Generalized fractional Ostrowski type inequalities for hiegher order derivaatives

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Abstract: In this paper, we present general fractional representation formulae for a function in terms of the fractional Riemann-Liouville integrals of different orders of the function and its ordinary derivatives. We also use these Montgomery identities to establish some new Ostrowski type integral inequalities.

Keywords: Ostrowski inequality, Riemann-Liouville fractional integral, fractional inequality.

1 Introduction

Fractional calculus derivative was interoduced in the 17th century. In recent years, the interest in fractional calculus has been growing continually owing to its benefical applications in a large number of areas of sciences such as electromagnetic waves, visco-elastic systems, quantum evolution of complex systems, diffusion waves, physics, engineering, finance, social sciences mathematical biology and chaos theory . Furthermore, fractional integrals and derivatives are used both in theoretical field and in diverse fields ranging from biological sciences and economics to physical sciences and engineering. The interested reader is able to look over the references [2]-[9] and [11] for fractional theory.

In 2009, Anastassiou [1] gave the fractional version of Ostrowski inequality. Zeki [14] gave a generalization of Anastassiou [1]. What is more, some authors provided novel inequalities with the help of Riemann-Liouville Fractional integration in [12] and [15]. Further, some inequalities involving Liouville–Caputo Fractional derivatives are obtained and its applications to special means of real numbers are given in [13]. There is not much work done in this direction and needs to be explored as fractional Ostrowski inequality is expected to have applications in many areas in the same way as its counterpart [10].

Anastassiou used the following lemma to prove his inequality in [1].

Lemma 1.*Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I^0 with $a,b \in I$ $(a<b)$ and $f' \in L_1[a,b]$, then*

$$f(x) = \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha} J_a^{\alpha}(f(b)) - J_a^{\alpha-1}(P_2(x,b)f(b)) + J_a^{\alpha}\left(P(x,b)f'(b)\right).$$

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From our results, some classical Ostrowski's inequalities can be deduced as special cases. Now we will give a definition and a result which is useful in understanding our derivations and results. They will also help in connecting our work with available literature.

Definition 1. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ is defined as*

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt \tag{1}$$

where

$$J_a^0 f(x) = f(x).$$

In this study, we establish some identity involving Riemann-Liouville fractional integral for functions which are one, two, three, four, five and n times differentiable Then we obtain two Ostrowski type inequalities using identity given in this work.

2 Derivation of some new identities

In this section, we will state and prove our main results. But before we do that we will prove a useful identity with the help of the following kernel.

Theorem 1.*Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping. Let $\omega(x,.) : [a,b] \to \mathbb{R}$, be the fractional Peano type kernel which is given by*

$$\omega(x,t) = \begin{cases} \frac{1}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha)\left(t-\left(a+h\frac{b-a}{2}\right)\right), & a \leq t \leq x \\ \frac{1}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha)\left(t-\left(b-h\frac{b-a}{2}\right)\right), & x < t \leq b \end{cases} \tag{2}$$

for all $x \in \left[a+h\frac{b-a}{2}, b-h\frac{b-a}{2}\right]$ and $h \in [0,1]$, then the following identity holds:

$$J_a^\alpha\left(\omega(x,b)f'(b)\right) = (1-h)f(x) + \frac{h}{2}(b-x)^{1-\alpha}(b-a)^{\alpha-1}f(a) - \frac{(b-x)^{1-\alpha}}{b-a}\Gamma(\alpha)J_a^\alpha(f(b)) + J_a^{\alpha-1}(\omega(x,b)f(b)). \tag{3}$$

for $\alpha \geq 1$.

Proof. Using (1) and (2), we find that

$$\begin{aligned} J_a^\alpha\left(\omega(x,b)f'(b)\right) =& \frac{1}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}\omega(x,t)f'(t)dt \\ =& \frac{(b-x)^{1-\alpha}}{b-a}\int_a^x (b-t)^{\alpha-1}\left(t-\left(a+h\frac{b-a}{2}\right)\right)f'(t)dt \\ &+ \frac{(b-x)^{1-\alpha}}{b-a}\int_x^b (b-t)^{\alpha-1}\left(t-\left(b-h\frac{b-a}{2}\right)\right)f'(t)dt \\ =& \frac{(b-x)^{1-\alpha}}{b-a}\left[(1-h)f(x) + \frac{h}{2}(b-x)^{1-\alpha}(b-a)^{\alpha-1}f(a)\right. \\ &\left. - \frac{(b-x)^{1-\alpha}}{b-a}\int_a^b (b-t)^{\alpha-1}f(t)dt + \frac{(\alpha-1)}{\Gamma(\alpha)}\int_a^b \omega(x,b)(b-t)^{\alpha-2}f(t)dt\right] \end{aligned} \tag{4}$$

After simplification, we get desired identity (3).

We continue with the following theorem.

Theorem 2. *Let $\alpha > 2$, $x \in \left[a+h\frac{b-a}{2}, b-h\frac{b-a}{2}\right]$, $f:[a,b]\to\mathbb{R}$ be twice differentiable with $f'':[a,b]\to\mathbb{R}$ integrable on $[a,b]$, then we obtain*

$$\begin{aligned}J_a^{\alpha}\left(\omega(x,b)f''(b)\right) =& (1-h)(\alpha-1)(b-x)^{-1}f(x) - 2\frac{(b-x)^{1-\alpha}}{b-a}\Gamma(\alpha)J_a^{\alpha-1}(f(b)) + J_a^{\alpha-2}(\omega(x,b)f(b)) \\ &+\frac{h}{2}\left[(b-x)^{1-\alpha}(b-a)^{\alpha-1}f'(a)+(\alpha-1)(b-x)^{1-\alpha}(b-a)^{\alpha-2}f(a)\right] \\ &+(b-x)^{1-\alpha}(b-a)^{\alpha-2}f(a)-(1-h)f'(x). \end{aligned} \tag{5}$$

Proof. Let $\alpha > 2$, and there exist $f'':[a,b]\to\mathbb{R}$ integrable on $[a,b]$, then we have

$$\begin{aligned}J_a^{\alpha}\left(\omega(x,b)f''(b)\right) =& \frac{1}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}\omega(x,t)f''(t)dt \\ =& \frac{(b-x)^{1-\alpha}}{b-a}\left[(b-x)^{\alpha-1}\left(x-\left(a+h\frac{b-a}{2}\right)\right)f'(x)+\frac{h(b-a)^{\alpha}f'(a)}{2}\right. \\ &+(\alpha-1)\left(x-\left(a+h\frac{b-a}{2}\right)\right)(b-x)^{\alpha-2}f(x)+\frac{h}{2}(\alpha-1)(b-a)^{\alpha-1}f(a) \\ &-(b-x)^{\alpha-1}\left(x-\left(b-h\frac{b-a}{2}\right)\right)f'(x)-\int_a^b (b-t)^{\alpha-1}f'(t)dt \\ &-(\alpha-1)(b-x)^{\alpha-2}\left(x-\left(b-h\frac{b-a}{2}\right)\right)f(x)-(\alpha-1)\int_a^b (b-t)^{\alpha-2}f(t)dt \\ &+(\alpha-1)(\alpha-2)\left\{\int_a^x \left(t-\left(a+h\frac{b-a}{2}\right)\right)(b-t)^{\alpha-3}f(t)dt\right. \\ &\left.\left.\int_x^b (b-t)^{\alpha-3}\left(t-\left(b-h\frac{b-a}{2}\right)\right)f(t)dt\right\}\right]. \end{aligned}$$

Applying elementary analysis operations, we obtain the inequality (5).

Remark. Under the same assumptions of Theorem 2 with $h=0$, we have the inequality

$$\begin{aligned}f(x) =& \frac{(b-x)^{2-\alpha}}{\alpha-1}\left[-(b-a)^{\alpha-2}f(a)+\frac{2}{b-a}\Gamma(\alpha)J_a^{\alpha-1}(f(b))\right. \\ &\left.-(b-x)^{\alpha-1}J_a^{\alpha-2}(p_1(x,b)f(b))+(b-x)^{\alpha-1}J_a^{\alpha}\left(p_1(x,b)f''(b)\right)+(b-x)^{\alpha-1}f'(x)\right].\end{aligned}$$

Now, we establish the following identity for functions which are three times differentiable.

Theorem 3. *Let* $\alpha > 3$, $x \in \left[a+h\frac{b-a}{2}, b-h\frac{b-a}{2}\right]$, $f : [a,b] \to \mathbb{R}$ *be three times differentiable with* $f''' : [a,b] \to \mathbb{R}$ *integrable on* $[a,b]$, *then we have*

$$\begin{aligned} J_a^{\alpha}\left(\omega(x,b) f'''(b)\right) = & (1-h)\left[f''(x) + (\alpha-1)(b-x)^{-1} f'(x) + (\alpha-1)(\alpha-2)(b-x)^{-2} f(x)\right] \\ & + \frac{h}{2}(b-x)^{1-\alpha}\left[(b-a)^{\alpha-1} f''(a) + (\alpha-1)(b-a)^{\alpha-2} f'(a)\right. \\ & \left. + (\alpha-1)(\alpha-2)(b-a)^{\alpha-3} f(a)\right] \\ & + (b-x)^{1-\alpha}(b-a)^{\alpha-2} f'(a) + 2(\alpha-1)(b-x)^{1-\alpha}(b-a)^{\alpha-3} f(a) \\ & - 3\frac{(b-x)^{1-\alpha}}{b-a}\Gamma(\alpha) J_a^{\alpha-2}(f(b)) + J_a^{\alpha-3}(\omega(x,b) f(b)). \end{aligned} \tag{6}$$

*Proof.*Let $\alpha > 3$, and there exist $f''' : [a,b] \to \mathbb{R}$ integrable on $[a,b]$, then we have

$$\Gamma(\alpha) J_a^{\alpha}\left(\omega(x,b) f'''(b)\right) = \frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} \omega(x,t) f'''(t) dt. \tag{7}$$

If we use the equality (7) and also apply similar methods in the proof of Theorem 2, we obtain the equality (6). Hence, the proof is completed.

Remark. If we choose $h = 0$ in Theorem 3, then we get

$$\begin{aligned} f(x) = & \frac{(b-x)^3}{(\alpha-1)(\alpha-2)}\left[(b-x)^{-1} J_a^{\alpha}\left(\omega(x,b) f'''(b)\right) - (b-x)^{-2} f''(x) - (\alpha-1)(b-x)^{-2} f'(x)\right. \\ & - (b-x)^{-\alpha}(b-a)^{\alpha-2} f'(a) - 2(\alpha-1)(b-x)^{-\alpha}(b-a)^{\alpha-3} f(a) \\ & \left. + \frac{3(b-x)^{-\alpha}}{b-a}\Gamma(\alpha) J_a^{\alpha-2}(f(b)) - (b-x)^{-1} J_a^{\alpha-3}(\omega(x,b) f(b))\right]. \end{aligned}$$

Theorem 4. *Let* $\alpha > 4$, $x \in \left[a+h\frac{b-a}{2}, b-h\frac{b-a}{2}\right]$, $f : [a,b] \to \mathbb{R}$ *be four times differentiable with* $f^{(iv)} : [a,b] \to \mathbb{R}$ *integrable on* $[a,b]$, *then*

$$\begin{aligned} J_a^{\alpha}\left(\omega(x,b) f^{(iv)}(b)\right) = & J_a^{\alpha-4}(\omega(x,b) f(b)) - 4\frac{(b-x)^{1-\alpha}}{b-a}\Gamma(\alpha) J_a^{\alpha-3}(f(b)) \\ & + \Gamma(\alpha)(1-h)\left[\frac{1}{\Gamma(\alpha-3)}(b-x)^{-3} f(x) + \frac{1}{\Gamma(\alpha-2)}(b-x)^{-2} f'(x)\right. \\ & \left. + \frac{1}{\Gamma(\alpha-1)}(b-x)^{-1} f''(x) + \frac{1}{\Gamma(\alpha)} f'''(x)\right] \\ & + \Gamma(\alpha)\frac{h}{2}(b-x)^{1-\alpha}(b-a)^{\alpha}\left[\frac{1}{\Gamma(\alpha)}(b-a)^{-1} f'''(a) + \frac{1}{\Gamma(\alpha-1)}(b-a)^{-2} f''(a)\right. \\ & \left. + \frac{1}{\Gamma(\alpha-2)}(b-a)^{-3} f'(a) + \frac{1}{\Gamma(\alpha-3)}(b-a)^{-4} f(a)\right] \\ & + (b-a)^{\alpha}(b-x)^{1-\alpha}\Gamma(\alpha)\left[\frac{1}{\Gamma(\alpha)}(b-a)^{-2} f''(a) + \frac{2}{\Gamma(\alpha-1)}(b-a)^{-3} f'(a)\right. \\ & \left. + \frac{3}{\Gamma(\alpha-2)}(b-a)^{-4} f(a)\right] \end{aligned} \tag{8}$$

Proof. Let $\alpha > 4$, and there exist $f^{(iv)}\colon [a,b] \to \mathbb{R}$ integrable on $[a,b]$, then we have

$$J_a^{\alpha}\left(\omega(x,b)f^{(iv)}(b)\right) = \frac{1}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}\omega(x,t)f^{(iv)}(t)dt. \tag{9}$$

Using similar methods in the proof of Theorem 2 and the equality (9), we deduce the equality (8) which completes the proof.

Remark. If we take $h=0$ in the equality (8), then we have

$$\begin{aligned} f(x) =& \Gamma(\alpha-3)(b-x)^3\left[\frac{1}{\Gamma(\alpha)}J_a^{\alpha}\left(\omega(x,b)f^{(iv)}(b)\right)\right. \\ & - \frac{1}{\Gamma(\alpha)}J_a^{\alpha-4}(\omega(x,b)f(b)) + \frac{4(b-x)^{1-\alpha}}{b-a}J_a^{\alpha-3}(f(b)) \\ & - \frac{1}{\Gamma(\alpha-2)}(b-x)^{-2}f'(x) - \frac{1}{\Gamma(\alpha-1)}(b-x)^{-1}f''(x) - \frac{1}{\Gamma(\alpha)}f'''(x) \\ & \left. - (b-a)^{\alpha}(b-x)^{1-\alpha}\left\{\frac{1}{\Gamma(\alpha)}(b-a)^{-2}f''(a) + \frac{2(b-a)^{-3}}{\Gamma(\alpha-1)}f'(a) + \frac{3(b-a)^{-4}}{\Gamma(\alpha-2)}f(a)\right\}\right]. \end{aligned}$$

Theorem 5. *Let here $\alpha > 5$, $x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right]$, $f\colon [a,b] \to \mathbb{R}$ be five times differentiable with $f^{(v)}\colon [a,b] \to \mathbb{R}$ integrable on $[a,b]$, then*

$$\begin{aligned} J_a^{\alpha}\left(\omega(x,b)f^{(v)}(b)\right) =& J_a^{\alpha-5}(\omega(x,b)f(b)) - 5\frac{(b-x)^{1-\alpha}}{b-a}\Gamma(\alpha)J_a^{\alpha-4}(f(b)) \\ & + (1-h)\Gamma(\alpha)\left[\frac{(b-x)^{-4}}{\Gamma(\alpha-4)}f(x) + \frac{(b-x)^{-3}}{\Gamma(\alpha-3)}f'(x) + \frac{(b-x)^{-2}}{\Gamma(\alpha-2)}f''(x)\right. \\ & \left. + \frac{(b-x)^{-1}}{\Gamma(\alpha-1)}f'''(x) + \frac{1}{\Gamma(\alpha)}f^{(iv)}(x)\right] \\ & + (b-a)^{\alpha}(b-x)^{1-\alpha}\Gamma(\alpha)\frac{h}{2}\left[\frac{(b-a)^{-1}}{\Gamma(\alpha)}f^{(iv)}(a) + \frac{(b-a)^{-2}}{\Gamma(\alpha-1)}f'''(a)\right. \\ & \left. + \frac{(b-a)^{-3}}{\Gamma(\alpha-2)}f''(a) + \frac{(b-a)^{-4}}{\Gamma(\alpha-3)}f'(a) + \frac{(b-a)^{-5}}{\Gamma(\alpha-4)}f(a)\right] \\ & + (b-a)^{\alpha}(b-x)^{1-\alpha}\Gamma(\alpha)\left[\frac{(b-a)^{-2}}{\Gamma(\alpha)}f'''(a) + \frac{2(b-a)^{-3}}{\Gamma(\alpha-1)}f''(a)\right. \\ & \left. + \frac{3(b-a)^{-4}}{\Gamma(\alpha-2)}f'(a) + \frac{4(b-a)^{-5}}{\Gamma(\alpha-3)}f(a)\right] \end{aligned} \tag{10}$$

Proof. Using similar methods in the proof of Theorem 2, we obtain the inequality (10). Hence, the proof is completed.

Remark. Under the same assumptions of Theorem 5 with $h=0$, we have

$$\begin{aligned} f(x) =& \Gamma(\alpha-4)(b-x)^4 \left[\frac{1}{\Gamma(\alpha)} J_a^{\alpha} \left(\omega(x,b) f^{(v)}(b) \right) \right. \\ & - \frac{1}{\Gamma(\alpha)} J_a^{\alpha-5} \left(\omega(x,b) f(b) \right) + \frac{5(b-x)^{1-\alpha}}{(b-a)} J_a^{\alpha-4} (f(b)) \\ & - \frac{(b-x)^{-3}}{\Gamma(\alpha-3)} f'(x) - \frac{(b-x)^{-2}}{\Gamma(\alpha-2)} f''(x) - \frac{(b-x)^{-1}}{\Gamma(\alpha-1)} f'''(x) - f^{(iv)}(x) \\ & - (b-a)^{\alpha} (b-x)^{1-\alpha} \left\{ \frac{(b-a)^{-2}}{\Gamma(\alpha)} f'''(a) + \frac{2(b-a)^{-3}}{\Gamma(\alpha-1)} f''(a) \right. \\ & \left. \left. + \frac{3(b-a)^{-4}}{\Gamma(\alpha-2)} f'(a) + \frac{4(b-a)^{-5}}{\Gamma(\alpha-3)} f(a) \right\} \right]. \end{aligned}$$

Similarly, we obtain the following identity for higher order derivatives.

Theorem 6. *Let $\alpha > n$, $n \in N$, $x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right]$, $f : [a,b] \to \mathbb{R}$ be n- times differentiable with $f^{(n)}$: $[a,b] \to \mathbb{R}$ integrable on $[a,b]$, then we have*

$$\begin{aligned} J_a^{\alpha} \left(\omega(x,b) f^{(n)}(b) \right) =& (1-h) \sum_{k=0}^{n-1} \frac{\Gamma(\alpha)(b-x)^{-(n-k-1)}}{\Gamma(\alpha-(n-k-1))} f^{(k)}(x) + \frac{h}{2}(b-x)^{1-\alpha} \sum_{k=0}^{n-1} \frac{\Gamma(\alpha)(b-a)^{\alpha-n+k}}{\Gamma(\alpha-(n-k-1))} f^{(k)}(a) \\ & + (b-x)^{1-\alpha} \Gamma(\alpha) \sum_{k=0}^{n-1} \frac{k(b-a)^{\alpha-k-1}}{\Gamma(\alpha-k+1)} f^{(n-k-1)}(a) \\ & - \frac{(b-x)^{1-\alpha}}{b-a} \Gamma(\alpha) n J_a^{\alpha-n+1} f(b) + J_a^{\alpha-n} \left(\omega(x,b) f(b) \right). \end{aligned} \tag{11}$$

Proof. By integrating by parts, we find that

$$\begin{aligned} J_a^{\alpha} \left(\omega(x,b) f^{(n)}(b) \right) =& \frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} \omega(x,t) f^{(v)}(t) dt \\ =& \frac{(b-x)^{1-\alpha}}{b-a} \left[\int_a^x (b-t)^{\alpha-1} \left(t - \left(a + h\frac{b-a}{2} \right) \right) f^{(n)}(t) dt \right. \\ & \left. + \int_x^b (b-t)^{\alpha-1} \left(t - \left(b - h\frac{b-a}{2} \right) \right) f^{(n)}(t) dt \right] \\ =& (1-h) f^{(n-1)}(x) + \frac{h}{2}(b-x)^{1-\alpha}(b-a)^{\alpha-1} f^{(n-1)}(a) \\ & - \frac{(b-x)^{1-\alpha}}{b-a} \Gamma(\alpha) J_a^{\alpha} f^{(n-1)}(b) + J_a^{\alpha-1} \left(\omega(x,b) f^{(n-1)}(b) \right). \end{aligned}$$

Now we observe that

$$\begin{aligned} J_a^{\alpha-1} \left(\omega(x,b) f^{(n-1)}(b) \right) =& (\alpha-1)(1-h)(b-x)^{-1} f^{(n-2)}(x) \\ & + \frac{h}{2}(\alpha-1)(b-x)^{1-\alpha}(b-a)^{\alpha-2} f^{(n-2)}(a) \\ & - \frac{(b-x)^{1-\alpha}}{b-a} \Gamma(\alpha) J_a^{\alpha-1} f^{(n-2)}(b) + J_a^{\alpha-2} \left(\omega(x,b) f^{(n-2)}(b) \right) \end{aligned}$$

and

$$\begin{aligned}J_a^{\alpha-2}\left(\omega(x,b)f^{(n-2)}(b)\right) &= (\alpha-1)(\alpha-2)(1-h)(b-x)^{-2}f^{(n-3)}(x)\\ &+\frac{h}{2}(\alpha-1)(\alpha-2)(b-x)^{1-\alpha}(b-a)^{\alpha-3}f^{(n-3)}(a)\\ &-\frac{(b-x)^{1-\alpha}}{b-a}\Gamma(\alpha)J_a^{\alpha-2}f^{(n-3)}(b)+J_a^{\alpha-3}\left(\omega(x,b)f^{(n-3)}(b)\right)\\ &\dots\end{aligned}$$

$$\begin{aligned}J_a^{\alpha-(n-1)}\left(\omega(x,b)f'(b)\right) &= (\alpha-1)(\alpha-2)\cdots(\alpha-(n-1))(1-h)(b-x)^{-(n-1)}f(x)\\ &+\frac{h}{2}(\alpha-1)(\alpha-2)\cdots(\alpha-(n-1))(b-x)^{1-\alpha}(b-a)^{\alpha-n}f(a)\\ &-\frac{(b-x)^{1-\alpha}}{b-a}\Gamma(\alpha)J_a^{\alpha-(n-1)}f(b)+J_a^{\alpha-n}\left(\omega(x,b)f(b)\right).\end{aligned}$$

So, we obtain the identity

$$\begin{aligned}J_a^{\alpha}\left(\omega(x,b)f^{(n)}(b)\right) =&(1-h)\sum_{k=0}^{n-1}\frac{\Gamma(\alpha)}{\Gamma(\alpha-(n-k-1))}(b-x)^{-(n-k-1)}f^{(k)}(x)\\ &+\frac{h}{2}(b-x)^{1-\alpha}\sum_{k=0}^{n-1}\frac{\Gamma(\alpha)(b-a)^{\alpha-n+k}}{\Gamma(\alpha-(n-k-1))}f^{(k)}(a)\\ &-\frac{(b-x)^{1-\alpha}}{b-a}\Gamma(\alpha)\sum_{k=0}^{n-1}J_a^{\alpha-k}f^{(n-k-1)}(b)+J_a^{\alpha-n}\left(\omega(x,b)f(b)\right).\end{aligned} \tag{12}$$

Also, we have

$$\sum_{k=0}^{n-1}J_a^{\alpha-k}f^{(n-k-1)}(b) = -\sum_{k=1}^{n-1}\frac{k(b-a)^{\alpha-k}}{\Gamma(\alpha-k+1)}f^{(n-k-1)}(a)+nJ_a^{\alpha-n+1}f(b). \tag{13}$$

If we substitute the equality (13) in (12), we obtain desired equality (11) which completes the proof.

Remark. If we choose $h=0$ in theorem 6, we get

$$\begin{aligned}J_a^{\alpha}\left(\omega(x,b)f^{(n)}(b)\right) =&\sum_{k=0}^{n-1}\frac{\Gamma(\alpha)(b-x)^{-(n-k-1)}}{\Gamma(\alpha-(n-k-1))}f^{(k)}(x)+(b-x)^{1-\alpha}\Gamma(\alpha)\sum_{k=1}^{n-1}\frac{k(b-a)^{\alpha-k-1}}{\Gamma(\alpha-k+1)}f^{(n-k-1)}(a)\\ &-\frac{(b-x)^{1-\alpha}}{b-a}\Gamma(\alpha)nJ_a^{\alpha-n+1}f(b)+J_a^{\alpha-n}\left(\omega(x,b)f(b)\right).\end{aligned}$$

3 An Ostrowski type fractional inequality

We now use the integral equality (3) developed in the previous section, to obtain Ostrowski type inequality involving Riemann-Liouville fractional integral.

Theorem 7. *Let* $f:[a,b]\to\mathbb{R}$ *be differentiable on* (a,b) *such that* $f'(b)\in L_1(a,b)$*, where* $a<b$*. If* $|f'(x)|\le M$ *for every* $x\in[a,b]$ *and* $\alpha\ge 1$*, then the following Ostrowski fractional inequalities hold:*

$$\left|(1-h)(b-x)^0 f(x)-\frac{(b-x)^{1-\alpha}}{b-a}\Gamma(\alpha)J_a^{\alpha}(f(b))+J_a^{\alpha-1}(\omega(x,b)f(b))+\frac{h}{2}(b-x)^{1-\alpha}(b-a)^{\alpha-1}f(a)\right| \tag{14}$$
$$\le\frac{M(b-x)^{1-\alpha}}{b-a}\left\{-\frac{(b-a)^{\alpha+1}}{\alpha+1}\left(\left(1-\frac{h}{2}\right)^{\alpha+1}-1\right)+\frac{(b-a)^{\alpha+1}}{\alpha}\left(\left(1-\frac{h}{2}\right)^{\alpha+1}-\left(1-\frac{h}{2}\right)\right)\right.$$
$$+\frac{1}{\alpha+1}\left[(b-x)^{\alpha+1}-(b-a)^{\alpha+1}\left(1-\frac{h}{2}\right)^{\alpha+1}\right]-\frac{1}{\alpha}(b-a)\left[\left(1-\frac{h}{2}\right)(b-x)^{\alpha}-(b-a)^{\alpha}\left(1-\frac{h}{2}\right)^{\alpha+1}\right]$$
$$-\frac{1}{\alpha+1}\left(\left(\frac{b-a}{2}\right)^{\alpha+1}-(b-x)^{\alpha+1}\right)-\frac{1}{\alpha+1}\left(\frac{b-a}{2}\right)^{\alpha+1}(h^{\alpha+1}-1)+\left(h\frac{b-a}{2\alpha}\right)\left(\left(\frac{b-a}{2}\right)^{\alpha}-(b-x)^{\alpha}\right)$$
$$\left.+\frac{h}{\alpha}\left(\frac{b-a}{2}\right)^{\alpha+1}(h^{\alpha}-1)+\frac{1}{\alpha(\alpha+1)}\left(h\frac{b-a}{2}\right)^{\alpha+1}\right\}$$

or

$$\left|(1-h)(b-x)^0 f(x)-\frac{(b-x)^{1-\alpha}}{b-a}\Gamma(\alpha)J_a^{\alpha}(f(b))+J_a^{\alpha-1}(\omega(x,b)f(b))+\frac{h}{2}(b-x)^{1-\alpha}(b-a)^{\alpha-1}f(a)\right|$$
$$\le\frac{M(b-x)^{1-\alpha}}{b-a}\left\{-\frac{1}{\alpha+1}\left[\left(1-\frac{h}{2}\right)^{\alpha+1}-1\right](b-a)^{\alpha+1}+\frac{1}{\alpha}\left[\left(1-\frac{h}{2}\right)^{\alpha+1}-\left(1-\frac{h}{2}\right)\right](b-a)^{\alpha+1}\right.$$
$$+\frac{1}{\alpha+1}(b-a)^{\alpha+1}\left[\left(\frac{1}{2}\right)^{\alpha+1}-\left(1-\frac{h}{2}\right)^{\alpha+1}\right]-\frac{(b-a)^{\alpha+1}}{\alpha}\left[\left(1-\frac{h}{2}\right)\left(\frac{1}{2}\right)^{\alpha}-\left(1-\frac{h}{2}\right)^{\alpha+1}\right] \tag{15}$$
$$+\frac{1}{\alpha+1}\left[(b-x)^{\alpha+1}-\left(\frac{b-a}{2}\right)^{\alpha+1}\right]-\frac{b-a}{\alpha}\left(1-\frac{h}{2}\right)\left[(b-x)^{\alpha}-\left(\frac{b-a}{2}\right)^{\alpha}\right]$$
$$\left.-\frac{1}{\alpha+1}\left[\left(h\frac{b-a}{2}\right)^{\alpha+1}-(b-x)^{\alpha+1}\right]+\left(h\frac{b-a}{2\alpha}\right)\left[\left(h\frac{b-a}{2}\right)^{\alpha}-(b-x)^{\alpha}\right]+\frac{1}{\alpha(\alpha+1)}\left(h\frac{b-a}{2}\right)^{\alpha+1}\right\}.$$

Proof. From Lemma , we get

$$\left|(1-h)(b-x)^0 f(x)-\frac{(b-x)^{1-\alpha}}{b-a}\Gamma(\alpha)J_a^{\alpha}(f(b))+J_a^{\alpha-1}(\omega(x,b)f(b))+\frac{h}{2}(b-x)^{1-\alpha}(b-a)^{\alpha-1}f(a)\right|$$
$$=\left|J_a^{\alpha}(\omega(x,b)f'(b))\right|$$

From (1) and (4), we get

$$\frac{1}{\Gamma(\alpha)}\left|\int_a^b(b-t)^{\alpha-1}\omega(x,t)f'(t)dt\right|\le\frac{1}{\Gamma(\alpha)}\int_a^b(b-t)^{\alpha-1}|\omega(x,t)|\left|f'(t)\right|dt\le\frac{M}{\Gamma(\alpha)}\int_a^b(b-t)^{\alpha-1}|\omega(x,t)|dt$$
$$\le\frac{M(b-x)^{1-\alpha}}{b-a}\left[\int_a^x\left|(b-t)^{\alpha-1}\left(t-\left(a+h\frac{b-a}{2}\right)\right)\right|dt\right. \tag{16}$$
$$\left.+\int_x^b\left|(b-t)^{\alpha-1}\left(\left(b-h\frac{b-a}{2}\right)-t\right)\right|dt\right]$$

We have two cases:

(i) For $x \in \left[a+h\frac{b-a}{2}, \frac{a+b}{2}\right]$ and $h \in [0,1]$

$$\int_a^x \left|(b-t)^{\alpha-1}\left(t-\left(a+h\frac{b-a}{2}\right)\right)\right| dt + \int_x^b \left|(b-t)^{\alpha-1}\left(\left(b-h\frac{b-a}{2}\right)-t\right)\right| dt \tag{17}$$
$$= \left[\int_a^{a+h\frac{b-a}{2}} (b-t)^{\alpha-1}\left(a+h\frac{b-a}{2}-t\right) dt + \int_{a+h\frac{b-a}{2}}^{x} (b-t)^{\alpha-1}\left(t-\left(a+h\frac{b-a}{2}\right)\right) dt\right.$$
$$+ \int_x^{\frac{a+b}{2}} (b-t)^{\alpha-1}\left(b-h\frac{b-a}{2}-t\right) dt$$
$$\left. + \int_{\frac{a+b}{2}}^{b-h\frac{b-a}{2}} (b-t)^{\alpha-1}\left(b-h\frac{b-a}{2}-t\right) dt + \int_{b-h\frac{b-a}{2}}^{b} (b-t)^{\alpha-1}\left(t-\left(b-h\frac{b-a}{2}\right)\right) dt\right]$$
$$= -\frac{(b-a)^{\alpha+1}}{\alpha+1}\left[\left(1-\frac{h}{2}\right)^{\alpha+1}-1\right] + \frac{(b-a)^{\alpha+1}}{\alpha}\left[\left(1-\frac{h}{2}\right)^{\alpha+1}-\left(1-\frac{h}{2}\right)\right]$$
$$+ \frac{1}{\alpha+1}\left[(b-x)^{\alpha+1}-(b-a)^{\alpha+1}\left(1-\frac{h}{2}\right)^{\alpha+1}\right]$$
$$-\frac{1}{\alpha}\left[\left(1-\frac{h}{2}\right)(b-a)(b-x)^{\alpha}-(b-a)^{\alpha+1}\left(1-\frac{h}{2}\right)^{\alpha+1}\right]$$
$$-\frac{1}{\alpha+1}\left[\left(\frac{b-a}{2}\right)^{\alpha+1}-(b-x)^{\alpha+1}\right] + \frac{1}{\alpha}\left(h\frac{b-a}{2}\right)\left[\left(\frac{b-a}{2}\right)^{\alpha}-(b-x)^{\alpha}\right]$$
$$-\frac{1}{\alpha+1}\left[\left(h\frac{b-a}{2}\right)^{\alpha+1}-\left(\frac{b-a}{2}\right)^{\alpha+1}\right] + \frac{1}{\alpha}\left(h\frac{b-a}{2}\right)\left[\left(h\frac{b-a}{2}\right)^{\alpha}-\left(\frac{b-a}{2}\right)^{\alpha}\right]$$
$$-\frac{1}{\alpha+1}\left(h\frac{b-a}{2}\right)^{\alpha+1} + \frac{1}{\alpha}\left(h\frac{b-a}{2}\right)^{\alpha+1}$$

Now, using (16) and (17), we get (14). For $h=0$, we get Anastassiou's result [1].

(ii) For $x \in \left[\frac{a+b}{2}, b-h\frac{b-a}{2}\right]$ and $h \in [0,1]$

$$\int_a^x \left|(b-t)^{\alpha-1}\left(t-\left(a+h\frac{b-a}{2}\right)\right)\right| dt + \int_x^b \left|(b-t)^{\alpha-1}\left(\left(b-h\frac{b-a}{2}\right)-t\right)\right| dt \tag{18}$$
$$= \left[\int_a^{a+h\frac{b-a}{2}} (b-t)^{\alpha-1}\left(a+h\frac{b-a}{2}-t\right) dt + \int_{a+h\frac{b-a}{2}}^{\frac{a+b}{2}} (b-t)^{\alpha-1}\left(t-\left(a+h\frac{b-a}{2}\right)\right) dt\right.$$
$$+ \int_{\frac{a+b}{2}}^{x} (b-t)^{\alpha-1}\left(t-\left(a+h\frac{b-a}{2}\right)\right) dt$$

$$+\int_{x}^{b-h\frac{b-a}{2}}(b-t)^{\alpha-1}\left(b-h\frac{b-a}{2}-t\right)dt+\int_{b-h\frac{b-a}{2}}^{b}(b-t)^{\alpha-1}\left(t-\left(b-h\frac{b-a}{2}\right)\right)dt\Bigg]$$

$$=-\frac{(b-a)^{\alpha+1}}{\alpha+1}\left(\left(1-\frac{h}{2}\right)^{\alpha+1}-1\right)+\frac{(b-a)^{\alpha+1}}{\alpha}\left(\left(1-\frac{h}{2}\right)^{\alpha+1}-\left(1-\frac{h}{2}\right)\right)$$

$$+\frac{1}{\alpha+1}(b-a)^{\alpha+1}\left(\left(\frac{1}{2}\right)^{\alpha+1}-\left(1-\frac{h}{2}\right)^{\alpha+1}\right)-\frac{1}{\alpha}(b-a)^{\alpha+1}\left(\left(1-\frac{h}{2}\right)\left(\frac{1}{2}\right)^{\alpha}-\left(1-\frac{h}{2}\right)^{\alpha+1}\right)$$

$$+\frac{1}{\alpha+1}\left((b-x)^{\alpha+1}-\left(\frac{b-a}{2}\right)^{\alpha+1}\right)-\frac{1}{\alpha}\left(1-\frac{h}{2}\right)(b-a)\left((b-x)^{\alpha}-\left(\frac{b-a}{2}\right)^{\alpha}\right)$$

$$-\frac{1}{\alpha+1}\left(\left(h\frac{b-a}{2}\right)^{\alpha+1}-(b-x)^{\alpha+1}\right)+\frac{1}{\alpha}h\frac{b-a}{2}\left(\left(h\frac{b-a}{2}\right)^{\alpha}-(b-x)^{\alpha}\right)$$

$$-\frac{1}{\alpha+1}\left(h\frac{b-a}{2}\right)^{\alpha+1}+\frac{1}{\alpha}\left(h\frac{b-a}{2}\right)^{\alpha+1}$$

Hence, we get (15).

For $h=0$, we get Anastassiou's result [1].

4 Another Ostrowski type fractional inequality

In this section, we deduce Ostrowski type inequality for twice differentiable mappings by using the identity (5).

Theorem 8. *Let $f:[a,b]\to\mathbb{R}$ be differentiable on (a,b) such that $f''(b)\in L_1(a,b)$, where $a<b$. If $|f''(x)|\leq\|f''\|_\infty$ for every $x\in[a,b]$ and $\alpha\geq 1$, then the following Ostrowski fractional inequalities hold:*

$$\Bigg|(1-h)(\alpha-1)(b-x)^{-1}f(x)-2\frac{(b-x)^{1-\alpha}}{b-a}\Gamma(\alpha)J_a^{\alpha-1}(f(b))+J_a^{\alpha-2}(\omega(x,b)f(b)) \tag{19}$$

$$+\frac{h}{2}\left[(b-x)^{1-\alpha}(b-a)^{\alpha-1}f'(a)+(\alpha-1)(b-x)^{1-\alpha}(b-a)^{\alpha-2}f(a)\right]$$

$$+(b-x)^{1-\alpha}(b-a)^{\alpha-2}f(a)+(1-h)f'(x)\Bigg|$$

$$\leq\|f''\|_\infty\frac{(b-x)^{1-\alpha}}{b-a}\Bigg\{-\frac{(b-a)^{\alpha+1}}{\alpha+1}\left(\left(1-\frac{h}{2}\right)^{\alpha+1}-1\right) \tag{20}$$

$$+\frac{(b-a)^{\alpha+1}}{\alpha}\left(\left(1-\frac{h}{2}\right)^{\alpha+1}-\left(1-\frac{h}{2}\right)\right) \tag{21}$$

$$+\frac{1}{\alpha+1}\left((b-x)^{\alpha+1}-(b-a)^{\alpha+1}\left(1-\frac{h}{2}\right)^{\alpha+1}\right) \tag{22}$$

$$-\frac{1}{\alpha}\left(\left(1-\frac{h}{2}\right)(b-a)(b-x)^{\alpha}-(b-a)^{\alpha+1}\left(1-\frac{h}{2}\right)^{\alpha+1}\right) \tag{23}$$

$$-\frac{1}{\alpha+1}\left(\left(\frac{b-a}{2}\right)^{\alpha+1}-(b-x)^{\alpha+1}\right)+\frac{1}{\alpha}\left(h\frac{b-a}{2}\right)\left(\left(\frac{b-a}{2}\right)^{\alpha}-(b-x)^{\alpha}\right) \tag{24}$$

$$-\frac{1}{\alpha+1}\left(\left(h\frac{b-a}{2}\right)^{\alpha+1}-\left(\frac{b-a}{2}\right)^{\alpha+1}\right)+\frac{1}{\alpha}\left(h\frac{b-a}{2}\right)\left(\left(h\frac{b-a}{2}\right)^{\alpha}-\left(\frac{b-a}{2}\right)^{\alpha}\right)$$
$$\left.-\frac{1}{\alpha+1}\left(h\frac{b-a}{2}\right)^{\alpha+1}+\frac{1}{\alpha}\left(h\frac{b-a}{2}\right)^{\alpha+1}\right\}$$

or

$$\Bigg|(1-h)(\alpha-1)(b-x)^{-1}f(x)-2\frac{(b-x)^{1-\alpha}}{b-a}\Gamma(\alpha)J_a^{\alpha-1}(f(b))+J_a^{\alpha-2}(\omega(x,b)f(b)) \tag{25}$$
$$+\frac{h}{2}\left[(b-x)^{1-\alpha}(b-a)^{\alpha-1}f'(a)+(\alpha-1)(b-x)^{1-\alpha}(b-a)^{\alpha-2}f(a)\right]$$
$$+(b-x)^{1-\alpha}(b-a)^{\alpha-2}f(a)+(1-h)f'(x)\Bigg|$$
$$\le\|f''\|_\infty\frac{(b-x)^{1-\alpha}}{b-a}\left\{-\frac{1}{\alpha+1}\left(\left(1-\frac{h}{2}\right)^{\alpha+1}-1\right)(b-a)^{\alpha+1}\right.$$
$$+\frac{1}{\alpha}\left(\left(1-\frac{h}{2}\right)^{\alpha+1}-\left(1-\frac{h}{2}\right)\right)(b-a)^{\alpha+1}$$
$$+\frac{1}{\alpha+1}(b-a)^{\alpha+1}\left(\left(\frac{1}{2}\right)^{\alpha+1}-\left(1-\frac{h}{2}\right)^{\alpha+1}\right)$$
$$-\frac{1}{\alpha}(b-a)^{\alpha+1}\left(\left(1-\frac{h}{2}\right)\left(\frac{1}{2}\right)^{\alpha}-\left(1-\frac{h}{2}\right)^{\alpha+1}\right)$$
$$+\frac{1}{\alpha+1}\left((b-x)^{\alpha+1}-\left(\frac{b-a}{2}\right)^{\alpha+1}\right)-\frac{1}{\alpha}\left(1-\frac{h}{2}\right)(b-a)\left((b-x)^{\alpha}-\left(\frac{b-a}{2}\right)^{\alpha}\right)$$
$$-\frac{1}{\alpha+1}\left(\left(h\frac{b-a}{2}\right)^{\alpha+1}-(b-x)^{\alpha+1}\right)+\frac{1}{\alpha}\left(h\frac{b-a}{2}\right)\left(\left(h\frac{b-a}{2}\right)^{\alpha}-(b-x)^{\alpha}\right)$$
$$\left.-\frac{1}{\alpha+1}\left(h\frac{b-a}{2}\right)^{\alpha+1}+\frac{1}{\alpha}\left(h\frac{b-a}{2}\right)\left(h\frac{b-a}{2}\right)^{\alpha}\right\}.$$

Proof. From (5), we get

$$\Bigg|(1-h)(\alpha-1)(b-x)^{-1}f(x)-2\frac{(b-x)^{1-\alpha}}{b-a}\Gamma(\alpha)J_a^{\alpha-1}(f(b))+J_a^{\alpha-2}(\omega(x,b)f(b))$$
$$+\frac{h}{2}\left[(b-x)^{1-\alpha}(b-a)^{\alpha-1}f'(a)+(\alpha-1)(b-x)^{1-\alpha}(b-a)^{\alpha-2}f(a)\right]$$
$$+(b-x)^{1-\alpha}(b-a)^{\alpha-2}f(a)+(1-h)f'(x)\Bigg|=\left|J_a^{\alpha}\left(\omega(x,b)f''(b)\right)\right|.$$

Now from (1) and (14), we get

$$\frac{1}{\Gamma(\alpha)}\left|\int_a^b(b-t)^{\alpha-1}\omega(x,t)f''(t)dt\right|\le\frac{1}{\Gamma(\alpha)}\int_a^b(b-t)^{\alpha-1}|\omega(x,t)|\left|f''(t)\right|dt$$

$$\leq \frac{\|f''\|_\infty}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} |\omega(x,t)|\, dt \leq \|f''\|_\infty \frac{(b-x)^{1-\alpha}}{b-a} \left[\int_a^x \left| (b-t)^{\alpha-1} \left(t - \left(a + h\frac{b-a}{2} \right) \right) \right| dt \right.$$
$$\left. + \int_x^b \left| (b-t)^{\alpha-1} \left(\left(b - h\frac{b-a}{2} \right) - t \right) \right| dt \right]$$

$$\leq \|f''\|_\infty \frac{(b-x)^{1-\alpha}}{b-a} \left\{ -\frac{(b-a)^{\alpha+1}}{\alpha+1} \left(\left(1-\frac{h}{2}\right)^{\alpha+1} - 1 \right) \right.$$
$$+ \frac{(b-a)^{\alpha+1}}{\alpha} \left(\left(1-\frac{h}{2}\right)^{\alpha+1} - \left(1-\frac{h}{2}\right) \right)$$
$$+ \frac{1}{\alpha+1} \left((b-x)^{\alpha+1} - (b-a)^{\alpha+1} \left(1-\frac{h}{2}\right)^{\alpha+1} \right)$$
$$- \frac{1}{\alpha} \left[\left(1-\frac{h}{2}\right)(b-a)(b-x)^{\alpha} - (b-a)^{\alpha+1} \left(1-\frac{h}{2}\right)^{\alpha+1} \right]$$
$$- \frac{1}{\alpha+1} \left(\left(\frac{b-a}{2}\right)^{\alpha+1} - (b-x)^{\alpha+1} \right) + \frac{1}{\alpha} \left(h\frac{b-a}{2} \right) \left(\left(\frac{b-a}{2}\right)^{\alpha} - (b-x)^{\alpha} \right)$$
$$- \frac{1}{\alpha+1} \left(\left(h\frac{b-a}{2}\right)^{\alpha+1} - \left(\frac{b-a}{2}\right)^{\alpha+1} \right) + \frac{1}{\alpha} \left(h\frac{b-a}{2} \right) \left(\left(h\frac{b-a}{2}\right)^{\alpha} - \left(\frac{b-a}{2}\right)^{\alpha} \right)$$
$$\left. - \frac{1}{\alpha+1} \left(h\frac{b-a}{2} \right)^{\alpha+1} + \frac{1}{\alpha} \left(h\frac{b-a}{2} \right)^{\alpha+1} \right\}$$
$$\leq \|f''\|_\infty \frac{(b-x)^{1-\alpha}}{b-a} \left\{ -\frac{1}{\alpha+1} \left(\left(1-\frac{h}{2}\right)^{\alpha+1} - 1 \right) (b-a)^{\alpha+1} \right.$$
$$+ \frac{1}{\alpha} \left(\left(1-\frac{h}{2}\right)^{\alpha+1} - \left(1-\frac{h}{2}\right) \right) (b-a)^{\alpha+1} + \frac{1}{\alpha+1} (b-a)^{\alpha+1} \left(\left(\frac{1}{2}\right)^{\alpha+1} - \left(1-\frac{h}{2}\right)^{\alpha+1} \right)$$
$$- \frac{1}{\alpha} (b-a)^{\alpha+1} \left(\left(1-\frac{h}{2}\right) \left(\frac{1}{2}\right)^{\alpha} - \left(1-\frac{h}{2}\right)^{\alpha+1} \right)$$
$$+ \frac{1}{\alpha+1} \left((b-x)^{\alpha+1} - \left(\frac{b-a}{2}\right)^{\alpha+1} \right) - \frac{1}{\alpha} \left(1-\frac{h}{2}\right)(b-a) \left((b-x)^{\alpha} - \left(\frac{b-a}{2}\right)^{\alpha} \right)$$
$$- \frac{1}{\alpha+1} \left(\left(h\frac{b-a}{2}\right)^{\alpha+1} - (b-x)^{\alpha+1} \right) + \frac{1}{\alpha} \left(h\frac{b-a}{2} \right) \left(\left(h\frac{b-a}{2}\right)^{\alpha} - (b-x)^{\alpha} \right)$$
$$\left. - \frac{1}{\alpha+1} \left(h\frac{b-a}{2} \right)^{\alpha+1} + \frac{1}{\alpha} \left(h\frac{b-a}{2} \right) \left(h\frac{b-a}{2} \right)^{\alpha} \right\}$$

Hence proved (19) and (25).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] G. Anastassiou, M.R. Hooshmandas, A. Ghasemi and F. Moftakharzadeh, Montgomery identities for fractional integrals and related fractional inequalities, Journal of Inequalities in Pure and Applied Mathematics, 10(4), 2009, Art. 97, 6 pp.

[2] E. Ahmed, A.S. Elgazzar, On Fractional Order Differential Equations Model for Nonlocal Epidemics, Physica A, Vol.379(2007),pp.607-614.

[3] W. M. Ahmad, R. El-Khazali, Fractional-order Dynamical Models of Love, Chaos Solitons and Fract., Vol.33(2007),pp.1367-1375.

[4] R. L. Bagley, R.A. Calico, Fractional order State Equations for the Control of Viscoelastically Damped Structures, J. Guid. Control Dyn., Vol.14(1991),pp.304-311.

[5] A. M. A El-Sayed, Fractional-order Diffusion-wave Equation, Int. J. Theor. Phys., Vol.35(1996),pp.311-322.

[6] D. Kusnezov, A. Bulgac, G.D. Dang, Quantum Levy Processes and Fractional Kinetics, Phys. Rev. Lett., Vol.82(1999),pp.1136-1139.

[7] N. Laskin, Fractional Market Dynamics, Physica A, Vol.287(2000),pp.482-492.

[8] A. E. Matouk, Stability Conditions, Hyperchaos and Control in a Novel Fractional Order Hyperchaotic System, Phys. Lett. A, Vol.373(2009),pp.2166-2173.

[9] D. S. Mitrinović, J. E. Pecarić and A. M. Fink, Inequalities for Functions and Their Integrals and Derivatives, Kluwer Academic Publishers, Dordrecht, 1994.

[10] A. Ostrowski, Uber die Absolutabweichung einer di erentienbaren Funktionen von ihren Integralimittelwert, Comment. Math. Hel. 10(1938), 226-227.

[11] I. Podlubny, Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. New York: Academic Press; 1999.

[12] A. Qayyum, I. Faye, M. Shoaib, On new generalized inequalities via Riemann-Liouville Fractional integration, Journal of Fractional Calculus and Applications, Vol. 6(1) Jan. 2015, pp. 91-100.

[13] B. Samet and H. Aydi, On Some Inequalities Involving Liouville–Caputo Fractional Derivatives and Applications to Special Means of Real Numbers, Mathematics, 6, 193, (2018), 9 pages.

[14] M. Z. Sarikaya and H. Yaldiz, On weighted montogomery identities for Riemann-Luville fractional integral, Konuralp Journal of Mathematics, Vol.1(2013), No. 1 pp. 48–53 .

[15] M. Z. Sarikaya and H. Ogunmez, On new inequalities via Riemann-Liouville Fractional integration, Abstract and Applied Analysis, Volume 2012, Article ID 428983, 10 pages.