integral equation (5). If the equation in (7) is integral twice

$$\begin{aligned}
x(t)=&-t\int_0^1 2(1-s)\left(-2s^3+2s^{\frac{5}{2}}+\frac{5}{2}s^2-\frac{3}{2}s^{\frac{3}{2}}-4\right)ds\\
&+t\int_0^{1/2}(1-2s)\left(-2s^3+2s^{\frac{5}{2}}+\frac{5}{2}s^2-\frac{3}{2}s^{\frac{3}{2}}-4\right)ds\\
&+\int_0^t(t-s)\left(-2s^3+2s^{\frac{5}{2}}+\frac{5}{2}s^2-\frac{3}{2}s^{\frac{3}{2}}-4\right)ds\\
&+t\int_0^{\frac{1}{2}}2[1-\frac{1}{16}(1+\sqrt{1+16\sigma})^2]\frac{1}{16}(1+\sqrt{1+16\sigma})^2\frac{1+\sqrt{1+16\sigma}}{\sqrt{1+16\sigma}}x(\sigma)d\sigma\\
&-t\int_0^{\frac{1}{2}-\frac{\sqrt{2}}{4}}[1-\frac{1}{8}(1+\sqrt{1+16\sigma})^2]\frac{1}{16}(1+\sqrt{1+16\sigma})^2\frac{1+\sqrt{1+16\sigma}}{\sqrt{1+16\sigma}}x(\sigma)d\sigma\\
&-\int_0^{t-\frac{\sqrt{t}}{2}}[t-\frac{1}{16}(1+\sqrt{1+16\sigma})^2]\frac{1}{16}(1+\sqrt{1+16\sigma})^2\frac{1+\sqrt{1+16\sigma}}{\sqrt{1+16\sigma}}x(\sigma)d\sigma
\end{aligned}$$

and

$$\begin{aligned}
x(t)=&2,872933230t-2t^2-0,1714285714t^{\frac{7}{2}}+0,2083333333t^4+0,1269841270t^{\frac{9}{2}}\\
&-0,1t^5+\frac{t}{8}\int_0^{\frac{1}{2}}\left[3+4\sigma-16\sigma^2+\frac{3+28\sigma-80\sigma^2}{\sqrt{1+16\sigma}}\right]x(\sigma)d\sigma\\
&+\frac{t}{8}\int_0^{\frac{1}{2}-\frac{\sqrt{2}}{4}}\left[1+4\acute{o}-16\sigma^2+\frac{1+4\sigma-80\sigma^2}{\sqrt{1+16\sigma}}\right]x(\sigma)d\sigma\\
&-\frac{1}{16}\int_0^{t-\frac{\sqrt{t}}{2}}\left[(4t-1)+(16t-12)\sigma-16\sigma^2+\frac{(4t-1)+(48t-20)\sigma-80\sigma^2}{\sqrt{1+16\sigma}}\right]x(\sigma)d\sigma
\end{aligned}$$

obtained. Here,[1].

$$
\begin{aligned}
h(t) =&2,872933230t-2t^2-0,1714285714t^{\frac{7}{2}}+0,2083333333t^4+0,1269841270t^{\frac{9}{2}}-0,1t^5\\
K_1(\sigma) =&\frac{1}{8}(3+4\sigma-16\sigma^2+\frac{3+28\sigma-80\sigma^2}{\sqrt{1+16\sigma}})\\
K_2(\sigma) =&-\frac{1}{8}(1+4\sigma-16\sigma^2+\frac{1+4\sigma-80\sigma^2}{\sqrt{1+16\sigma}})\\
K(t,\sigma) =&-\frac{1}{16}((4t-1)+(16t-12)\sigma-16\sigma^2+\frac{(4t-1)+(48t-20)\sigma-80\sigma^2}{\sqrt{1+16\sigma}}).
\end{aligned}
$$

3 Properties of CAS wavelets

a and b are two parameters,

$$\psi_{a,b}(t)=|a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right),\quad a,b\in R\ ,\ a\neq 0$$

equality is wavelet function. here a and b parameters $a=a_0^{-k},\ b=nb_0a_0^{-k}$, $a_0>1$, $b_0>0$ n and k positive integers, we have the following family of discrete wavelets;

$$\psi_{k,n}(t)=|a_0|^{\frac{k}{2}}\psi(a_0^k t-nb_0)$$

Where $\psi_{k,n}(t)\in L^2(R)$. In particular, when $a_0=2,\ \ b_0=1$ then $\psi_{k,n}(t)$ forms an orthonormal basis [6].

$$CAS_m(t)=\cos(2m\pi t)+\sin(2m\pi t)$$

$$\psi_{nm}(t)=\begin{cases} 2^{\frac{k}{2}}CAS_m\left(2^k t-n\right), & \frac{n}{2^k}\leq t<\frac{n+1}{2^k}\\ 0\ , & \text{other cases}\end{cases}$$

where $n=0,1,2,\ldots,2^k-1$, k can assume any nonnegative integer, m is any integer [10].

3.1 Function approximation

$$u(t)\in L^2[0,1)$$

$$u(t)=\sum_{n=1}^{\infty}\sum_{m\in Z}c_{nm}\psi_{nm}(t)$$

where,

$$c_{nm}=<u(t),\ \psi_{nm}(t)> \tag{8}$$

In which denotes the inner product. If the infinite series in eq.(8) is truncated, then eq.(8) can be written as

$$u(t)\cong\sum_{n=1}^{2^k}\sum_{m=-M}^{M}c_{nm}\psi_{nm}(t)=C^T\Psi(t)$$

where $M\in Z$, C and $ø(t)$ are $2^k(2M+1)\times 1$ matrices given by

$$C=[c_{1(-M)}\ ,\ c_{1(-M+1)}\ ,\ c_{1(-M+2)}\ ,\ldots,c_{2(-M)},\ \ldots,c_{2^k(-M)},\ldots,c_{2^k(M)}]^T$$

$$C = [c_1,\ c_2,\ \ldots, c_{2^k}]^T$$
$$\Psi(t) = [\psi_{1(-M)}\ ,\ \psi_{1(-M+1)}\ ,\ \psi_{1(-M+2)}\ ,\ldots, \psi_{2(-M)},\ \ldots, \psi_{2^k(-M)}, \ldots, \psi_{2^k(M)}]^T.$$

3.2 Solution of the Fredholm integral equations

$$y(x) = f(x) + \int_0^T K(x,t)y(t)dt \tag{9}$$

Eq.(9) is a fredholm integral equations.

$$y(x) = C^T \Psi(x)$$
$$f(x) = d^T \Psi(x)$$

and

$$K(x,t) = \Psi(x)^T K \Psi(x)$$

where C and $\Psi(x)$ are known functions, $K(x,t)$ is kernel function, K is $2^k(2M+1)\times 2^k(2M+1)$ matrices where the elements of K calculated as follows ,

$$K = \int_0^1 \int_0^1 \Psi_{ni}(x)\Psi_{lj}(t)K(x,t)dtdx$$

where $n = 1,\ldots,2^k$, $i = -M,\ldots,M$,$l = 1,\ldots,2^k$, $j = -M,\ldots,M$
then

$$C^T\Psi(x) = d^T\Psi(x) + \lambda \int_0^1 \Psi(x)^T K\Psi(x)\Psi(x)^T C dt$$

$$\Psi(x)^T C = \Psi(x)^T d + \lambda \Psi(x)^T KC \tag{10}$$

eq.(10) is a linear systems interms of C and the answer is

$$C = (I-K)^{-1}d$$

where **I** is identity matrix.

3.3 Solution of the Volterra integral equations

$$x(t) = f(t) + \int_0^t K(t,s)x(s)ds \tag{11}$$

Eq.(11) is a volterra integral equattions.

$$K^*(t,s) = \begin{cases} K(t,s)\ ,\quad 0\le t\le T\ ,\ 0\le s\le T \\ \quad 0\ ,\ 0\le t\le T\ ,\ t\le s\le T \end{cases}$$

the property

$$x(t) = f(t) + \int_0^T K^*(t,s)x(s)ds$$

Fredholm integral equation is obtained [2].

3.4 Solution of the (1) integral equations

$$x''(t)+tx\left(t-\tfrac{1}{2}\sqrt{t}\right)=-2t^3+2t^{\frac{5}{2}}+\tfrac{5}{2}t^2-\tfrac{3}{2}t^{\frac{3}{2}}-4\,,(0\le t\le 1)$$
$$x(t)=0\,,\ \left(-\tfrac{1}{16}\le t\le 0\right)\ ,\ x(1)=x(\tfrac{1}{2})$$

Fredholm - Volterra integral equation equivalent to the (7) problem

$$\begin{aligned}x(t)&=2,872933230t-2t^2-0,1714285714t^{\frac{7}{2}}+0,2083333333t^4+0,1269841270t^{\frac{9}{2}}\\&-0,1t^5+\frac{t}{8}\int_0^{\frac{1}{2}}\left[3+4\sigma-16\sigma^2+\frac{3+28\sigma-80\sigma^2}{\sqrt{1+16\sigma}}\right]x(\sigma)\,d\sigma\\&+\frac{t}{8}\int_0^{\frac{1}{2}-\frac{\sqrt{2}}{4}}\left[1+4\sigma-16\sigma^2+\frac{1+4\sigma-80\sigma^2}{\sqrt{1+16\sigma}}\right]x(\sigma)\,d\sigma\\&-\frac{1}{16}\int_0^{t-\frac{\sqrt{t}}{2}}\left[(4t-1)+(16t-12)\sigma-16\sigma^2+\frac{(4t-1)+(48t-20)\sigma-80\sigma^2}{\sqrt{1+16\sigma}}\right]x(\sigma)\,d\sigma\end{aligned}$$

where,

$$f(x)=2,872933230x-2x^2-0,1714285714x^{\frac{7}{2}}+0,2083333333x^4+0,1269841270x^{\frac{9}{2}}-0,1x^5$$

$$K_1(x)=\frac{1}{8}(3+4x-16x^2+\frac{3+28x-80x^2}{\sqrt{1+16x}})$$

$$K_2(x)=-\frac{1}{8}(1+4x-16x^2+\frac{1+4x-80x^2}{\sqrt{1+16x}})$$

$$K(x,t)=(4x-1)+(16x-12)t-16t^2+\frac{(4x-1)+(48x-20)t-80t^2}{\sqrt{1+16t}}$$

approximate solution

$$y(x)=C^T\Psi(x)$$
$$f(x)=d^T\Psi(x)$$

and

$$d=[-0,163928\ \ 0,296835\ \ -0,132907\ \ -0,276517\ \ 0,728253\ \ 0,219904]^T$$

$$\frac{t}{8}\int_0^{\frac{1}{2}}\left[3+4\sigma-16\sigma^2+\frac{3+28\sigma-80\sigma^2}{\sqrt{1+16\sigma}}\right]x(\sigma)\,d\sigma$$

for the first integral equation,

$$K=\int_0^1\int_0^1\Psi_{ni}(x)\Psi_{lj}(t)K(x,t)dtdx$$

$$K_1=\begin{bmatrix}0&-0,29232&0&0&-0,29232&0\\0&-1,0156&0&0&-1,0156&0\\0&0,38861&0&0&0,38861&0\\0&-0,23447&0&0&-0,23447&0\\0&-2,7792&0&0&-2,7792&0\\0&0,24657&0&0&0,24657&0\end{bmatrix}$$

and

$$C=(I-K)^{-1}d$$

$$C=[-0,22642\ \ 7,9712\times10^{-2}\ \ -4,9826\times10^{-2}\ \ -0,32665\ \ 0,13408\ \ 0,27262]^T$$

$$y_1(x) = 1.\,0972 sin(12.\,566x) - 0.46708 cos(12.\,566x) + 0.30235$$

$$\frac{t}{8}\int_0^{\frac{1}{2}-\frac{\sqrt{2}}{4}}\left[1+4\sigma-16\sigma^2+\frac{1+4\sigma-80\sigma^2}{\sqrt{1+16\sigma}}\right]x(\sigma)\,d\sigma$$

and then

$$K_2 = \begin{bmatrix} 0 & 0,35460 & 0 & 0 & 0,35460 & 0 \\ 0 & 1,4844 & 0 & 0 & 1,4844 & 0 \\ 0 & -0,45695 & 0 & 0 & -0,45695 & 0 \\ 0 & 0,28969 & 0 & 0 & 0,28969 & 0 \\ 0 & 3,6247 & 0 & 0 & 3,6247 & 0 \\ 0 & -0,30332 & 0 & 0 & -0,30332 & 0 \end{bmatrix}$$

$$C = (I-K)^{-1}d = [-0.25239 \;\; -7.\,3478\times10^{-2} \;\; -1.\,8917\times10^{-2} \;\; -0.34879 \;\; -0.1760.29557]^T$$

$$y_2(x) = \; 1.\,2414 sin(12.\,566x) - 0.45895 cos(12.\,566x) - 0.35282$$

$$\frac{1}{16}\int_0^{t-\frac{\sqrt{t}}{2}}\left[(4t-1)+(16t-12)\sigma-16\sigma^2+\frac{(4t-1)+(48t-20)\sigma-80\sigma^2}{\sqrt{1+16\sigma}}\right]x(\sigma)\,d\sigma$$

$$\text{K=}\begin{bmatrix} -4,9563\times10^{-3} & -3,7302\times10^{-2} & 5,4378\times10^{-3} & -4,3944\times10^{-3} & -6,7278\times10^{-2} & 4,516\times10^{-3} \\ 0,19287 & 0,85938 & -0,24556 & 0,15865 & 2,0237 & -0,16585 \\ 4,9563\times10^{-3} & 3,7302\times10^{-2} & -5,4378\times10^{-3} & 4,3944\times10^{-3} & 6,7278\times10^{-2} & -4,516\times10^{-3} \\ -4,9563\times10^{-3} & -3,7302\times10^{-2} & 5,4378\times10^{-3} & -4,3944\times10^{-3} & -6,7278\times10^{-2} & 4,516\times10^{-3} \\ 0,16173 & 0,625 & -0,21139 & 0,13104 & 1,6010 & -0,13747 \\ 4,9563\times10^{-3} & 3,7302\times10^{-2} & -5,4378\times10^{-3} & 4,3944\times10^{-3} & 6,7278\times10^{-2} & -4,516\times10^{-3} \end{bmatrix}$$

$$C = [-0.11499 \;\; -0.92852 \;\; -0.18185 \;\; -0.22757 \;\; -0.19045 \;\; 0.17098]^T$$

$$y_3(x) = \; 0.46908 sin(12.\,566x) - 0.49983 cos(12.\,566x) - 1.\,5825$$

therefore

$$\begin{aligned} y(x) =& 1.2414 sin(12.56x) - 0.45895 cos(12.56x) - 0.26234 + 0.46908 sin(12.56x) \\ & - 0.49983 cos(12.56x) - 1.5825 + 1.0972 sin(12.56x) \\ & - 0.46708 cos(12.56x) + 0.30235, \\ y(x) =& 2.8077 sin(12.\,56x) - 1.4259\, cos(12.56x) - 1.5425 \end{aligned}$$

$y(x)$ is the approximate solution.

4 Conclusion

The problem of delayed variable boundary value has already been solved by different methods. This problem was first transformed into the Fredholm-Volterra integral equation. Then the CAS wavelet methodwas applied and the approximate solution was found. So alternatif solution is found.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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