Communication in Mathematical Modeling and Applications
http://ntmsci.com/cmma

Notes on some p-valent functions

Zeliha Karahüseyin[1], *Melike Aydoğan*[2] *and Shigeyoshi Owa*3

[1]Istanbul Kultur University, Turkey
[2]Istanbul Technical University, Istanbul, Turkey
[3]University of Alba Iulia, Alba Iulia, Romania

Received: 30 May 2019, Accepted: 29 August 2019
Published online: 30 September 2019

Abstract: Let $\mathscr{A}_p$ be the class of analytic functions

$$f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots$$

in the open unit disk $\mathbb{U}$. We introduce a subclass $\mathscr{A}_p(j,\alpha)$ of $\mathscr{A}_p$ using some inequality for $f(z) \in \mathscr{A}_p$. The object of the present paper is to consider some interesting properties for $f(z)$ concerning with the class $\mathscr{A}_p(j,\alpha)$.

Keywords: Analytic function, p-valent function, Fejér-Riesz inequality.

1 Introduction

Let $\mathscr{A}_p$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1,2,3,\cdots\}) \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For functions $f(z)$ in the class $\mathscr{A}_p$, we say that $f(z) \in \mathscr{A}_p(j,\alpha)$ if it satisfies

$$\left|\arg\left(\frac{f^{(j)}(z)}{z^{p-j}}\right)\right| < \alpha\left(1+\frac{1}{\pi}\log j\right) \quad (z \in \mathbb{U}) \tag{2}$$

for some real α $(0 < \alpha \leq \frac{\pi}{2})$ and $j = 1,2,3,\cdots,p$. If we take $j = p$ and $\alpha = \frac{\pi}{2}$ in (2), then the inequality (2) can be written by

$$\left|\arg f^{(p)}(z)\right| < \frac{\pi}{2}\left(1+\frac{1}{\pi}\log p\right) \quad (z \in \mathbb{U}). \tag{3}$$

The above inequality (3) was considered by Nunokawa [2]. In his paper [2], we know that $f(z) \in \mathscr{A}_p$ is p-valent in $\mathbb{U}$ if $f(z)$ satisfies (3). Recently, Nunokawa, Cho, Kwon and Sokol published their paper [4] applying the inequality (3). Also, Nunokawa [3] showed that if $f(z) \in \mathscr{A}_p$ satisfies

$$|\arg f^{(p)}(z)| < \frac{3}{4}\pi \quad (z \in \mathbb{U}), \tag{4}$$

then $f(z)$ is p-valent in $\mathbb{U}$. To discuss our problem for the class $\mathscr{A}_p(j,\alpha)$, we have to recall here the following lemma due to Fejér and Riesz [1] (or Tsuji [5]).

* Corresponding author e-mail: karahuseyinzeliha@gmail.com

Lemma 1. *Let a function $f(z)$ be analytic in $|z| \leq 1$. Then $f(z)$ satisfies the following inequality*

$$\int_{-1}^{1} |f(z)|^q dz \leq \frac{1}{2} \int_{|z|=1} |f(z)|^q |dz| \quad (q > 0), \tag{5}$$

where the above integral on the left hand side is considered along the real axis.

If we make a change of variables in Lemma 1, then the inequality (5) can be change that

$$\int_{-r}^{r} |f(\rho e^{i\theta})|^q d\rho \leq \frac{r}{2} \int_{0}^{2\pi} |f(re^{i\theta})|^q d\theta. \tag{6}$$

2 Properties of functions

Our first result for $f(z) \in \mathscr{A}_p(j,\alpha)$ is given in the following theorem.

Theorem 1. *If a function $f(z) \in \mathscr{A}_p$ satisfies*

$$\left| \frac{zf^{(j+1)}(z)}{f^{(j)}(z)} \right| \leq \frac{\alpha}{\pi}\left(1 + \frac{1}{\pi}\log j\right) - 2(p-j) \quad (z \in \mathbb{U}) \tag{7}$$

for some real α $(0 < \alpha \leq \frac{\pi}{2})$ and $j = 1,2,3,\cdots,p$, then $f(z) \in \mathscr{A}_p(j,\alpha)$.

Proof. We note that

$$\log\left(\frac{f^{(j)}(z)}{z^{p-j}}\right) = \log\left|\frac{f^{(j)}(z)}{z^{p-j}}\right| + i\arg\left(\frac{f^{(j)}(z)}{z^{p-j}}\right) \tag{8}$$

and

$$\begin{aligned} \log\left(\frac{f^{(j)}(z)}{z^{p-j}}\right) &= \int_0^z \left(\log\left(\frac{f^{(j)}(t)}{t^{p-j}}\right)\right)' dt = \int_0^z (\log f^{(j)}(t))' dt - \int_0^z (\log t^{p-j})' dt \\ &= \int_0^z \frac{f^{(j+1)}(t)}{f^{(j)}(t)} dt - (p-j)\int_0^z \frac{1}{t} dt. \end{aligned} \tag{9}$$

This gives us that

$$\begin{aligned} \left|\arg\left(\frac{f^{(j)}(z)}{z^{p-j}}\right)\right| &= \left|\mathrm{Im}\int_0^z \frac{f^{(j+1)}(t)}{f^{(j)}(t)} dt - (p-j)\arg(z)\right| \\ &\leq \left|\mathrm{Im}\int_0^r \frac{f^{(j+1)}(\rho e^{i\theta})}{f^{(j)}(\rho e^{i\theta})} e^{i\theta} d\rho\right| + (p-j)|\arg(z)| \\ &\leq \int_0^r \left|\mathrm{Im}\left(\frac{e^{i\theta} f^{(j+1)}(\rho e^{i\theta})}{f^{(j)}(\rho e^{i\theta})}\right)\right| d\rho + 2(p-j)\pi \\ &< \int_{-r}^r \left|\mathrm{Im}\left(\frac{e^{i\theta} f^{(j+1)}(\rho e^{i\theta})}{f^{(j)}(\rho e^{i\theta})}\right)\right| d\rho + 2(p-j)\pi \\ &\leq \int_{-r}^r \left|\frac{f^{(j+1)}(\rho e^{i\theta})}{f^{(j)}(\rho e^{i\theta})}\right| d\rho + 2(p-j)\pi, \end{aligned} \tag{10}$$

where $z = re^{i\theta}$, $0 \le r < 1$, $0 \le \rho \le r$, and $0 \le \theta \le 2\pi$. Applying Lemma 1 with (6), we obtain that

$$\begin{aligned} \left|\arg\left(\frac{f^{(j)}(z)}{z^{p-j}}\right)\right| &< \frac{r}{2}\int_0^{2\pi}\left|\frac{f^{(j+1)}(re^{i\theta})}{f^{(j)}(re^{i\theta})}\right| d\theta + 2(p-j)\pi \\ &= \frac{1}{2}\int_0^{2\pi}\left|\frac{re^{i\theta}f^{(j+1)}(re^{i\theta})}{f^{(j)}(re^{i\theta})}\right| d\theta + 2(p-j)\pi \\ &\le \frac{1}{2}\left(\frac{\alpha}{\pi}\left(1+\frac{1}{\pi}\log j\right) - 2(p-j)\right)\int_0^{2\pi} d\theta + 2(p-j)\pi \\ &= \alpha\left(1+\frac{1}{\pi}\log j\right). \end{aligned} \tag{11}$$

This shows us that $f(z) \in \mathscr{A}_p(j,\alpha)$.

Letting $j = p$ and $\alpha = \frac{\pi}{2}$ in Theorem 1 , we have

Corollary 1. *If $f(z) \in \mathscr{A}_p$ satisfies*

$$\left|\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right| \le \frac{1}{2}\left(1+\frac{1}{\pi}\log p\right) \quad (z \in \mathbb{U}), \tag{12}$$

then

$$|\arg f^{(p)}(z)| < \frac{1}{2}\left(1+\frac{1}{\pi}\log p\right) \quad (z \in \mathbb{U}). \tag{13}$$

Remark. Corollary 1 is given by Nunokawa, Cho, Kwon, and Sokol [4], recently.

Next, we derive

Theorem 2. *If a function $f(z) \in \mathscr{A}_p$ satisfies*

$$\left|\frac{zf^{(j+1)}(z)}{f^{(j)}(z)}\right| \le \frac{\alpha}{\pi}\left(1+\frac{1}{\pi}\log j\right)\operatorname{Re}\left(\frac{1+(2\beta-1)z}{1-z}\right) - 2(p-j) \quad (z \in \mathbb{U}) \tag{14}$$

for some real α $(0 < \alpha \le \frac{\pi}{2})$, $j = 1,2,3,\cdots,p$, and for some real β $(0 < \beta \le 1)$, then $f(z) \in \mathscr{A}_p(j,\alpha)$.

Proof. It follows from the proof of Theorem 1 that

$$\begin{aligned} \left|\arg\left(\frac{f^{(j)}(z)}{z^{p-j}}\right)\right| &< \frac{1}{2}\int_0^{2\pi}\left|\frac{re^{i\theta}f^{(j+1)}(re^{i\theta})}{f^{(j)}(re^{i\theta})}\right| d\theta + 2(p-j)\pi \\ &\le \frac{1}{2}\int_0^{2\pi}\left\{\frac{\alpha}{\pi}\left(1+\frac{1}{\pi}\log j\right)\operatorname{Re}\left(\frac{1+(2\beta-1)re^{i\theta}}{1-re^{i\theta}}\right) - 2(p-j)\right\} d\theta \\ &\quad + 2(p-j)\pi \\ &= \frac{\alpha}{2\pi}\left(1+\frac{1}{\pi}\log j\right)\int_0^{2\pi}\operatorname{Re}\left(\frac{1+(2\beta-1)re^{i\theta}}{1-re^{i\theta}}\right) d\theta \\ &= \frac{\alpha}{2\pi}\left(1+\frac{1}{\pi}\log j\right)\int_0^{2\pi}\operatorname{Re}\left(1-2\beta+\frac{2\beta}{1-re^{i\theta}}\right) d\theta \\ &= \alpha\left(1+\frac{1}{\pi}\log j\right). \end{aligned} \tag{15}$$

This means that $f(z) \in \mathscr{A}_p(j,\alpha)$.

Making $j = p$ and $\alpha = \frac{\pi}{2}$ in Theorem 2, we obtain

Corollary 2. *If $f(z) \in \mathscr{A}_p$ satisfies*

$$\left|\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right| \leq \frac{1}{2}\left(1+\frac{1}{\pi}\log p\right)\mathrm{Re}\left(\frac{1+(2\beta-1)z}{1-z}\right) \quad (z \in \mathbb{U}) \tag{16}$$

for some real β $(0 < \beta \leq 1)$, then

$$|\arg f^{(p)}(z)| < \frac{\pi}{2}\left(1+\frac{1}{\pi}\log p\right) \quad (z \in \mathbb{U}), \tag{17}$$

that is, that $f(z)$ is p-valent in $\mathbb{U}$.

Remark. If we take $\beta = 1$ in Corollary 2, then we have the result due to Nunokawa, Cho, Kwon and Sokol [4].

Letting $\beta = \frac{1}{2}$ in Theorem 2, we see

Corollary 3. *If $f(z) \in \mathscr{A}_p$ satisfies*

$$\left|\frac{zf^{(j+1)}(z)}{f^{(j)}(z)}\right| \leq \frac{\alpha}{\pi}\left(1+\frac{1}{\pi}\log j\right)\mathrm{Re}\left(\frac{1}{1-z}\right) - 2(p-j) \quad (z \in \mathbb{U}) \tag{18}$$

for some real α $(0 < \alpha \leq \frac{\pi}{2})$ and $j = 1, 2, 3, \cdots, p$, then $f(z) \in \mathscr{A}_p(j, \alpha)$.

Remark. If we make $\alpha = \frac{\pi}{2}$ and $j = p$ in Corollary 3, then we have the result by Nunokawa, Cho, Kwon and Sokol [4].

3 Case of $j = 1$ for $\mathscr{A}_p(j, \alpha)$

Let us consider the special case of $j = 1$ in Section 2.

Corollary 4. *If $f(z) \in \mathscr{A}_p$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{\alpha}{\pi} - 2(p-1) \quad (z \in \mathbb{U}) \tag{19}$$

for some real α $(0 < \alpha \leq \frac{\pi}{2})$ and $j = 1, 2, 3, \cdots, p$, then $f(z) \in \mathscr{A}_p(1, \alpha)$. Further, if $p = 1$ in (19), then we have

$$|\arg f'(z)| < \alpha \quad (z \in \mathbb{U}). \tag{20}$$

Thus $f(z)$ is univalent in $\mathbb{U}$.

From Theorem 2, we obtain

Corollary 5. *If $f(z) \in \mathscr{A}_p$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{\alpha}{\pi}\mathrm{Re}\left(\frac{1+(2\beta-1)z}{1-z}\right) - 2(p-1) \quad (z \in \mathbb{U}) \tag{21}$$

for some real α $(0 < \alpha \leq \frac{\pi}{2})$ and for some real β $(0 < \beta \leq 1)$, then $f(z) \in \mathscr{A}_p(1, \alpha)$. Further, if $p = 1$ in (21), then we have

$$|\arg f'(z)| < \alpha \quad (z \in \mathbb{U}) \tag{22}$$

Thus $f(z)$ is univalent in $\mathbb{U}$.

Finally, we derive

Theorem 3. *If $f(z) \in \mathscr{A}_1$ satisfies*

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right| \leq \frac{\alpha}{\pi}\mathrm{Re}\left(\frac{1+(2\beta-1)z}{1-z}\right) \quad (z \in \mathbb{U}) \tag{23}$$

for some real α $(0<\alpha\leq\frac{\pi}{2})$ and for some real β $(0<\beta\leq 1)$, then

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \alpha \quad (z \in \mathbb{U}). \tag{24}$$

Proof. Note that

$$\begin{aligned}\log\left(\frac{zf'(z)}{f(z)}\right) &= \int_0^z \left(\log\left(\frac{tf'(t)}{f(t)}\right)\right)' dt \\ &= \int_0^z \left(\frac{1}{t}+\frac{f''(t)}{f'(t)}-\frac{f'(t)}{f(t)}\right) dt.\end{aligned} \tag{25}$$

This gives us that

$$\begin{aligned}\left|\arg\frac{zf'(z)}{f(z)}\right| &= \left|\mathrm{Im}\log\left(\frac{zf'(z)}{f(z)}\right)\right| \\ &= \left|\mathrm{Im}\int_0^z \left(\frac{1}{t}+\frac{f''(t)}{f'(t)}-\frac{f'(t)}{f(t)}\right) dt\right| \\ &= \left|\mathrm{Im}\int_0^r \left(\frac{1}{\rho e^{i\theta}}+\frac{f''(\rho e^{i\theta})}{f'(\rho e^{i\theta})}-\frac{f'(\rho e^{i\theta})}{f(\rho e^{i\theta})}\right) e^{i\theta} d\rho\right| \\ &\leq \int_0^r \left|\mathrm{Im}\left(\frac{1}{\rho e^{i\theta}}+\frac{f''(\rho e^{i\theta})}{f'(\rho e^{i\theta})}-\frac{f'(\rho e^{i\theta})}{f(\rho e^{i\theta})}\right)\right| d\rho \\ &\leq \int_{-r}^r \left|\frac{1}{\rho e^{i\theta}}+\frac{f''(\rho e^{i\theta})}{f'(\rho e^{i\theta})}-\frac{f'(\rho e^{i\theta})}{f(\rho e^{i\theta})}\right| d\rho,\end{aligned} \tag{26}$$

where $z = re^{i\theta}$, $0 \leq r < 1$, $0 \leq \rho \leq r$, and $0 \leq \theta \leq 2\pi$.

Applying Lemma 1 with (6), we see that

$$\begin{aligned}\left|\arg\frac{zf'(z)}{f(z)}\right| &\leq \frac{r}{2}\int_0^{2\pi} \left|\frac{1}{re^{i\theta}}+\frac{f''(re^{i\theta})}{f'(re^{i\theta})}-\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \left|1+\frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}-\frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})}\right| d\theta \\ &< \frac{1}{2}\int_0^{2\pi} \frac{\alpha}{\pi}\mathrm{Re}\left(\frac{1+(2\beta-1)re^{i\theta}}{1-re^{i\theta}}\right) d\theta \\ &= \frac{\alpha}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{1+(2\beta-1)re^{i\theta}}{1-re^{i\theta}}\right) d\theta \\ &= \alpha.\end{aligned} \tag{27}$$

This completes the proof of the theorem.

If we take $\beta = 1$ and $\beta = \frac{1}{2}$ in Theorem 3, then we have

Corollary 6. *If* $f(z) \in \mathscr{A}_1$ *satisfies*

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right| \leq \frac{\alpha}{\pi}\text{Re}\left(\frac{1+z}{1-z}\right) \quad (z \in \mathbb{U}) \tag{28}$$

for some real α $(0 < \alpha \leq \frac{\pi}{2})$, *then*

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \alpha \quad (z \in \mathbb{U}). \tag{29}$$

Corollary 7. *If* $f(z) \in \mathscr{A}_1$ *satisfies*

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right| \leq \frac{\alpha}{\pi}\text{Re}\left(\frac{1}{1-z}\right) \quad (z \in \mathbb{U}) \tag{30}$$

for some real α $(0 < \alpha \leq \frac{\pi}{2})$, *then*

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \alpha \quad (z \in \mathbb{U}). \tag{31}$$

Remark. If we consider $\alpha = \frac{\pi}{2}$ in Corollary 7, then we obtain the result due to Nunokawa, Cho, Kwon and Sokol [4].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] Fejér L, Riesz F. Aber einige funktionentheoretische Ungleichungen. Math. Z. 1921; 11: 305-314.
[2] Nunokawa M. On the theory of multivalent functions. PanAmerican Math. J. 1996; 6, 87-96.
[3] Nunokawa M. A note on multivalent functions. Tsukuba J. Math. 1989; 13: 453-455.
[4] Nunokawa M, Cho N.E., Kwon O.S., Sokol J. On the starlikeness of p-valent functions. Turkish J. Math. 2019; 43: 143-150.
[5] Tsuji M. Complex Function Theory. Tokyo, Japanese: Maki Book Comp., 1968.