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Stability analysis of generalized Ebola Hemorrhagic Fever model

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Abstract: In this paper, we present a generalized Ebola Hemorrhagic Fever model in Caputo sense, which is assumed to have a constant size of the total population over the period of the disease. We show that this model possesses non-negative solutions as desired in any population dynamics. The stability of different equilibria of this model are discussed in detail. Natural-Adomian Decomposition method (N-ADM) is used to compute an analytical solution of the system of nonlinear fractional differential equations governing the problem. The results are compared with the results obtained by the classical Runge-Kutta method in the case of integer-order derivatives.

Keywords: Fractional differential equations, Ebola Hemorrhagic Fever model, stability, The Natural-Adomian Decomposition method.

1 Introduction

Ebola is a deadly virus that attacks healthy cells and replicates itself in a host's body. There exist five Ebola viruses according to the International Committee on Taxonomy of Viruses currently: Ebola virus (EBOV), Sudan virus (SUDV), Reston virus (RESTV),Taï Forest virus (TAFV), and Bundibugyo virus (BDBV). Ebola is an unusual nevertheless fatal virus that causes bleeding inside and outside the body [1]. Ebola virus is transmitted initially to human by contact with an infected animal's body fluid. A mathematical description of the spread of Ebola virus based on the basic SEIR model has been carried out, e.g. in [2,3]. Discrete SEIR time models to Ebola epidemics are available in [4]. In [5] transmission Dynamics of Ebola Virus is studied. In [6] the properties of SEIR models with respect to Ebola Virus,the basic SEIR demographic effects and numerical simulation of Ebola Virus are discussed. To describe the Ebola Hemorrhagic Fever, A. Atangana and E. F. D. Goufo proposed the following nonlinear system of ordinary differential equations [7].

$$\begin{cases} \frac{dS(t)}{dt} = -iS(t)I(t) + \gamma R(t) - \beta N, \\ \frac{dI(t)}{dt} = iS(t)I(t) - dI(t) - rI(t), \\ \frac{dR(t)}{dt} = rI(t) - \gamma R(t), \\ \frac{dD(t)}{dt} = dI(t) + \beta N. \end{cases} \tag{1}$$

In this model the total population N consists of four types where, at time t, $S(t)$ is the number of susceptible individuals, $I(t)$ is the number of infected individuals, able to spread the disease by contact with susceptible, $R(t)$ is the number of

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recovery individuals and $D(t)$ is the total death population. Moreover, the rate of death caused by natural death and other diseases is factored out to be β. i, r, γ and d are to be rate of infection by Ebola, rate of recovery, rate of susceptibility, and rate of death by Ebola, respectively.

The fractional order extension of the Ebola model have been studied in [7,8] and [9]. The reason of using fractional differential equations (FDEs) is that FDEs are naturally related to systems with memory which exists in most biological system. Also they show the realistic biphasic decline behavior of infection of diseases but at a slower rate. In our work, we consider fractional-order for the system (1), where $D^{\alpha}S, D^{\alpha}I, D^{\alpha}R$ and $D^{\alpha}D$ are the derivatives of $S(t), I(t), R(t)$ and $D(t)$ respectively, of arbitrary order α (where $0<\alpha\leq 1$) in the sense of Caputo (see e.g. [10]), then our system is described by the following set of fractional order differential equations,

$$\begin{cases} D^{\alpha}S(t) = -iS(t)I(t) + \gamma R(t) - \beta N, \\ D^{\alpha}I(t) = iS(t)I(t) - dI(t) - rI(t), \\ D^{\alpha}R(t) = rI(t) - \gamma R(t), \\ D^{\alpha}D(t) = dI(t) + \beta N, \end{cases} \tag{2}$$

subject to the initial conditions in Table 1. The motivation of this paper is to find analytical solution for the generalized Ebola Hemorrhagic Fever model in the sense of Caputo by using the N-ADM. The rest of the paper is organized as follow. In Section 2, a brief review of the fractional calculus and definitions of Natural, Laplace transform and Mittag-Leffler function is presented. In Section 3, we apply the Natural-Adomian Decomposition method for obtaining the solution of the generalized Ebola Hemorrhagic Fever model. In Section 4, we show that the model (2) possesses a unique solution which is non-negative. Section 5 is devoted to study the equilibrium points and the stability analysis of our model (2). Numerical simulations are represented graphically and discussed in Section 6.

2 Preliminary

Here, we present some necessary definitions and notations related to fractional calculus (see e.g. [10]) and the Natural transform [11,12,13]. The most commonly used definitions are Riemann-Liouville and Caputo.

Definition 1. *The Riemann-Liouville fractional integeration of order α is defined as:*

$$(J^{\alpha}_{t_0}f)(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s)ds,\ \alpha>0,\ t>t_0,$$
$$(J^{0}_{t_0}f)(t) = f(t).$$

The Riemann-Liouville derivative has certain disadvantages such that the fractional derivative of a constant is not zero. Therefore, we will make use of Caputo's definition owing to its convenience for initial conditions of the fractional differential equations.

Definition 2. *Riemann-Liouville and Caputo fractional derivatives of order α can be defined respectively as:*

$$D^{\alpha}_{*}f(t) = D^{n}(J^{n-\alpha}f(t)),$$
$$D^{\alpha}f(t) = J^{n-\alpha}(D^{n}f(t)),$$

where $n-1<\alpha\leq n,\ n\in\mathbb{N}$, f is a given function, and $\Gamma(\cdot)$ denotes the gamma function. It is known that $(D^{\alpha}_{t_0}f)(t)\rightarrow f'(t)$ as $\alpha\rightarrow 1$. Now, we recall the definitions of Natural transform, Laplace transform of Caputo's derivative and Mittag-Leffler function in two arguments.

Definition 3. *Over the set of functions,*

$$A=\{f(t):\exists M,\tau_1,\tau_2>0,|f(t)|<Me^{|t|/\tau_j},\ if\, t\in(-1)^j x[0,\infty)\}.$$

The Natural transform of $f(t)$ is defined by

$$\mathscr{N}\{f(t)\}=R(s,u)=\int_0^{\infty}f(ut)\,e^{-st}dt,\ u>0,\ s>0,$$

where $R(s,u)$ is the Natural transform of the time function $f(t)$.

Theorem 1. *If $\mathscr{N}\{f(t)\}$ is the natural transform of the function $f(t)$, then the natural transform of the fractional derivative of order α is defined as:*

$$\mathscr{N}\{D^{\alpha}(f(t))\}=\frac{s^{\alpha}}{u^{\alpha}}R(s,u)-\sum_{k=0}^{n-1}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}f^{(k)}(0)$$

Definition 4.

$$\mathscr{L}\{D^{\alpha}f(t),s\}=s^{\alpha}F(s)-\sum_{i=0}^{n-1}s^{\alpha-i-1}f^{(i)}(0),\ \ (n-1<\alpha\leq n);\ n\in\mathbb{N}.$$

$$E_{a,b}(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(an+b)},\ a>0,\ b>0.$$

3 The Natural-Adomian Decomposition Method (N-ADM)

Consider the fractional-order Ebola model (2) subject to the initial condition in Table 1. The nonlinear term in this model Eqs. (2) is $S\,I$ and $i,\beta,\gamma,d,\ r$ are known constants. For $\alpha=1$ the fractional order model converts to the classical Ebola model (see e.g.[7]). Applying the Natural transform on both sides of Eqs. (2)

$$\begin{cases}\mathscr{N}\{D^{\alpha}(S)\}=-i\,\mathscr{N}\{S(t)I(t)\}+\gamma\,\mathscr{N}\{R(t)\}-\beta N\mathscr{N}\{1\},\\[2ex] \mathscr{N}\{D^{\alpha}(I)\}=i\,\mathscr{N}\{S(t)I(t)\}-d\,\mathscr{N}\{I(t)\}-r\,\mathscr{N}\{I(t)\},\\[2ex] \mathscr{N}\{D^{\alpha}(R)\}=r\,\mathscr{N}\{I(t)\}-\gamma\,\mathscr{N}\{R(t)\},\\[2ex] \mathscr{N}\{D^{\alpha}(D)\}=d\,\mathscr{N}\{I(t)\}+\beta N\mathscr{N}\{1\},\end{cases}\tag{3}$$

using property of the Natural transform, we get

$$\begin{cases} \frac{s^\alpha}{u^\alpha}\mathscr{N}\{S\}-\frac{s^{\alpha-1}}{u^\alpha}S(0)=-i\,\mathscr{N}\{S(t)I(t)\}+\gamma\,\mathscr{N}\{R(t)\}-\beta N\mathscr{N}\{1\}, \\ \frac{s^\alpha}{u^\alpha}\mathscr{N}\{I\}-\frac{s^{\alpha-1}}{u^\alpha}I(0)=i\,\mathscr{N}\{S(t)I(t)\}-d\,\mathscr{N}\{I(t)\}-r\,\mathscr{N}\{I(t)\}, \\ \frac{s^\alpha}{u^\alpha}\mathscr{N}\{R\}-\frac{s^{\alpha-1}}{u^\alpha}R(0)=r\,\mathscr{N}\{I(t)\}-\gamma\,\mathscr{N}\{R(t)\}, \\ \frac{s^\alpha}{u^\alpha}\mathscr{N}\{D\}-\frac{s^{\alpha-1}}{u^\alpha}D(0)=d\,\mathscr{N}\{I(t)\}+\beta N\mathscr{N}\{1\}, \end{cases} \tag{4}$$

using initial condition from Table 1

$$\begin{cases} \mathscr{N}\{S\}=\frac{900}{s}-i\,\frac{u^\alpha}{s^\alpha}\mathscr{N}\{S(t)I(t)\}+\gamma\frac{u^\alpha}{s^\alpha}\mathscr{N}\{R(t)\}-\beta N\frac{u^\alpha}{s^\alpha}\mathscr{N}\{1\}, \\ \mathscr{N}\{I\}=\frac{10}{s}+i\,\frac{u^\alpha}{s^\alpha}\mathscr{N}\{S(t)I(t)\}-d\frac{u^\alpha}{s^\alpha}\mathscr{N}\{I(t)\}-r\frac{u^\alpha}{s^\alpha}\,\mathscr{N}\{I(t)\}, \\ \mathscr{N}\{R\}=r\frac{u^\alpha}{s^\alpha}\,\mathscr{N}\{I(t)\}-\gamma\frac{u^\alpha}{s^\alpha}\,\mathscr{N}\{R(t)\}, \\ \mathscr{N}\{D\}=d\frac{u^\alpha}{s^\alpha}\,\mathscr{N}\{I(t)\}+\beta N\frac{u^\alpha}{s^\alpha}\mathscr{N}\{1\}. \end{cases} \tag{5}$$

The method assumes the solution as an infinite series:

$$S=\sum_{k=0}^{\infty}S_k,\quad I=\sum_{k=0}^{\infty}I_k,\quad R=\sum_{k=0}^{\infty}R_k,\quad D=\sum_{k=0}^{\infty}D_k. \tag{6}$$

The nonlinearity SI is decomposed as

$$SI=\sum_{k=0}^{\infty}A_k, \tag{7}$$

where A_k so-called Adomian Polynomials given as,

$$A_k=\frac{1}{k!}\frac{d^k}{d\lambda^k}\left[\sum_{j=0}^{k}\lambda^j S_j\sum_{j=0}^{k}\lambda^j I_j\right]\Bigg|_{\lambda=0}. \tag{8}$$

Substituting from Eqs. (8), (7) and (6) into (5) the result is,

$$\begin{cases} \mathscr{N}\{S_0\}=\frac{900}{s}-\beta N\frac{u^\alpha}{s^{\alpha+1}}, \\ \mathscr{N}\{I_0\}=\frac{10}{s}, \\ \mathscr{N}\{R_0\}=0, \\ \mathscr{N}\{D_0\}=\beta N\frac{u^\alpha}{s^{\alpha+1}}, \end{cases} \tag{9}$$

$$\begin{cases} \mathscr{N}\{S_1\} = -i\, \frac{u^\alpha}{s^\alpha} \mathscr{N}\{A_0\} + \gamma \frac{u^\alpha}{s^\alpha} \mathscr{N}\{R_0\}, \\ \mathscr{N}\{I_1\} = i\, \frac{u^\alpha}{s^\alpha} \mathscr{N}\{A_0\} - d\frac{u^\alpha}{s^\alpha} \mathscr{N}\{I_0\} - r\frac{u^\alpha}{s^\alpha} \mathscr{N}\{I_0\}, \\ \mathscr{N}\{R_1\} = r\frac{u^\alpha}{s^\alpha} \mathscr{N}\{I_0\} - \gamma\frac{u^\alpha}{s^\alpha} \mathscr{N}\{R_0\}, \\ \mathscr{N}\{D_1\} = d\frac{u^\alpha}{s^\alpha} \mathscr{N}\{I_0\}, \end{cases} \tag{10}$$

$$\vdots$$

$$\begin{cases} \mathscr{N}\{S_{k+1}\} = -i\, \frac{u^\alpha}{s^\alpha} \mathscr{N}\{A_k\} + \gamma \frac{u^\alpha}{s^\alpha} \mathscr{N}\{R_k\}, \\ \mathscr{N}\{I_{k+1}\} = i\, \frac{u^\alpha}{s^\alpha} \mathscr{N}\{A_k\} - d\frac{u^\alpha}{s^\alpha} \mathscr{N}\{I_k\} - r\frac{u^\alpha}{s^\alpha} \mathscr{N}\{I_k\}, \\ \mathscr{N}\{R_{k+1}\} = r\frac{u^\alpha}{s^\alpha} \mathscr{N}\{I_k\} - \gamma\frac{u^\alpha}{s^\alpha} \mathscr{N}\{R_k\}, \\ \mathscr{N}\{D_{k+1}\} = d\frac{u^\alpha}{s^\alpha} \mathscr{N}\{I_k\}. \end{cases}$$

The aim is to study the mathematical behavior of the solution $S(t)$, $I(t)$, $R(t)$, $D(t)$ for the different values of α. By applying the inverse Natural transform to both sides of Eqs.(9) the values of S_0, I_0, R_0, D_0 are obtained. Substituting these values of A_0, I_0, R_0 into Eqs.(10), the first component S_1, I_1, R_1, D_1 are obtained. The other terms of $S_2, S_3, \dots,$ $I_2, I_3, \dots$ $R_2, R_3, \dots$ and $D_2, D_3, \dots$ can be calculated recursively in a similar way and we can write the solution,

$$S(t) = S_0 + S_1 + \dots, I(t) = I_0 + I_1 + \dots, R(t) = R_0 + R_1 + \dots, D(t) = D_0 + D_1 + \dots. \tag{11}$$

4 Non-negative solutions

Let $\mathbb{R}^4_+ = \{X \in \mathbb{R}^4 | X \geq 0\}$ and $X(t) = (S(t), I(t), R(t), D(t))^T$, we now prove the main theorem.

Theorem 2. *There is a unique solution $X(t) = (S(t), I(t), R(t), D(t))^T$ for model (2) at $t \geq 0$ (where, $t_0 = 0$) and the solution will remain in $\mathbb{R}^4_+$.*

Proof. From Theorem 3.1 and Remark 3.2 of [14], we know that the solution on $(0, \infty)$ is existent and unique. Now, we will show that the feasible region $\mathbb{R}^4_+$ is positively invariant (non-negative solutions). Rearranging the following equation (for the recovery population)

$$D^\alpha R(t) + \gamma R(t) = rI(t),$$

and we assume that $g(t) = rI$ is a constant function of time. Then we get the fractional order differential equation representing the recovery population as follows:

$$D^\alpha R(t) + \gamma R(t) = g(t). \tag{12}$$

Solving equation (12) using Laplace transform (from Definition 4) method [10] and taking the initial condition to be zero (to simplify), we have the following solution

$$R(t) = \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\gamma (t-\tau)^\alpha) g(\tau) d\tau \geq 0,$$

where $0 < \alpha < 1,\ \gamma > 0$ and $E_{a,b}(x)$ is the two-parameter Mittag-Leffler function (see Definition 4). For $S(t),\ I(t)$ and $D(t)$ by the same way we have $S(t), I(t), D(t) \geq 0$, hence proved that the solution $X(t)$ is positive invariant.

5 The stability of the equilibrium points

We frist evaluate the equilibrium points or steady states of the following system of the FDEs.

$$\begin{cases} D^{\alpha}S(t) = -iS(t)I(t) + \gamma R(t) - \beta N, \\ D^{\alpha}I(t) = iS(t)I(t) - dI(t) - rI(t), \\ D^{\alpha}R(t) = rI(t) - \gamma R(t). \end{cases} \tag{13}$$

To evaluate the equilibrium points, let

$$\begin{cases} D^{\alpha}S = 0, \\ D^{\alpha}I = 0, \\ D^{\alpha}R = 0, \end{cases} \tag{14}$$

then, the system (13) has two equilibrium points

1. At disease-free equilibrium: We now consider the equations below and solve for the values S and R, since at this point there is no infection, thus from (14)

$$-iSI + \gamma R - \beta N = 0, \tag{15}$$

$$iSI - (d + r)I = 0, \tag{16}$$

$$rI - \gamma R = 0. \tag{17}$$

From equation (16), we have $I = 0$, $S = \frac{d+r}{i}$ and by substituting in equation (17). Then disease-free equilibrium (DFE) of the system (2) is

$$E^0 = (S_{eq}, I_{eq}, R_{eq})_{I=0} = \left(\frac{d+r}{i},\ 0,\ 0 \right).$$

2. At endemic equilibrium: We now consider the case where there is infection, thus from equation (16) $S = \frac{d+r}{i}$ by substituting in equations (17) and (15), then we have

$$E^* = (S_{eq}, I_{eq}, N_{eq})_{I \neq 0} = (S^*, I^*, N^*) = \left(\frac{d+r}{i},\ \frac{-\beta N}{d},\ \frac{-r\beta N}{d\gamma} \right).$$

We can note that the equilibrium points are the same for both integer and fractional system. But the stability region of the fractional-order system with order α, which is illustrated in Figure 1 (where σ, ω refer to the real and imaginary parts of the eigenvalues, respectively, and $j = \sqrt{-1}$), is greater than the stability region of the integer order case (see e.g.[15]). Therefore, we will now drive analytically the stability of different equilibria of the model (2). For E^0 , we have the following theorem,

Theorem 3. *The disease free equilibria E^0 of the system (2) is local stable.*

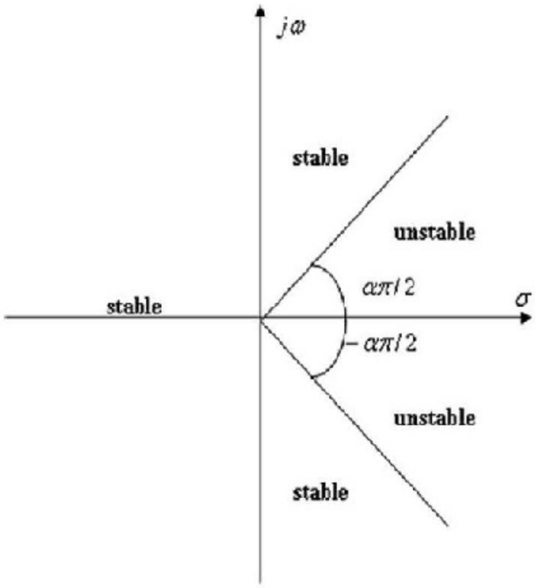

Fig. 1: Stability region of the fractional-order system.

Proof. Determining the Jacobian matrix of the system (2) at E^0 we have,

$$J_{E^0} = \begin{bmatrix} 0 & -(d+r) & \gamma \\ 0 & 0 & 0 \\ 0 & r & -\gamma \end{bmatrix}$$

The eigenvalues of J_{E^0} are

$$\lambda_1 = \lambda_2 = 0,\ \lambda_3 = -\gamma < 0.$$

In this case, where the eigenvalue is negative for the steady state E^0, then it will be linearly (locally) stable (see e.g. [16]).

For E^*, we have the following theorem,

Theorem 4. *The endemic equilibrium point E^* of the system (2) is local unstable.*

Proof. The Jacobian matrix evaluated at the endemic equilibrium gives

$$J_{E^*} = \begin{bmatrix} B & -(d+r) & \gamma \\ -B & 0 & 0 \\ 0 & r & -\gamma \end{bmatrix} \text{ where } \quad B = \frac{i\beta N}{d},$$

and its eigenvalues can be obtained by solving the following characteristic equation

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \tag{18}$$

with a_1, a_2 and a_3 being

$$a_1 = \gamma - B,\ a_2 = -B\,(\gamma + d + r),\ a_3 = -i\beta N\gamma,$$

where $a_1, a_2, a_3 < 0$, then from Descartes' rule of signs it is clear that the characteristic equation (18) has one change. Accordingly, there is at least one positive real root. Now set $w = -\lambda$,

$$-w^3 + a_1 w^2 - a_2 w + a_3 = 0.$$

There are now two sign changes and thus at most two real negative root for λ (see e.g. [16]). Then the steady state E^* of the system is (locally) unstable.

6 Numerical results and discussion

The N-ADM provides an analytical approximate solution in terms of an infinite power series. For numerical results, the following values, for parameters, are considered [7].

Parameter	$S(0)$	$I(0)$	$R(0)$	$D(0)$	N	β	r	i	γ	d
Values	900	10	0	0	1000	0.01	0.4	0.01	0.02	0.6

Table 1: Parameters values.

The first few components of N-ADM solution $S(t)$, $I(t)$, $R(t)$ and $D(t)$ are calculated. We computed the first three terms of the N-ADM series solution for the system (2). We present one of them as follows,

$$S_1 = \frac{-9000i}{\Gamma(\alpha+1)} t^{\alpha} + \frac{10\beta Ni}{\Gamma(2\alpha+1)} t^{2\alpha},\ I_1 = \frac{(9000i - 10(d+r))}{\Gamma(\alpha+1)} t^{\alpha} - \frac{10\beta Ni}{\Gamma(2\alpha+1)} t^{2\alpha},$$

$$R_1 = \frac{10r}{\Gamma(\alpha+1)} t^{\alpha},\ D_1 = \frac{10d}{\Gamma(\alpha+1)} t^{\alpha}.$$

thus, the N-ADM series solution of the system (2) can be given by Eqs.11. With the values of initial conditions and parameters in Table 1. The approximate solutions displayed in Figs. 2-3 with different value of fractional order $0 < \alpha \leq 1$ and it is clear that the number of susceptible decrease in the beginning of time interval while the number of infected, recovery, total death population increases.

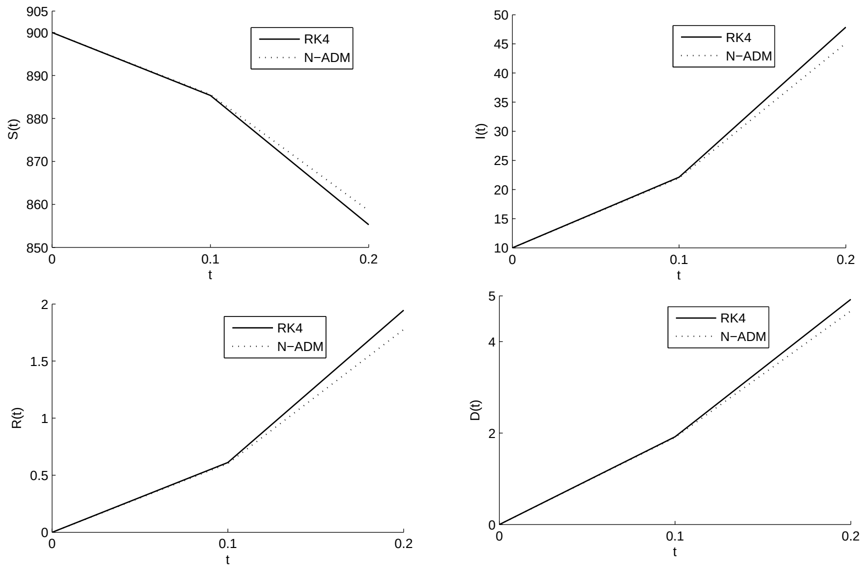

Fig. 2: The numerical results for $S(t)$, $I(t)$, $R(t)$ and $D(t)$ at $\alpha = 1$ compared with RK4.

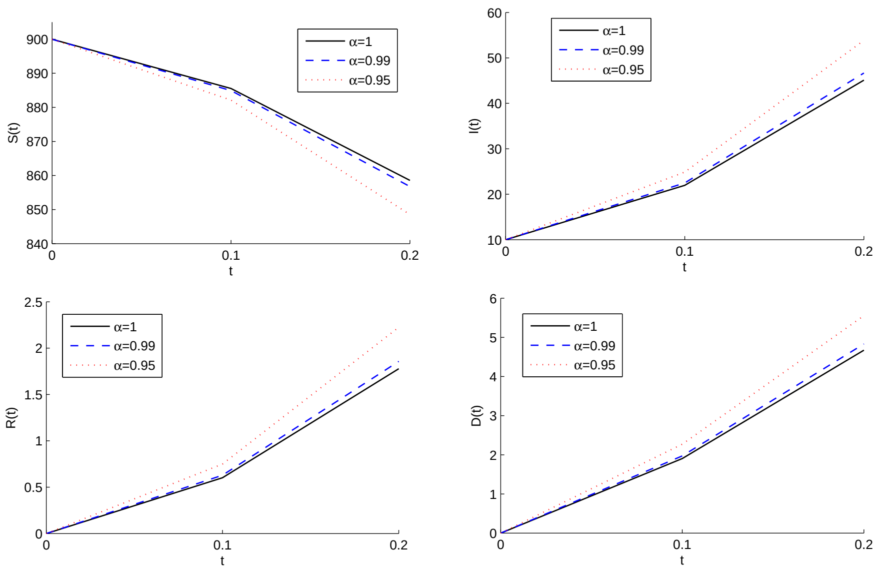

Fig. 3: The numerical results for $S(t)$, $I(t)$, $R(t)$ and $D(t)$ at different values of α.

7 Conclusion

In this work we have introduced a generalized of Ebola Hemorrhagic Fever model in Caputo sense, describing the spread of a fatal disease in a given population. We obtained the non-negative solutions of the fractional model by Laplace transform. The disease free equilibria of the given model is locally stable and the endemic equilibrium point is locally unstable. Using the Natural-Adomian Decomposition method for solving the generalized of Ebola Hemorrhagic Fever model in Caputo sense. The comparison for some different values of α has been obtained.

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