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Ag-convex functions

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Abstract: In this paper, the concept of Ag-convex function is given the first time in the literature. Some inequalities of Hadamard's type for Ag-convex functions are given. Some special cases are discussed.

Keywords: Convex function, Ag-convex, Hermite-Hadamard inequality.

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1 Introduction

It is well known that convexity theory plays a central and fundamental role in the fields of mathematical finance, economics, engineering, management sciences, and optimization theory. In recent years, the concept of convexity has been extended and generalized in several directions using the novel and innovative ideas; see, for example, [1,3,4,5,6] and the references therein.

Definition 1. *A function* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *is said to be convex if the inequality*

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y),$$

is valid for all $x, y \in I$ *and* $t \in [0,1]$. *If this inequality reverses, then* f *is said to be concave on interval* $I \neq \emptyset$. *This definition is well known in the literature. Denote by* $C(I)$ *the set of the convex functions on the interval* I.

Definition 2. $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a convex function defined on the interval* I *of real numbers and* $a, b \in I$ *with* $a < b$. *The following inequality,*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions.

In [2], Cristescu obtained the following integral inequalities for products of convex functions.

Theorem 1. *Let* f *and* g *be real-valued, nonnegative and convex functions on* $[a,b]$. *Then*

$$\begin{aligned}&\frac{3}{2}\frac{1}{(b-a)^2}\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)\,g(f(tx+(1-t)y))\,dtdydx,\\ &\leq \frac{1}{b-a}\int_a^b\int_0^1 f(x)g(x)dx+\frac{1}{8}\left[M(a,b)+N(a,b)\right],\end{aligned} \tag{2}$$

and,

$$\begin{aligned}&\frac{3}{b-a}\int_a^b\int_0^1 f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)dtdx,\\ &\leq \frac{1}{b-a}\int_a^b f(x)g(x)dx+\frac{1}{2}\left[M(a,b)+N(a,b),\right]\end{aligned} \tag{3}$$

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where,

$$M(a,b)=f(a)g(a)+f(b)g(b),$$

and,

$$N(a,b)=f(a)g(b)+f(b)g(a).$$

The main purpose of this paper is to give a new class of convex functions called as Ag-convex function and establish both the Hermite-Hadamard type integral inequalities and new inequalities related to the products of Ag-convex functions. The results obtained in special cases are reduced to the results obtained in the literature.

2 Main result for Ag-convex functions

Definition 3. *Let $I\subset\mathbb{R}$ be an interval, $f:I\to\mathbb{R}$, $g:J\to\mathbb{R}$, $J\supset f(I)$. f is said to be Ag-convex if the inequality,*

$$f\left(tx+(1-t)y\right)\leq tg\left(f(x)\right)+(1-t)g\left(f(y)\right), \tag{4}$$

is valid for all $x,y\in I$ and $t\in[0,1]$. Denote by $AgC(I)$ the set of the Ag-convex functions on the interval I.

If the function g satisfies the condition $g(x)\leq x, x\in f(I)$, then the function f is also convex.

Proposition 1. *Let $I\subset\mathbb{R}$ be an interval, $f:I\to\mathbb{R}$, $g:J\to\mathbb{R}$ and $J\supset f(I)$. If f is Ag-convex, then $y\leq g(y)$ for every $y\in f(I)$.*

Proof. Let $y\in f(I)$ be arbitrary. Then, there exists a $x\in I$ such that $y=f(x)$. If we take $a\in I\backslash\{x\}$ as a constant, then since the function f is Ag-convex, for every $t\in[0,1]$

$$f\left(tx+(1-t)a\right)\leq t\left(gof\right)(x)+(1-t)\left(gof\right)(a).$$

For $t=1$, $f(x)\leq g\left(f(x)\right)$, that is $y\leq g(y)$. This show us that $y\leq g(y)$ for every $y\in f(I)$.

Remark. (i) According to the Preposition 1, every convex function is Ag-convex function. Really, for every $t\in[0,1]$ and every $a,b\in I$,

$$\begin{aligned}f\left(ta+(1-t)b\right)&\leq tf(a)+(1-t)f(b),\\&\leq tg\left(f(a)\right)+(1-t)g\left(f(b)\right),\\&\leq t\left(gof\right)(a)+(1-t)\left(gof\right)(b).\end{aligned}$$

This inequalities show that $C(I)\subseteq AgC(I)$.

(ii) But, the above is not always true. That is, every Ag *-convex function may not be convex function. For example,*let $f:(-\infty,0)\to\mathbb{R}$, $f(x)=\frac{1}{x}$, $g:\mathbb{R}\to\mathbb{R}, g(x)=-x$. For all $x,y\in(-\infty,0)$, since,

$$\frac{1}{tx+(1-t)y}\leq -t\frac{1}{x}-(1-t)\frac{1}{y},$$

the function f is Ag-convex. But this function is not convex on $(-\infty,0)$.
(iii) It is obvious that $AgC(I)=C(I)\Leftrightarrow g(x)=x$.

Theorem 2. *Let $c\in[0,\infty)$. If f is Ag-convex function and g is linear, then cf is Ag-convex function.*

Proof. For $c\in[0,\infty)$,

$$\begin{aligned}(cf)\left(tx+(1-t)y\right)&\leq c\left[tg(f(x))+(1-t)g(f(y))\right],\\&=tg\left(cf(x)\right)+(1-t)g(cf(y)),\\&=t\left(go(cf)\right)(x)+(1-t)\left(go(cf)\right)(y).\end{aligned}$$

This completes the proof of theorem.

Theorem 3. *If the functions f,h are Ag-convex and g is linear, then $f+h$ is Ag-convex function.*

Proof. For $x,y \in I$ and $t \in [0,1]$,

$$\begin{aligned}(f+h)(tx+(1-t)y) &= f((tx+(1-t)y)) + h((tx+(1-t)y)),\\ &\leq [tg(f(x)) + (1-t)g(f(y))],\\ &+[tg(h(x)) + (1-t)g(h(y))],\\ &= t[g(f(x)) + g(h(x))] + (1-t)[g(f(y)) + g(h(y))],\\ &= tg(f(x)+h(x)) + (1-t)g(f(y)+h(y)),\\ &= t(go(f+h))(x) + (1-t)(go(f+h))(y).\end{aligned}$$

This completes the proof of theorem.

Theorem 4. *If the function f is Ag-convex and monotone increasing, and h is convex, then foh is Ag-convex function.*

Proof. For $x,y \in I$ and $t \in [0,1]$,

$$\begin{aligned}(foh)(tx+(1-t)y) &= f(h(tx+(1-t)y)),\\ &\leq f(th(x) + (1-t)h(y)),\\ &\leq tg(f(h(x))) + (1-t)g(f(h(y))),\\ &\leq t(go(f(h)))(x) + (1-t)(go(f(h)))(y).\end{aligned}$$

This completes the proof of theorem.

Theorem 5. *Let $f,h : I \to \mathbb{R}$ are both nonnegative, monotone (increasing or decreasing) and $g : J \to \mathbb{R}$, $J \supset f(I)$, is monotone (increasing or decreasing) and satisfies the condition $g(u)g(v) \leq g(uv)$. If f,h are Ag-convex function, then fh is Ag-convex function.*

Proof. If $x \leq y$ (the case $y \leq x$ runs in the same fashion) then,

$$[g(f(x)) - g(f(y))][g(h(y)) - g(h(x))] \leq 0,$$

which implies,

$$g(f(x))g(h(y)) + g(f(y))g(h(x)) \leq g(f(x))g(h(x)) + g(f(y))g(h(y)). \tag{5}$$

On the other hand for $x,y \in I$ and $t \in [0,1]$,

$$\begin{aligned}(fh)(tx+(1-t)y) &= f(tx+(1-t)y)h(tx+(1-t)y),\\ &\leq [tg(f(x)) + (1-t)g(f(y))][tg(h(x)) + (1-t)g(h(y))],\\ &= t^2 g(f(x))g(h(x)) + t(1-t)g(f(x))g(h(y)),\\ &+t(1-t)g(f(y))g(h(x)) + (1-t)^2 g(f(y))g(h(y)).\end{aligned}$$

Using now (5), we obtain,

$$\begin{aligned}&(fh)(tx+(1-t)y),\\ &\leq t^2 g(f(x))g(h(x)) + (1-t)^2 g(f(y))g(h(y)),\\ &+t(1-t)[g(f(x))g(h(x)) + g(f(y))g(h(y))],\\ &\leq t[t+(1-t)]g(f(x))g(h(x)) + (1-t)[t+(1-t)]g(f(y))g(h(y)),\\ &= tg(f(x))g(h(x)) + (1-t)g(f(y))g(h(y)),\\ &\leq tg(fh)(x) + (1-t)g(fh)(y).\end{aligned}$$

This completes the proof of theorem.

3 Hermite-Hadamard inequality forAg-convex functions

Theorem 6. *Let $f : I \to \mathbb{R}$ be a Ag-convex function, $g : J \to \mathbb{R}$, $J \supset f(I)$, $a, b \in I$ with $a < b$ and $gof \in L[a,b]$. The following inequality,*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b (gof)(u)du,$$

holds.

Proof. By the Ag-convexity of the function f, we have,

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{[ta+(1-t)b]+[(1-t)a+tb]}{2}\right), \\ &\leq \frac{1}{2}(gof)(ta+(1-t)b)+\frac{1}{2}(gof)((1-t)a+tb). \end{aligned}$$

Now, if we take integral the last inequality on $t \in [0,1]$, we get,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b (gof)(u)du.$$

Remark. If we take $g(x) = x$ in the Theorem 6, then we obtain the left side of the Hermite-hadamard inequality for the convex functions.

Theorem 7. *Let $f : I \to \mathbb{R}$ be a Ag-convex function, $g : J \to \mathbb{R}$, $J \supset f(I)$, $a, b \in I$ with $a < b$ and $f \in L[a,b]$. The following inequality,*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{(gof)(a)+(gof)(b)}{2},$$

holds.

Proof. By using Ag-convexity of f and changing variable as $x = ta+(1-t)b$,

$$\begin{aligned} \int_0^1 f(ta+(1-t)b)dt &= \frac{1}{b-a}\int_a^b f(x)dx, \\ &\leq \int_0^1 [tg(f(a))+(1-t)(f(b))]\,dt, \\ &= \frac{(gof)(a)+(gof)(b)}{2}. \end{aligned}$$

This completes the proof of theorem.

Remark. If we take $g(x) = x$ in the Theorem 7, then we obtain the right side of the Hermite-hadamard inequality for the convex functions.

Theorem 8. *Let f and h be real-valued, nonnegative and Ag-convex functions on interval $[a,b]$. Then, the following inequalities,*

$$\begin{aligned} &\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)h(tx+(1-t)y)dtdydx, \\ &\leq \frac{2(b-a)}{3}\int_a^b (gof)(x)(goh)(x), \\ &+\frac{1}{3}\left(\int_a^b (gof)(x)dx\right)\left(\int_a^b (goh)(y)dy\right). \end{aligned} \tag{6}$$

and,

$$\begin{aligned}
&\int_a^b\int_0^1 f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)h\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)dtdx,\\
&\leq \frac{1}{3}\int_a^b (gof)(x)(goh)(x)dx,\\
&+\frac{b-a}{3}(gof)\left(\frac{a+b}{2}\right)(goh)\left(\frac{a+b}{2}\right),\\
&+\frac{1}{6}(goh)\left(\frac{a+b}{2}\right)\int_a^b (gof)(x)dx,\\
&+\frac{1}{6}(gof)\left(\frac{a+b}{2}\right)\int_a^b (goh)(x)dx, \qquad (7)
\end{aligned}$$

are valid for all $x,y\in[a,b]$ *and* $t\in[0,1]$.

Proof. Since both functions f and g are Ag-convex, for every two points $x,y\in[a,b]$ and $t\in[0,1]$, the following inequalities are valid,

$$\begin{aligned}
f(tx+(1-t)y) &\leq t(gof)(x)+(1-t)(gof)(y),\\
h(tx+(1-t)y) &\leq t(goh)(x)+(1-t)(goh)(y).
\end{aligned}$$

Multiplying the above inequalities, we have the following,

$$\begin{aligned}
&f(tx+(1-t)y)h(tx+(1-t)y),\\
&\leq t^2(gof)(x)(goh)(x)+(1-t)^2(gof)(y)(goh)(y),\\
&+t(1-t)[(gof)(x)(goh)(y)+(gof)(y)(goh)(x)].
\end{aligned}$$

Both sides of the above inequality are integrable with respect to t on the interval $[0,1]$, together with the known properties of the Ag-convex functions. Then, integrating this inequality over $[0,1]$, we have,

$$\begin{aligned}
&\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)h(tx+(1-t)y)dtdydx, \qquad (8)\\
&\leq \frac{2(b-a)}{3}\int_a^b (gof)(x)(goh)(x),\\
&+\frac{1}{3}\left(\int_a^b (gof)(x)dx\right)\left(\int_a^b (goh)(y)dy\right).
\end{aligned}$$

Let's prove the inequality (7). Ag-convexity of the two functions f and g gives us the inequalities,

$$\begin{aligned}
f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) &\leq t(gof)(x)+(1-t)(gof)\left(\frac{a+b}{2}\right),\\
h\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) &\leq t(goh)(x)+(1-t)(goh)\left(\frac{a+b}{2}\right).
\end{aligned}$$

As above, multipling the above inequalities, one obtains,

$$\begin{aligned}
&f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)h\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right), \qquad (9)\\
&\leq t^2(gof)(x)(goh)(x)+(1-t)^2(gof)\left(\frac{a+b}{2}\right)(goh)\left(\frac{a+b}{2}\right),\\
&+t(1-t)\left[(gof)(x)(goh)\left(\frac{a+b}{2}\right)+(gof)\left(\frac{a+b}{2}\right)(goh)(x)\right].
\end{aligned}$$

Similar to the proof of the first inequality, integrating both sides of (9) over the interval $[0,1]$, we find the following inequality,

$$\begin{aligned}
&\int_0^1 f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) h\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) dt, \\
\leq &\frac{1}{3}\left[(gof)(x)(goh)(x)+(gof)\left(\frac{a+b}{2}\right)(goh)\left(\frac{a+b}{2}\right)\right], \\
&+\frac{1}{6}\left[(gof)(x)(goh)\left(\frac{a+b}{2}\right)+(gof)\left(\frac{a+b}{2}\right)(goh)(x)\right].
\end{aligned} \tag{10}$$

Now, using the Ag-convexity of the functions f and h and integrating both sides of (10) over the interval $[a,b]$, we have,

$$\begin{aligned}
&\int_a^b\int_0^1 f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) h\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) dtdx, \\
\leq &\frac{1}{3}\int_a^b (gof)(x)(goh)(x)\,dx, \\
&+\frac{b-a}{3}(gof)\left(\frac{a+b}{2}\right)(goh)\left(\frac{a+b}{2}\right), \\
&+\frac{1}{6}(goh)\left(\frac{a+b}{2}\right)\int_a^b (gof)(x)\,dx, \\
&+\frac{1}{6}(gof)\left(\frac{a+b}{2}\right)\int_a^b (goh)(x)\,dx.
\end{aligned} \tag{11}$$

This completes the proof.

Remark. If the function gof is convex in Theorem 8, then we have,

$$\begin{aligned}
&\frac{3}{2}\frac{1}{(b-a)^2}\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)h(tx+(1-t)y)dtdydx, \\
\leq &\frac{1}{b-a}\int_a^b (gof)(x)(goh)(x)dx+\frac{1}{8}[M(a,b)+N(a,b)],
\end{aligned} \tag{12}$$

and,

$$\begin{aligned}
&\frac{3}{b-a}\int_a^b\int_0^1 f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) h\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) dtdx, \\
\leq &\frac{1}{b-a}\int_a^b (gof)(x)(goh)(x)dxdx+\frac{1}{2}[M(a,b)+N(a,b)],
\end{aligned} \tag{13}$$

are valid for all $x,y\in[a,b]$ and $t\in[0,1]$, where,

$$\begin{aligned}
M(a,b)&=(gof)(a)(goh)(a)+(gof)(b)(goh)(b), \\
N(a,b)&=(gof)(a)(goh)(b)+(gof)(b)(goh)(a).
\end{aligned}$$

Remark. If we take $g(x)=x$ in the inequalities (12) and (13), then we obtain the inequalities (2) and (3) of the Theorem 1. That is, our results reduce to the results in [2].

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