Journal of Abstract and Computational Mathematics
http://www.ntmsci.com/jacm

Chebyshev wavelet solution of nonlinear ordinary differential equations with delays

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Received: 11 July 2019, Accepted: 19 August 2019
Published online: 02 October 2019.

Abstract: The purpose of this paper is to illustrate the use of the Chebyshev wavelet method in the solution of high-order nonlinear ordinary differential equations with variable, proportional and constant delays. The main advantage of using Chebyshev polynomials lies in the orthonormality property, which enables a decrease in the computational cost and runtime. The other advantage is that it does not require domain discretization. The application of the method transforms the nonlinear equation to a system of algebraic equations. The method is applied to five differential equations up to sixth order, and the results are compared with the exact solutions and other numerical solutions when available. The accuracy of the method is presented in terms of absolute errors. The numerical results demonstrate that the method is accurate, effectual and simple to apply.

Keywords: Chebyshev Wavelets, Nonlinear Ordinary Differential Equations, Variable Delay, Proportional Delay.

1 Introduction

Many physical phenomena are modeled by using both the present and the past states of the model. Hence, differential equations with time delays are needed in the modeling of real life situations. Applications of these equations can be seen in many areas such as human body control and multibody control systems, electric circuits, dynamical behaviour of a system in fluid mechanics, chemical engineering [1], spread of bacteriophage infection [2], population dynamics, epidemiology, physiology, immunology, neural networks and cell kinetics [3].

Several numerical techniques are introduced to find approximate solutions of nonlinear differential equations with proportional, constant and variable delays. These methods can be listed as: Aboodh transformation method [4], Adomian decomposition method [5], Power series method [6], Decomposition method [7], Differential transform method [8], Hermite wavelet based method [9], Variational iteration method [10,11], Power and Padé series based method [12], Spectral method [13], Variable multistep methods [14], Quasilinearization technique [15], Runge-Kutta-Fehlberg methods [16], Polynomial least squares method [17], Homotopy perturbation method [18], Variational approach [19], First Boubaker polynomial approach [20]. A new multi-step technique with differential transform method (DTM) [21], Legendre-Gauss collocation method [22], Runge-Kutta method with Hermite interpolation [23], Legendre wavelet method [24], and Lucas polynomial approach [25].

In the present study, our aim is to use Chebyshev wavelet method to solve the initial value problem in the form

$$\sum_{i=0}^{6}\sum_{j=0}^{J} P_{ij}(t)y^{(i)}(t-\tau_{ij}(t))+\sum_{p=0}^{2}\sum_{q=0}^{p} R_{pq}(t)y^{(p)}(\alpha_{pq}t)y^{(q)}(\beta_{pq}t)=g(t) \quad ,J\leq 6 \tag{1}$$

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subject to the initial conditions

$$y^{(i)}(0) = \lambda_i,\ i = 0, 1, ..., 5 \tag{2}$$

where $P_{ij}(t)$, $R_{pq}(t)$, $g(t)$ and the variable delays $\tau_{ij}(t)$ are given continuous functions on the interval $0 \leq t \leq 1$, α_{pq} and β_{pq} are given constants to express proportional delays. We note that the delays are nonnegative for $t \geq 0$. The case $\tau_{ij}(t) = a$ refers to constant delay.

Chebyshev wavelet method is applied to many problems such as one-dimensional heat equation [26], integro-differential form of Lane-Emden type [27], Abel's integral equation [28], advection-dispersion equation [29], nonlinear Fredholm-Hammerstein integral equation [30], porous catalysts model [31], nonlocal BVP [32], Fredholm integral equation [33], Ito-Volterra integral equation [34], ODE with non analytic solution [35], linear ODE [36], MHD squeezing flow [37], telegraph equation [38]. To the best knowledge of author, this is the first application to high-order nonlinear ODE with constant, proportional and variable delays.

In the next section, we describe the fundamentals of the wavelets and Chebyshev wavelets which is required for the representation of the differential equation and its solution.

2 Chebyshev Polynomials

Chebyshev polynomials of second kind, usually denoted by $U_m(x)$, are m-th degree polynomials defined by [39]

$$U_m(x) = \frac{\sin(m+1)\theta}{\sin\theta},\ \text{where}\, x = \cos\theta.$$

Here θ is in the interval $[0,\pi]$, corresponding to the range of $x \in [-1,1]$. These polynomials constitute a sequence of polynomials $\{U_m(x), m = 0, 1, 2, \ldots\}$ which can be constructed recursively by

$$U_{m+1}(x) = 2xU_m(x) - U_{m-1}(x),\ \text{for}\, m = 1, 2, 3, \ldots$$

starting with $U_0(x) = 1$ and $U_1(x) = 2x$.

Chebyshev polynomials constitute an orthogonal set with respect to weight function $w(x) = \sqrt{1-x^2}$, that is

$$\int_{-1}^{1} w(x)U_m(x)U_n(x)dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{\pi}{2}, & \text{if } m = n \end{cases}$$

When the interval of interest is $[0,1]$, the shifted Chebyshev polynomials $\tilde{U}_m(t)$ can be defined by means of a substitution $x = 2t-1$, such that $\tilde{U}_m(t) = U_m(2t-1)$. Due to this transformation, all results and properties should be converted to their shifted forms. Hence, the shifted Chebyshev polynomials are obtained by the recurrence relation

$$\begin{aligned} \tilde{U}_0(t) &= 1 \\ \tilde{U}_1(t) &= 2(2t-1) \\ \tilde{U}_{m+1}(t) &= 2(2m-1)\tilde{U}_m(t) - \tilde{U}_{m-1}(t),\ \text{for}\, m > 1, \end{aligned}$$

and orthogonal over the interval $[0,1]$ with respect to the weight function $\tilde{w}(t) = \sqrt{t-t^2}$, that is

$$\int_0^1 \tilde{w}(t)\tilde{U}_m(t)\tilde{U}_n(t)dt = \begin{cases} 0, & \text{if } m \neq n \\ \frac{\pi}{8}, & \text{if } m = n. \end{cases}$$

3 Chebyshev Wavelets and properties

A family of functions constructed by dilation and translation of a single function

$$\psi_{d,e}(t) = |d|^{-1/2}\psi\left(\frac{t-e}{d}\right),\ d,e \in R,\ d \neq 0,$$

is referred as wavelets (child wavelets). Here d is the dilation parameter, e is the translation parameter. Their generating function $\psi(t)$ is called mother wavelet.

If these parameters are restricted to be $d = d_0^{-k}$ and $e = ne_0d_0^{-k}$ for $d_0 > 1$, $e_0 > 0$ and for integers $k > 0$, n then we obtain the discrete wavelets which form an orthonormal basis in $L^2(R)$, in the case $d_0 = 2$, $e_0 = 1$. Chebyshev wavelets, Legendre wavelets, Haar wavelets are some of these discrete wavelets. Chebyshev wavelets like the others combine the basic properties of corresponding polynomials with a compact support, which gives it the advantage of being good at modeling localized features in applications [40]. Due to this and such advantages, Haar wavelet [41], Legendre wavelet [42,43], Chebyshev wavelet [44] are frequently used to solve varieties of differential and integral equations.

Second kind Chebyshev wavelets $\psi_{nm} = \psi_{nm}(k,n,m,t)$ with four arguments can be defined on $[0,1)$ for integers $k > 0$, n, using the shifted Chebyshev polynomials $\tilde{U}_m(t)$, of order m as

$$\psi_{nm}(t) = \begin{cases} \frac{1}{\sqrt{\pi}}2^{(k+3)/2}\tilde{U}_m(2^kt-n), & \text{if } \frac{n}{2^k} < t < \frac{n+1}{2^k}, \\ 0, & otherwise \end{cases}$$

where $m = 0,1,2,\ldots,M$ and $n = 0,1,2\ldots,2^k-1$ and t is normalized time. Here, the coefficient $\frac{1}{\sqrt{\pi}}2^{(k+3)/2}$ is used to get an orthonormal set of Chebyshev wavelets with respect to the dilated and translated weight function $\tilde{w}_n(t) = \tilde{w}(2^kt-n)$. Note that Chebyshev wavelets $\psi_{nm}(t)$ form a wavelet basis in $L^2(R)$, so that a square integrable function $f(t)$, defined in $[0,1]$ can be expanded by an infinite series of Chebyshev wavelets [44]

$$f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm}\psi_{nm}(t), \tag{3}$$

where $A_{nm} =< f(t), \psi_{nm}(t) >_{\tilde{w}_n} = \int_0^1 \tilde{w}_n(t)f(t)\psi_{nm}(t)dt$. If the series is truncated then

$$f(t) \approx \sum_{k=0}^{2^k-1}\sum_{m=0}^{M} A_{nm}\psi_{nm}(t) = \mathbf{A}^T\Psi(t) \tag{4}$$

where $\mathbf{A}$ and $\Psi(t)$ are $2^k(M+1)\times 1$ matrices in the form

$$\mathbf{A} = [A_{00}, A_{01}, \ldots, A_{0M}, A_{10}, \ldots, A_{1M}, A_{(2^k-1)0}, \ldots, A_{(2^k-1)M}]^T, \tag{5}$$

$$\Psi(t) = [\psi_{00}(t), \psi_{01}(t), \ldots, \psi_{0M}(t), \psi_{10}(t), \ldots, \psi_{1M}(t), \ldots, \psi_{(2^k-1)0}(t), \ldots, \psi_{(2^k-1)M}(t)]^T \tag{6}$$

3.1 Operational Matrix of Differentiation

The i-th derivative of the vector $\Psi(t)$, defined in Eq. (6) can be obtained by

$$\frac{d^i}{dt^i}\Psi(t) = \mathbf{D}^i\Psi(t), \tag{7}$$

where $\Psi(t)$ is the vector of Chebyshev wavelets, $\mathbf{D}^i$ is the i-th power of the $2^k(M+1)\times 2^k(M+1)$ operational matrix for differentiation $\mathbf{D}$, derived in [44] as

$$\mathbf{D} = \begin{pmatrix} \mathbf{F} & 0 & \dots & 0 \\ 0 & \mathbf{F} & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & \mathbf{F} \end{pmatrix} \tag{8}$$

where

$$\mathbf{F} = 2^{k+2}\begin{bmatrix} 0\;0\;0\;0\;0\;0 \cdots & 0 & 0 & \\ 1\;0\;0\;0\;0\;0 \cdots & 0 & 0 & \\ 0\;2\;0\;0\;0\;0 \cdots & 0 & 0 & \\ 1\;0\;3\;0\;0\;0 \cdots & 0 & 0 & \\ 0\;2\;0\;4\;0\;0 \cdots & 0 & 0 & \\ 1\;0\;3\;0\;5\;0 \cdots & 0 & 0 & \\ \vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots \dots & \vdots & \vdots & \\ 1\;0\;3\;0\;5\;0 \cdots & M & 0 & \}M \text{ is odd} \\ 0\;2\;0\;4\;0\;6 \cdots & M & 0 & \}M \text{ is even} \end{bmatrix}.$$

We note that F is a $(M+1)\times(M+1)$ submatrix whose components satisfy

$$F_{ab} = \begin{cases} 2^{k+2}b, & \text{a=2,...,(M+1), b=1,...,(a-1), and a+b is odd;} \\ 0, & \text{otherwise.} \end{cases}$$

3.2 Application of the Operational Matrix of Differentiation

Let $y(t)$ be the approximate solution of the given IVP Eqs. $(1)-(2)$. To find this solution, first we write it in terms of the series in Eq. (4), using Chebyshev wavelets as

$$y(t) = \sum_{k=0}^{2^k-1}\sum_{m=0}^{M} A_{nm}\psi_{nm}(t) = \mathbf{A}^T\Psi(t) \tag{9}$$

where A_{nm} are unknown coefficients to be determined.

Then, we use Eq. (7) and Eq. (8) to approximate the derivatives of $y(t)$ as follows, [43]

$$y^i(t) = \mathbf{A}^T\mathbf{D}^i\Psi(t), \quad \text{for } i = 1,...,6. \tag{10}$$

The differential equation (1) can be written by substituting $t-\tau_{ij}(t)$, $\alpha_{pq}t$ and $\beta_{pq}t$ in place of t in Eq. (10) as

$$\sum_{i=0}^{6}\sum_{j=0}^{J} P_{ij}(t)\mathbf{A}^T\mathbf{D}^{(i)}\Psi(t-\tau_{ij}(t)) \tag{11}$$
$$+\sum_{p=0}^{2}\sum_{q=0}^{p} R_{pq}(t)\left(\mathbf{A}^T\mathbf{D}^{(p)}\Psi(\alpha_{pq}t)\right)\left(\mathbf{A}^T\mathbf{D}^{(q)}\Psi(\beta_{pq}t)\right)=g(t),\;\; J\leq 6$$

In order to find the unknown coefficients A_{nm} in the vector $\mathbf{A}$, we need $2^k(M+1)$ equations. The first six equations are obtained from the initial conditions

$$y(0)=\mathbf{A}^T\Psi(0),$$
$$y^{(i)}(0)=\mathbf{A}^T\mathbf{D}^i\Psi(0),\; i=1,\dots,5$$

and $(2^k(M+1))-6$ equations are obtained by substituting the first $(2^k(M+1))-6$ roots of shifted Chebyshev polynomial $\tilde{U}_{(2^k(M+1))}(t)$ in Eq. (11). The resulting system of nonlinear equations are solved for the coefficients A_{nm} by MATLAB tools; then, the approximate solution in Eq. (9) is computed.

4 Convergence Analysis of Chebyshev Wavelets

In this section, we analyze the convergence of the Chebyshev wavelet expansion of a function $f(t)$ defined in $[0,1]$, and error estimation of the approximation by using the theorems in [33].

Theorem: Let $f(t)$ be a square integrable function in $[0,1]$ with a bounded first and second derivatives such that $|f'(t)|\leq S_1$ and $|f''(t)|\leq S_2$, then $f(t)$ can be expanded by the infinite series of Chebyshev wavelet basis as in Eq. (3) and the series converges to $f(t)$ uniformly.

Proof: The justification of this theorem can be done similar to [33]. We begin with

$$A_{nm}=<f(t),\psi_{nm}(t)>_{\tilde{w}_n}=\int_0^1 \tilde{w}_n(t)f(t)\psi_{nm}(t)dt,$$

and we firstly assume that $m>1$. Since, $\psi_{nm}(t)$ is nonzero in the subintervals $\left[\frac{n}{2^k},\frac{n+1}{2^k}\right]$,

$$A_{nm}=\frac{2^{(k+3)/2}}{\sqrt{\pi}}\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}}\tilde{w}(2^kt-n)f(t)\tilde{U}_{nm}(2^kt-n)dt.$$

Note that $\tilde{U}_{nm}(2^kt-n)$ is shifted Chebyshev polynomial such that

$$\tilde{U}_{nm}(2^kt-n)=U_{nm}(2^{k+1}t-2n-1).$$

Using the substitution $2^kt-n=\frac{\cos\theta+1}{2}$, and $U_{nm}(2^{k+1}t-2n-1)=U_{nm}(\cos\theta)=\frac{\sin(m+1)\theta}{\sin\theta}$, $\tilde{w}_n(\frac{\cos\theta+1}{2})=\frac{\sin\theta}{2}$, we get

$$A_{nm}=\frac{2^{-(k+1)/2}}{\sqrt{\pi}}\int_0^{\pi} f\left(\frac{\cos\theta+2n+1}{2^{k+1}}\right)\sin((m+1)\theta)\sin\theta d\theta$$
$$=\frac{2^{-(k+3)/2}}{\sqrt{\pi}}\int_0^{\pi} f\left(\frac{\cos\theta+2n+1}{2^{k+1}}\right)[\cos(m\theta)-\cos((m+2)\theta)]d\theta$$

Applying integration by parts gives

$$A_{nm}=\frac{2^{-(3k+5)/2}}{\sqrt{\pi}}\int_0^{\pi}[\frac{1}{m}\sin m\theta-\frac{1}{m+2}\sin(m+2)\theta]\sin(\theta)f'\left(\frac{\cos\theta+2n+1}{2^{k+1}}\right)d\theta \tag{12}$$

Repeating this process gives

$$A_{nm}=\frac{2^{-5(k+1)/2}}{4\sqrt{\pi}}\int_0^{\pi}\sigma_m(\theta)f''\left(\frac{\cos\theta+2n+1}{2^{k+1}}\right)\sin\theta d\theta$$

where

$$\begin{aligned}\sigma_m(\theta)=&\frac{1}{m}\left[\frac{1}{m-1}\sin(m-1)\theta-\frac{1}{m+1}\sin(m+1)\theta\right]\\&-\frac{1}{m+2}\left[\frac{1}{m+1}\sin(m+1)\theta-\frac{1}{m+3}\sin(m+3)\theta\right]\end{aligned}$$

Note that

$$\begin{aligned}|\sigma_m(\theta)\sin\theta|\le&\left|\frac{\sin\theta}{m}\left[\frac{1}{m-1}\sin(m-1)\theta-\frac{1}{m+1}\sin(m+1)\theta\right]\right|\\&+\left|\frac{\sin\theta}{m+2}\left[\frac{1}{m+1}\sin(m+1)\theta-\frac{1}{m+3}\sin(m+3)\theta\right]\right|\\&\le\left|\frac{1}{m}\right|\left(\left|\frac{1}{m-1}\right|+\left|\frac{1}{m+1}\right|\right)+\left|\frac{1}{m+2}\right|\left(\left|\frac{1}{m+1}\right|+\left|\frac{1}{m+3}\right|\right)\\&\le\frac{4}{(m-1)(m+3)}\\&<\frac{4}{(m-1)^2}\end{aligned}$$

Since $n+1\le 2^k$, $|A_{nm}|<\frac{S_2\sqrt{\pi}}{4\sqrt{2}}\frac{1}{(n+1)^{5/2}(m-1)^2}$, for $n\ge 0$ and $m>1$.

Now, if $m=1$, we use Eq. (12), then

$$|A_{n1}|<\frac{S_1\sqrt{\pi}}{3\sqrt{2}}\frac{1}{(n+1)^{3/2}}.$$

Thus, the infinite series $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_{nm}$ is absolutely convergent and note that if $m=0$, the sequence $\{\psi_{n0},\quad n=0,1,\ldots,2^k-1\}$ construct an orthogonal set of Haar scaling function which implies the convergence of $\sum_{n=0}^{\infty}A_{n0}\psi_{n0}(t)$, [45]. So, the convergence of the series $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_{nm}\psi_{nm}(t)$ is coming from

$$\begin{aligned}\left|\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_{nm}\psi_{nm}(t)\right|&\le\left|\sum_{n=0}^{\infty}A_{n0}\psi_{n0}(t)\right|+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}|A_{nm}||\psi_{nm}(t)|\\&\le\left|\sum_{n=0}^{\infty}A_{n0}\psi_{n0}(t)\right|+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}|A_{nm}|<\infty \text{ for } t\in[0,1].\end{aligned}$$

Lemma: The Chebyshev wavelet series expansion Eq. (3) of a continuous function $f(t)$ converges to a unique function

$f(t)$.

Proof: Assume that the infinite series of Chebyshev wavelet basis converges to the function $f(t)$ that is

$$f(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_{nm}\psi_{nm}(t),$$

where $A_{nm}=< g(t),\psi_{nm} >_{\tilde{w}_n}$. Now we multiply $f(t)$ by $\psi_{rs}(t)\tilde{w}_n(t)$ for fixed values of r and s then integration term by term gives the inner product

$$\begin{aligned}< f(t),\psi_{rs} >_{\tilde{w}_n}=\int_0^1 f(t)\psi_{rs}(t)\tilde{w}_n(t)dt &= \int_0^1\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_{nm}\psi_{nm}(t)\psi_{rs}(t)\tilde{w}_n(t)dt\\ &=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_{nm}\int_0^1\psi_{nm}(t)\psi_{rs}(t)\tilde{w}_n(t)dt\\ &=A_{rs}\\ &=< g(t),\psi_{rs} >_{\tilde{w}_n} .\end{aligned}$$

This implies $f(t)=g(t)$ which is the desired result.

These theorems stated above imply the convergence of the infinite series of Chebyshev wavelets $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_{nm}\psi_{nm}(t)$ to a unique function $f(t)$. If the only finite terms of the approximation are considered then the error bound $|f(t)-\sum_{n=0}^{2^k-1}\sum_{m=0}^{M}A_{nm}\psi_{nm}(t)|$ can be determined by the next theorem.

Theorem: Let $f(t)$ be i-times differentiable function on $[0,1]$, then there exists a mean error bound for the approximation of Chebyshev wavelets $\sum_{n=0}^{2^k-1}\sum_{m=0}^{M}A_{nm}\psi_{nm}(t)=\mathbf{A}^T\Psi(t)$ to $f(t)$ as follows [46]

$$||f(t)-\mathbf{A}^T\Psi(t)||\leq\frac{1}{i!2^{ik}}\sup_{t\in[0,1]}|f^{(i)}(t)|$$

where $\mathbf{A}$ and $\Psi(t)$ are defined in Eq. (5) and Eq. (6).

Proof: If we assume that the function $f(t)$, defined in $[0,1]$, is i-times continuously differentiable function then there exists an approximation of Chebyshev wavelets $\sum_{n=0}^{2^k-1}\sum_{m=0}^{M}A_{nm}\psi_{nm}(t)=\mathbf{A}^T\Psi(t)$ to $f(t)$ where $\mathbf{A}$ and $\Psi(t)$ are defined in Eqs. (5) and (6), the mean error bound for this approximation follows

$$||f(t)-A^T\Psi(t)||\leq\frac{1}{i!2^{ik}}\sup_{t\in[0,1]}|f^{(i)}(t)|$$

To show this inequality, the interval $[0,1]$ is divided into subintervals $[\frac{n}{2^k},\frac{n+1}{2^k}]$ and $f(t)$ is approximated by $i-$th degree polynomial $\mathbf{A}^T\Psi(t)$ in these subintervals. Then

$$\begin{aligned}||f(t)-\mathbf{A}^T\Psi(t)||^2&=\sum_{n=0}^{2^k-1}\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}}\tilde{w}_n(t)[f(t)-A^T\Psi(t)]^2dt\\ &\leq\sum_{n=0}^{2^k-1}\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}}\tilde{w}_n(t)[f(t)-f^*(t)]^2dt\end{aligned}$$

where $f^*(t)$ is the $i-$th order interpolation of $f(t)$ on these subintervals with the following error bound [46]

$$|f(t)-f^*(t)| \leq \frac{1}{i!2^{ik}} \sup_{t\in[0,1]} |f^{(i)}(t)|$$

Then

$$\begin{aligned}||f(t)-\mathbf{A}^T\Psi(t)||^2 &\leq \sum_{n=0}^{2^k-1}\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} \tilde{w}_n(t)[\frac{1}{i!2^{ik}} \sup_{t\in[0,1]} |f^{(i)}(t)|]^2 dt \\ &\leq \int_0^1 \tilde{w}_n(t)[\frac{1}{i!2^{ik}} \sup_{t\in[0,1]} |f^{(i)}(t)|]^2 dt \\ &= ||\frac{1}{i!2^{ik}} \sup_{t\in[0,1]} |f^{(i)}(t)|||^2\end{aligned}$$

Taking square roots of both sides gives the desired result.

5 Results and Discussion

In this section, we solve five problems to show that the application of the method results in either the exact solution or accurate numerical solutions. The first two problems are first and second order nonlinear differential equations with variable delays. The third problem is a second order equation with both constant and proportional delays. The last two problems are third and sixth order equations with proportional delays. We solve these problems using quite small values of M and compare the results with the exact and other numerical solutions. Comparisons are given in terms of absolute error tables.

5.1 Problem 1

As a first example we consider the first order nonlinear differential equation with variable delay t^2

$$\begin{cases} y'(t)+ty(t-t^2)+ty^2(t)=1+t^2, \quad 0\leq t\leq 1 \\ y(0)=0 \end{cases} \tag{13}$$

The analytical solution of the above problem is $y(t)=t$. We approximate the solution by taking $k=0$ and $M=1$ as

$$y(t)=\mathbf{A}^T\Psi(t)=\begin{bmatrix} A_{00} \; A_{01}\end{bmatrix}\begin{bmatrix}\psi_{00}(t)\\ \psi_{01}(t)\end{bmatrix}$$

where A_{00} and A_{01} are unknown coefficients, $\psi_{00}(t)=\frac{2\sqrt{2}}{\pi}$, and $\psi_{01}(t)=\frac{2\sqrt{2}}{\pi}(4t-2)$. Hence, $y(t)$ and $y(t-t^2)$ can be written as

$$\begin{aligned} y(t) &= A_{00}\tfrac{2\sqrt{2}}{\pi}+A_{01}\tfrac{2\sqrt{2}}{\pi}(4t-2) \\ y(t-t^2) &= A_{00}\tfrac{2\sqrt{2}}{\pi}+A_{01}\tfrac{2\sqrt{2}}{\pi}(4(t-t^2)-2). \end{aligned} \tag{14}$$

Following the procedure described in the previous section, we approximate $y'(t)$ as follows

$$y'(t)=\mathbf{A}^T\mathbf{D}\Psi(t)=\begin{bmatrix} A_{00} \; A_{01}\end{bmatrix}\begin{bmatrix}0 & 0\\ 4 & 0\end{bmatrix}\begin{bmatrix}\frac{2\sqrt{2}}{\pi}\\ \frac{2\sqrt{2}}{\pi}(4t-2)\end{bmatrix}=A_{01}\frac{8\sqrt{2}}{\pi}.$$

Once we substitute these approximations in Eq. (13), we get

$$A_{01}\frac{8\sqrt{2}}{\pi}+t(A_{00}\frac{2\sqrt{2}}{\pi}+A_{01}\frac{2\sqrt{2}}{\pi}(4(t-t^2)-2))+t(A_{00}\frac{2\sqrt{2}}{\pi}+A_{01}\frac{2\sqrt{2}}{\pi}(4t-2))^2-1-t^2=0. \quad (15)$$

Solution of the problem requires two equations. The first equation is obtained by inserting the initial condition $y(0)=0$ into Eq. (14)

$$A_{00}\frac{2\sqrt{2}}{\pi}+A_{01}(\frac{-4\sqrt{2}}{\pi})=0.$$

and the second equation is obtained by inserting the first root of the second order shifted Chebyshev polynomial, $t=0.75$, into Eq. (15)

$$A_{01}\tfrac{8\sqrt{2}}{\pi}+0.75(A_{00}\tfrac{2\sqrt{2}}{\pi}+A_{01}\tfrac{2\sqrt{2}}{\pi}(4(0.75-(0.75)^2)-2))$$
$$+0.75(A_{00}\tfrac{2\sqrt{2}}{\pi}+A_{01}\tfrac{2\sqrt{2}}{\pi}(4(0.75)-2))^2-1-(0.75)^2=0$$

Solving this 2×2 system of nonlinear equations using Newton's method results in

$$\mathbf{A}^T=\left[A_{00},A_{01}\right]=\left[0.313328,0.156664\right]$$

Thus, the solution of the given initial value problem is obtained as

$$y(t)=\mathbf{A}^T\Psi(t)=\left[0.313328,0.156664\right]\begin{bmatrix}\frac{2\sqrt{2}}{\pi}\\ \frac{2\sqrt{2}}{\pi}(4t-2)\end{bmatrix}=t,$$

which is the exact solution. This shows that the proposed method is capable of finding the analytical solution once it is in the form of a polynomial.

5.2 Problem 2:

The second example is a second order DE with variable delays t^2 and $\frac{-t}{2}$

$$y''(t)+y'(t-t^2)-t^2y(t+t/2)+(y'(t))^2-y'(t)y(t)=e^t+e^{t-t^2}-t^2e^{3t/2}$$

with the initial conditions

$$y(0)=y'(0)=1.$$

In this problem, the exact solution is $y(t)=e^t$. We solved this problem using $k=0$ and several values of M. Recall that M refers to the order of the approximated polynomial. Table 1 present the results for $M=2,3,4,$ and 5. The absolute error is calculated by $E_M(t_i)=|y(t_i)-y_M(t_i)|$ for each $t_i\in[0,1]$. We observe that increasing M results in a decrease in the absolute error as expected. We also observed that the error increases as t gets closer to 1 for each case.

Table 1: Comparison of the absolute errors for M=2,3,4 and 5

t_i	$E_2(t_i)$	$E_3(t_i)$	$E_4(t_i)$	$E_5(t_i)$
0.1	2.227358e-03	4.805855e-05	3.196217e-06	1.543342e-07
0.2	8.190349e-03	8.585366e-05	3.568915e-06	1.079453e-07
0.3	1.672568e-02	3.847302e-05	8.672288e-07	1.398890e-07
0.4	2.654773e-02	4.665166e-05	3.471759e-06	5.019619e-07
0.5	3.623565e-02	1.312229e-05	1.552754e-05	6.806498e-07
0.6	4.421917e-02	5.498689e-04	2.272381e-05	1.231363e-08
0.7	4.876286e-02	2.060809e-03	2.342189e-05	2.060657e-07
0.8	4.794880e-02	5.225680e-03	2.229323e-04	7.046704e-06
0.9	3.965732e-02	1.092593e-02	7.465659e-04	4.010630e-05
1	2.154587e-02	2.026593e-02	1.857031e-03	1.362319e-04

5.3 Problem 3:

The third problem is a second order nonlinear DE with both proportional and constant delays [11,17,18]

$$y''(t) - e^{-t}y'(t-1/5) - (t-1)y(t) + 2y(t/3) - y^2(t) = -\tfrac{1}{4}\sin^2(\tfrac{t}{3}) + \tfrac{4}{9}\sin(\tfrac{t}{3}) + \sin(\tfrac{t}{9}) -$$

$$-e^{-t}(\tfrac{1}{6}\cos(\tfrac{t}{3} - \tfrac{1}{15}) - \tfrac{1}{6}\sin(\tfrac{t}{2} - \tfrac{1}{10})) - \tfrac{1}{9}\cos^2(\tfrac{t}{2}) + \tfrac{1}{4}\cos(\tfrac{t}{2}) + \tfrac{2}{3}\cos(\tfrac{t}{6}) - \tfrac{1}{3}\sin(\tfrac{t}{3})\cos(\tfrac{t}{2}) +$$

$$+t(-\tfrac{1}{2}\sin(\tfrac{t}{3}) - \tfrac{1}{3}\cos(\tfrac{t}{2}))$$

with the mixed initial conditions

$$3y(0) + 6y'(0) = 2, \quad -2y(0) + y'(0) = -1/2$$

Exact solution of this problem is $y(t) = \frac{1}{2}\sin(\frac{t}{3}) + \frac{1}{3}\cos(\frac{t}{2})$. We solved this problem by taking $k = 0$ and $M = 4$ and the numerical solution is obtained as

$$y(t) = A_{00}\tfrac{2\sqrt{2}}{\pi} + A_{01}\tfrac{2\sqrt{2}}{\pi}(4t-2) + A_{02}\tfrac{2\sqrt{2}}{\pi}(16t^2 - 16t + 3)$$

$$+A_{03}\tfrac{2\sqrt{2}}{\pi}(64t^3 - 96t^2 + 40t - 4) + A_{04}\tfrac{2\sqrt{2}}{\pi}(256t^4 - 512t^3 + 336t^2 - 80t + 5)$$

where $A_{00} = 0.252614, A_{01} = 0.019263, A_{02} = -0.001755, A_{03} = -1.2992e-05, A_{04} = 2.1589e-06$.

Table 2 presents the comparison of absolute errors of the present method with variational iteration method (VIM) [11], polynomial least square method (PLSM) with a $5-th$ order polynomial [17] and Homotopy perturbation method (HPM) with $n = 3$ [18]. One can see that the present method has a better accuracy with a smaller degree polynomial.

Table 2: Comparison of the solutions in terms of absolute errors

t_i	**Present Method**	**VIM** [11]	**PLSM** [17]	**HPM** [18]
0.2	4.60e-09	5.00e-05	1.02e-08	2.44e-05
0.4	1.75e-08	1.00e-04	6.14e-08	6.11e-05
0.6	2.08e-08	5.00e-05	8.92e-08	8.60e-05
0.8	5.53e-08	1.00e-04	1.02e-07	9.66e-05
1	4.68e-08	5.00e-04	1.53e-07	1.18e-04

5.4 Problem 4:

The fourth problem is a third order equation with proportional delay [5,7]

$$y'''(t)+1-2y^2(\frac{t}{2})=0 \quad 0\le t\le 1$$

with the initial conditions

$$y(0)=0, \quad y'(0)=1, \quad y''(0)=0.$$

Analytical solution of this equation is $y(t)=sin(t)$. Comparison of the present method using $5-th$ and $6-th$ order polynomials, Decomposition method (DM) with $13-th$ order polynomial [7], and Adomian decomposition method (ADM) with $9-th$ order polynomial [5] are given in Table 3. The results in [5] were not given explicitly, so we derived this polynomial for comparison purposes. When the results are compared, one can see that approximately the same accuracy is obtained by using smaller degree polynomials with the present method. This shows that the present method is more efficient than the other two methods.

Table 3: Comparison of the absolute errors

t_i	**Present Method**	**DM** [7]	**ADM** [5]
0.1	7.28e-09	0.0	1.02e-15
0.2	2.10e-08	0.0	5.28e-13
0.3	2.73e-08	0.0	2.02e-11
0.4	7.87e-08	0.0	2.69e-10
0.5	2.04e-07	2.61e-09	2.00e-09
0.6	2.66e-08	1.04e-08	1.03e-08
0.7	2.01e-06	4.07e-08	4.12e-08
0.8	9.91e-06	1.38e-07	1.36e-07
0.9	3.19e-05	4.00e-07	3.92e-07
1	8.29e-05	1.03e-06	1.00e-06

5.5 Problem 5:

The final problem is a sixth order equation with again proportional delay

$$y^{vi}(t)=1-2y^2(\frac{t}{2})=0; \quad 0\le t\le 1$$

with the initial conditions

$$y(0)=1, \quad y'(0)=0, \quad y''(0)=-1, \quad y'''(0)=0, \quad y^{iv}(0)=1, \quad y^{v}(0)=0.$$

Exact solution of this equation is $y(t)=cos(t)$. We solved this problem by taking $k=0$ and $M=6$ and obtained the solution as

$$y(t) = A_{00}\frac{2\sqrt{2}}{\pi} + A_{01}\frac{2\sqrt{2}}{\pi}(4t-2) + A_{02}\frac{2\sqrt{2}}{\pi}(16t^2-16t+3)$$

$$+A_{03}\frac{2\sqrt{2}}{\pi}(64t^3-96t^2+40t-4)$$

$$+A_{04}\frac{2\sqrt{2}}{\pi}(256t^4-512t^3+336t^2-80t+5)$$

$$+A_{05}\frac{2\sqrt{2}}{\pi}(1024t^5-2560t^4+2304t^3-896t^2+140t-6)$$

$$+A_{06}\frac{2\sqrt{2}}{\pi}(4096t^6-12288t^5+14080t^4-7680t^3+2016t^2-224t+7)$$

where $A_{00} = 0.5329$, $A_{01} = -0.0736$, $A_{02} = -0.0169$, $A_{03} = 0.000771$, $A_{04} = 8.8193e-05$, $A_{05} = -2.5480e-06$ and $A_{06} = -2.1234e-07$.
Table 4 presents the comparison of the absolute errors obtained from the present method and Legendre wavelet method (LWM) [24] for each point $t_i \in [0,1]$. $M = 6$ is used in both methods. It is observed that the accuracy of the methods are similar and both of them agree very well with the exact solution. This shows that the wavelet method is very effective even for high-order nonlinear delay differential equations.

Table 4: Comparison of the absoulte errors taking M=6

t_i	**Present Method**	**LWM** [24]
0.1	7.57727214306669e-13	2.01616501271928e-13
0.2	9.09050612563078e-13	3.46787043525865e-11
0.3	8.92359297566259e-10	1.29772448342891e-09
0.4	1.21052460277937e-08	1.43828530196899e-08
0.5	8.08965713305909e-08	8.95849524562564e-08
0.6	3.67981882964941e-07	3.93925250197213e-07
0.7	1.30367234418838e-06	1.36909182280043e-06
0.8	3.86789826289924e-06	4.01366510993650e-06
0.9	1.00462337447871e-05	1.03417449105470e-05
1	2.35222648431455e-05	2.40783212364093e-05

6 Conclusions

In this study, Chebyshev wavelet method is used to solve high-order nonlinear differential equations with variable, proportional and constant delays. Fiveproblems are considered to show the efficiency of the proposed method. The application of the method to the first problem results in the exact solution. The other problems are solved numerically. These solutions are compared with the exact solutions and other numerical solutions, and presented in terms of absolute error tables. It is observed that the present method yields high-order accuracy even for small values of M and absolute errors are lesser than the other numerical solutions.

References

[1] Ha, P.*Analysis and Numerical Solutions of Delay Differential-Algebraic Equations*. PhD Thesis, pp. 4-9, Berlin (2015)
[2] Stephen, A.G., Kuang, Y.*A Delay Reaction-Diffusion Model of the spread of Bacteriophage Infection*. SIAM J. Appl. Math. 65(2), 550-566 (2015)
[3] Bocharova, G.A., Rihanb, F.A. *Numerical modelling in biosciences using delay differential equations*. J. Comput. Appl. Math. 125, 183-199 (2000)

[4] Aboodh, K.S., Farah, R.A., Almardy, I.A., Osman, A.K. *Solving delay differential equations by Aboodh transformation method*. IJAMSS. 7(1), 21-30 (2018)

[5] Evans, D.J., Raslan, K.R. *The Adomian decomposition method for solving delay differential equations*. Int. J. Comput. Math. 82(1), 49-54 (2005)

[6] Benhammouda, B., Vazquez-Leal, H., Hernandez-Martinez, L. *Procedure for exact solutions of nonlinear Pantograph delay differential equations*. British J. Math. Comp. Sci. 40(19), 2738-2751 (2014)

[7] Taiwo, O.A., Odetunde, O.S. *On the Numerical Approximation of Delay Differential Equations by a Decomposition Method*. Asian J. Math. Stat. 3(4), 237-243 (2014)

[8] Mirzaee, F., Latifi, L. *Numerical Solution of Delay Differential Equations by Differential Transform Method*. J. Sci. IAU. 20(78/2), 83-88 (2011)

[9] Shiralashetti, S.C., Hoogar, B.S., Kumbinarasaiah, S. *Hermite Wavelet Based Method for the Numerical Solution of Linear and Nonlinear Delay Differential Equations*. Int. J. Eng. Sci. Math. 6(8), 71-79 (2017)

[10] Khader, M.M. *Numerical and theoretical treatment for solving linear and nonlinear delay differential equations using variational iteration method*. Arab J. Math. Sci. 19(2), 243-256 (2013)

[11] Yıldırım, A. , Koccak, H., Tutkun, S. *Reliable analysis for delay differential equations arising in mathematical biology*. J. King Saud Univ-Sci. 24, 359-365 (2012)

[12] Karimi-Vanani, S., Aminataei, A. *On the numerical solution of nonlinear delay differential equations*. J. Concrete Applicable Math. 8(4), 568-576 (2010)

[13] Ali, I., Brunner, H., Tang, T. *A Spectral Method for Pantograoh-Type Delay Differential Equations and its Convergence Analysis*. J. Comp. Math. 27(2-39), 254-265 (2009)

[14] Martin, J.A., Garcia, O. *Variable Multistep Methods for Delay Differential Equations*. Math. Computer Model. 35, 241-257 (2002)

[15] Ravi-Kanth, A.S.V., Murali-Mohan-Kumar, P. *A Numerical Technique for Solving Nonlinear Singularly Perturbed Delay Differential Equations*. Math. Model. Anal. 23(1), 64-78 (2018)

[16] Oberle, H.J., Pesch, H.J. *Numerical Treatment of Delay Differential Equations by Hermite Interpolation*. Numer. Math. 37, 235-255 (1981)

[17] Căruntu, B., Bota, C.*Analytical Approximate Solutions for a General Class of Nonlinear Delay Differential Equations*. Sci. World J. 2014, 6 pages (2014)

[18] Shakeri, F., Dehghan, M.*Solution of delay differential equations via a homotopy perturbation method*. Math. Computer Model. 48, 486-498 (2008)

[19] Abdullah, Z.A. *Solution of Nonlinear Ordinary Delay Differential Equations Using Variational Approach*. J. Al-Nahrain Univ. 15(4), 183-187 (2012)

[20] Davaeifar, S., Rashidinia, J. *Solution of a system of delay differential equations of multipantograph type*. J. Taibah Univ. Sci. 11, 1141-1157 (2017)

[21] Benhammouda, B., Vazquez-Leal, H. *A new multi-step technique with differential transform method for analytical solution of some nonlinear variable delay differential equations*. SpringerPlus. 5:1723 (2016)

[22] Wang, Z.Q., Wang, L.L. *A Legendre-Gauss Collocation Method for Nonlinear Delay Differential Equations*. DCDSS-B. 13(3), 685-708 (2010)

[23] Ismail, F., Al-Khasawneh, R.A., Lwin, A.S., Suleiman, M. *Numerical Treatment of Delay Differential Equations by Runge-Kutta Method Using Hermite Interpolation*. Matematika. 18(2), 79-90 (2002)

[24] Gümgüm, S., Özdek, D.E, Özaltun, G. *Legendre Wavelet Solution of High Order Nonlinear Ordinary Delay Differential Equations*. Turk. J. Math. 43, 1339-1352 (2019)

[25] Gümgüm, S., Baykuş-Savaşaneril, N., Kürkçü, Ö.K., Sezer, M. *Lucas Polynomial solution of Nonlinear Differential Equations with Variable Delay*. Hacet. J. Math. Stat. DOI: 10.15672/hujms.460975, 1-12 (2019)

[26] Hooshmandasi, M.R, Heydari, M.H., Maalek, F.M. *Numerical solution of the one-dimensional Heat equation by using Chebyshev wavelets method*. Applied Computat. Mathemat. 1:122 (2012)

[27] Sahu, P.K., Saha-Ray, S. *Chebyshev wavelet method for numerical solutions of integro-differential form of Lane-Emden type*. Int. J. Wavelets. 15(2) (2017)

[28] Sohrabi S. *Comparison Chebyshev wavelets method with BPFs method for solving Abel's integral equation*. Ain Shams Eng. J. 2, 249-254 (2011)

[29] Heydari, M. H. *A new approach of the Chebyshev wavelets for the variable-order time fractional mobile-immobile advection-dispersion model*. arXiv:1605.06332v1 [math.NA], 13 pages (2016)

[30] Shamooshaky, M.M., Assari, P., Adibi, H. *The Numerical Solution of Nonlinear Fredholm–Hammerstein Integral Equations of the Second Kind Utilizing Chebyshev Wavelets*. J. Math. Computer Sci. 10, 235-246 (2014)

[31] Pirabaharan, P., Chandrakumar, R.D., Hariharan, G. *An Efficient Wavelet Based Approximation Method for Estimating the Concentration of Species and Effectiveness Factors in Porous Catalysts*. MATCH Commun. Math. Comput. Chem. 73, 705-727 (2015)

[32] Çelik, I.*Chebyshev Wavelet collocation method for solving a class of linear and nonlinear nonlocal boundary value problems*. Fundamental J. Math. Appl. 1 (1), 25-35 (2018)

[33] Adibi, H., Assari, P.: Chebyshev Wavelet method for numerical solution of Fredholm integral equations of the first kind. Math. Probl. Eng. 138408, 17 pages (2010)

[34] Iqbal, J., Abass, R. *Second Kind Chebyshev Wavelet Galerkin Method for Stochastic Ito-Volterra Integral Equations*. Mediterr. J. Math. 13, 2613-2631 (2016)

[35] Hosseini, S.Gh. *A New Operational Matrix of Derivative for Chebyshev Wavelets and its Applications in Solving Ordinary Differential Equations with Non analytic Solution*. Appl. Math. Sci. 5(51), 2537-2548 (2011)

[36] Babolian, E., Fattahzadeh, F. *Numerical solution of differential equations by using Chebyshev wavelet operational matrix of integration*. Appl. Math. Comput. 188, 417-426 (2007)

[37] Iqbal, M.A., Khan, U., Ali, A., Mohyud-Din, S.T. *Shifted Chebyshev wavelet-quasilinearization technique for MHD squeezing flow between two infinite plates and JefferyeHamel flows*. Egyptian J. Basic Appl. Sci. 2, 229-235 (2015)

[38] Heydari, M.H., Hooshmandasl, M.R., Maalek Ghaini, F.M. *A new approach of the Chebyshev wavelets method for partial differential equations with boundary conditions of the telegraph type*. Appl. Math. Model. 38, 1597-1606 (2014)

[39] Mason, J.C., Handscomb, D.C. *Chebyshev Polynomials*. Chapman & Hall/CRC, Boca Raton, Fla, USA (2003)

[40] Goswami, J.C., Chan, A.K. *Fundamentals of Wavelets*, Theory, Algorithms and Applications. John Wiley and Sons Inc., New York (2011)

[41] Gu,J.S., Jiang, W.S. *The Haar wavelets operational matrix of integration*. Int. J. Syst. Sci. 27, 623-628 (1996)

[42] Razzaghi, M., Yousefi, S. *Legendre wavelets operational matrix of integration*. Int. J. Syst. Sci. 32, 495-502 (2001)

[43] Mohammadi, F., Hosseini, M.M. *A new Legendre wavelet operational matrix of derivative and its applications in solving the singular ordinary differential equations*. J. Franklin Inst. 348, 1787-1796 (2011)

[44] Abd-Elhamed, W.M., Doha, E.H., Youssri, Y.H. *New spectral second kind Cheybyshev wavelets algorithms for solving linear and nonlinear second order differential equations involving singular and Bratu-type equations*. Abstr. Appl. Anal. 2013, 1-9 (2013)

[45] Cohen, A. *Numerical Analysis of Wavelet Methods*, JAI Press Inc., Netherlands (2003)

[46] Biazar, J., Ebrahimi, H. *Chebyshev wavelets approach for nonlinear systems of Volterra integral equations*. Comput. Math. Appl. 63(3), 608-616 (2012)