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A mathematical model applied to understand the dynamical behavior of predator Prey model

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Abstract: Predator-prey model is useful and often used in the environmental science field because they allow researchers both to observe the dynamic of animal populations and make predictions as to how they will develop over time. One of the most ecological applications of differential equations system is predator prey problem. In this paper, we will discuss about shark and fish Lotka-Volterra modified predator prey model in differential equation. The solution, existence, uniqueness and boundedness of the solution of the model are investigated. We also analyze about the steady state and stability criteria using Jacobian matrix method. Finally, numerical simulations are carried out to justify analytical results.

Keywords: Lotka-Volterra model, steady-state, stability, Jacobian matrix method, numerical simulations.

1 Introduction

A predator is an animal that consumes another animal. The prey is the animal which the predator consumes. Some illustrations of predator and prey are tiger and deer, crocodile and fish and fox and rabbit. The words ”predator” and ”prey” are almost always used to mean only animals that eat animals, but the same idea also applies to plants: squirrel and berry, rabbit and lettuce, goat and leaf. Predator and prey express simultaneously. The prey is part and parcel of the predator’s environment and the predator die down if it does not get enough nutrition, so it sprouts whatever is necessary in order to consume the prey: agility, larceny, disguise (to hide while approaching the prey), a proper sense of flavor, sight, or hearing (to find the prey), immunity to the prey’s toxic, toxic (to kill the prey) the right kind of mouth parts or digestive system, etc. Like, the predator is portion of the prey’s circle and the prey dies down if it is eaten by the predator, so it sprouts whatever is obligate to get off being eaten: agility, disguise (to hide from the predator), a good sense of flavor, sight or hearing (to detect the predator), tine, toxic (to spray when approached or bitten) etc.

Mathematical modeling of exploitation of biological resources is still a very interesting field of research. Mathematical modeling is frequently an evolving process. Regulated mathematical exploration can often conduct to better understanding of bio economic models. The unfasten imbalance in turn lead to the essential exchange. The eventual model may or may not be free of any important imbalance but the exploration of the eventual model can thus be expected to publish significant and nontrivial features of the system. In the last decades, self-interest has been rising resolutely in the scheming and perusing of mathematical models of population interactions. The principle of exponential growth for human population was first propounded by Malthus [15] an English clergyman and political economist in the first edition of his book entitled an essay on the principle of population published in 1798. Belgian mathematician Verhulst introduced logistic equation as a model for human population growth in 1838. He mentions this as a logistic growth.Due

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to insufficient census data he was incapable to test the validity of this model. A biologist Humberto D'Ancona did a statistical study on the numbers of each species sold at three main Italian ports in 1920. By his study of these fish species from 1914-1923, he came over a dramatic conclusion. He supposes that this predator prey relevance between the sharks, rays and fish were in their regular states outside of human interaction, namely before and after the war. An American mathematical biologist Lotka who discovered and improvement many of the same conclusions and models as Volterra and around the same time. Finally, Vito Volterra wrote down a simple pair of differential equation to describe this system, which are known as predator-prey models.

Some of the prey-predator models were discussed by Wangersky [24], Varma [23] Colinvaux [7], Freedman [8], Narayan [17]. A population model with time delay was proposed by Kapur [5]. Volterra formulated a distributed time delay model for prey - predator ecological models. Kapur[10] discussed the solution in the closed form for that model. Perko [22] described the differential equation and dynamical system. Beretta and Takeuchi [5] described the global stability of single-species diffusion Volterra models with continuous time delays. Also, in the last decades many researchers described the Dynamical Behavior of Discrete Prey-Predator system with Scavenger [19], a stage-structured predator-prey model with distributed maturation delay and harvesting [10]. In this paper, we will consider an environmental model containing two related populations-a prey population, such as fish, and a predator population, such as sharks. we modified the Lotka-Volterra predator prey model by adding natural death rate of predator prey then we cheek the stability of the predator prey model using Jacobian matrix and the stability is shown by time series and phase diagram.

2 The Mathematical Model

In this segment we first present the Lotka- Volterra predator-prey model then we modified this model by adding natural death of prey and predator.

2.1 Lotka- Volterra Model

As a simplest form, the interaction between a predator and prey Lotka- Volterra model in terms of a couple of ordinary differential equations [12] can be represented by

$$\frac{dx}{dt} = ax - bxy, \frac{dy}{dt} = -cx + dxy \tag{1}$$

where

$$x(0) = x_0, y(0) = y_0 \tag{2}$$

$$a, b, c, d > 0. \tag{3}$$

Here, x(t)and y(t)represents the number of prey (fish) and predator (shark) ,a is the reproduction rate of prey, b is the proportional to the number of prey that a predator can eat, d is the amount of energy that a prey supplies to the consuming predator, c is the death rate of predator.

2.2 Assumptions

The following basic assumption are important to modify the above model * Fish only die by natural causes and eaten by sharks.

* Shark only die by natural causes and when fish die.

* The interaction between shark and fish the growth rate of fish decrease and the growth rate of shark increase. Based on the assumptions, the proposed model is

$$\frac{dx}{dt} = ax - bxy - rx^2, \frac{dy}{dt} = -cy + dxy - \delta y^2 \tag{4}$$

where

$$x(0) = x_0, y(0) = y_0 \tag{5}$$

$$a, b, c, d, r, \delta > 0 \tag{6}$$

$$r\text{is the natural death rate of prey and} \tag{7}$$

$$\delta is \text{is the natural death rate of predator} \tag{8}$$

2.3 Solution of the Model

We recall that When both the prey and predator are present, the model equations are

$$\frac{dx}{dt} = ax - bxy - rx^2, \frac{dy}{dt} = -cy + dxy - \delta y^2 \tag{9}$$

Dividing these equation

$$\frac{dy}{dx} = \frac{y(-c + dx - \delta y)}{x(a - by - rx)} \tag{10}$$

$$\frac{(a - by - rx)dy}{x} = \frac{(-c + dx - \delta y)dx}{y} \tag{11}$$

$$\frac{(a - by)dy}{y} - \frac{(rx)dy}{y} = \frac{(-c + dx)dx}{x} - \frac{(\delta y)dx}{x} \tag{12}$$

Integrating on both sides, we get

$$alny - by - rxlny = dx - clnx - \delta ylnx + lnk \tag{13}$$

$$lny^a - by - lny^{rx} = dx - lnx^c - lnx^{\delta y} + lnk \tag{14}$$

$$ln(\frac{y^a x^c x^{\delta y}}{e^{by} e^{dx} y^{rx}}) \tag{15}$$

$$\frac{y^a x^c x^{\delta y}}{e^{by} e^{dx} y^{rx}} = k \tag{16}$$

where is a constant of integration and equation (16) is the solution of the model (1).

3 Mathematical Analysis of the Model

In this segment we observe the boundedness of the Model, find out the different equilibrium point and cheek the stability at different equilibrium Point.

3.1 Boundedness of the System

Proposition 1. *All the solutions of the system (1) are uniformly bounded.*

Proof. Define a positive definite function was

$$w = x + y \tag{17}$$

From (1)

$$\frac{d(x+y)}{dt} \leq (1 - \frac{rx}{a}) \tag{18}$$

For arbitrarily chosen η this implies to

$$\frac{dw}{dt} + \eta \leq \alpha x(1 - \frac{rx}{a} + \frac{\eta}{a}) \tag{19}$$

The above equation has a solution

$$w \leq \alpha x(1 - \frac{rx}{a} + \frac{\eta}{a}), t \to \infty \tag{20}$$

implying that the solution is bounded for

$$0 \leq w \leq \alpha x(1 - \frac{rx}{a} + \frac{\eta}{a}) \tag{21}$$

Therefore, all the solutions of the system (1) are uniformly bounded in the region

$$\psi = \left\{ (x,y)\varepsilon R_+^2 : w \leq \alpha x(1 - \frac{rx}{a} + \frac{\eta}{a}) + \zeta \right\} \text{ for all} \zeta > 0, t \to \infty. \tag{22}$$

3.2 Existence of Equilibrium Points

In this segment, we first obtain the existence of the fixed points of the ordinary differential equations (1). The system (1) shows the following four equilibrium points

$$*E_0 = (0,0) \text{this point is the trivial equilibrium point and always exists} \tag{23}$$

$$*E_1 = (0, -\frac{c}{\delta}) \tag{24}$$

this point is the axial fixed equilibrium point and always exists, also in the absence of prey the predator population decrease to the carrying capacity.

$$*E_2 = (\frac{a}{r}, 0) \tag{25}$$

this point is the axial fixed point always exists, also in the absence of predator the prey population grows to the carrying capacity.

$$*E_3 = (\frac{a\delta + bc}{bd + \delta r}, \frac{ad - cr}{bd + \delta r}) \tag{26}$$

this point is the is the positive equilibrium point exists in the xy plane

3.3 Stability Analysis

In this segment, we observe their stability by considerate the Eigen values for the variation matrix of (1) at every equilibrium point.

3.3.1 Linear stability analysis

The Jacobian matrix of the model (1) at state variable is given by

$$J(x,y) = \begin{bmatrix} -by - 2rx + a & -bx \\ dy & dx - 2\delta y - c \end{bmatrix} \tag{27}$$

The linearized stability technique for analyzing the local behavior of the non-linear system (1) is given in the following theorem.

Theorem 1. *Let $F(\lambda) = A\lambda^2 + B\lambda + C$. Then the following statement are true.*

(i) *If each root of the equation is negative sign, then the standing point of the system is locally asymptotically stable and standing point is also called a sink.*
(ii) *If at-least one of the roots of the equation is positive sign, then the standing point of the system is saddle.*
(iii) *If each root of the equation is positive sign, then the then the standing point of the system is locally unstable point and standing point is also called a source.*
(iv) *The standing point of the system is called hyperbolic if the root of the equation has absolute value equal being positive. If there exists a root of the equation with absolute value being negative, then the standing point is called Non hyperbolic*

Theorem 2. *The system (1) have four equilibrium points*

$$(1)\, E_0 = (0,0) \textit{which is a saddle point.} \tag{28}$$

$$(2)\, E_1 = (0, -\frac{c}{\delta}) \textit{which is a locally unstable point.} \tag{29}$$

$$(3)\, E_2 = (\frac{a}{r}, 0) \textit{which is a saddle point.} \tag{30}$$

$$(4)\, E_3 = (\frac{a\delta + bc}{bd + \delta r}, \frac{ad - cr}{bd + \delta r}) \textit{which is locally asymptotically stable.} \tag{31}$$

Proof. (i) The Jacobian matrix of (1) at the equilibrium point $E_0 = (0,0)$ is given as follows

$$J(E_0) = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix} \tag{32}$$

The Eigen values are

$$\lambda_1 = -a, \lambda_2 = -c \tag{33}$$

Here we see that the Eigen values are opposite sign. So the equilibrium point E_0 is called saddle point

(ii) The Jacobian matrix of (1) at the equilibrium point $E_1 = (0, -\frac{c}{\delta})$ is given as follows

$$J(E_1) = \begin{bmatrix} \frac{bc}{\delta} + a & 0 \\ \frac{dc}{\delta} & c \end{bmatrix} \tag{34}$$

The Eigen values are

$$\lambda_1 = \frac{a\delta + bc}{\delta}, \lambda_2 = c \tag{35}$$

Here we see that the Eigen values are positive sign. So the equilibrium point

$$E_1 \text{ is called locally unstable point.} \tag{36}$$

(iii) The Jacobian matrix of (1) at the equilibrium point $E_2 = (\frac{a}{r}, 0)$ is given as follows

$$J(E_2) = \begin{bmatrix} -a & \frac{-ba}{r} \\ 0 & \frac{da}{r} - c \end{bmatrix} \tag{37}$$

The Eigen values are

$$\lambda_1 = -a, \lambda_2 = \frac{ad - cr}{r} \tag{38}$$

Here we see that the Eigen values are opposite sign. So the equilibrium point

$$E_2 \text{ is called saddle point.} \tag{39}$$

(iv) The Jacobian matrix of (1) at the equilibrium point $E_3 = (\frac{a\delta+bc}{bd+\delta r}, \frac{ad-cr}{bd+\delta r})$ is given as follows is given as follows

$$J(E_3) = \begin{bmatrix} -\frac{b(ad-cr)}{bd+\delta r} - \frac{2r(a\delta+bc)}{bd+\delta r} + a & -\frac{b(a\delta+bc)}{bd+\delta r} \\ \frac{d(ad-cr)}{bd+\delta r} & \frac{d(a\delta+bc)}{bd+\delta r} - \frac{2\delta(ad-cr)}{bd+\delta r} - c \end{bmatrix} \tag{40}$$

After solving this Jacobian matrix, the Eigen values are

$$\lambda_1 = \frac{ad\delta + a\delta r + bcr - c\delta r - i\sqrt{q}}{2(bd+\delta r)}, \lambda_2 = \frac{ad\delta + a\delta r + bcr - c\delta r + i\sqrt{q}}{2(bd+\delta r)} \tag{41}$$

Where

$$q = \sqrt{WW + WP} \tag{42}$$

where

$$WW = \left(4a^2bd^2\delta - a^2d^2\delta^2 + a^2d\delta^2 r + a^2r^2\delta^2 - 4ab^2cd^2\right)$$

and

$$WP = \left(2abcd\delta r + 2abc\delta r^2 - 2acd\delta^2 r^2 + 2ac\delta^2 r^2 + 4b^2c^2dr + b^2c^2r^2 + c^2\delta^2 r^2\right).$$

The point E_3 is stable because all of the real part is negative. The imaginary part expose that it will be periodic.

4 Numerical Simulations

In this segment, we take on the numerical simulations of the prey-predator model (1) for the case of different parameter values response. In the result, we plot diagrams for the prey and predator system, the trivial and axial equilibrium points of the model (1) and which is shown by the time series and phase diagram. The plots have been generated using ode45 and which are shown in the Figures 1-10. We observe that the equilibrium point is locally unstable because both of

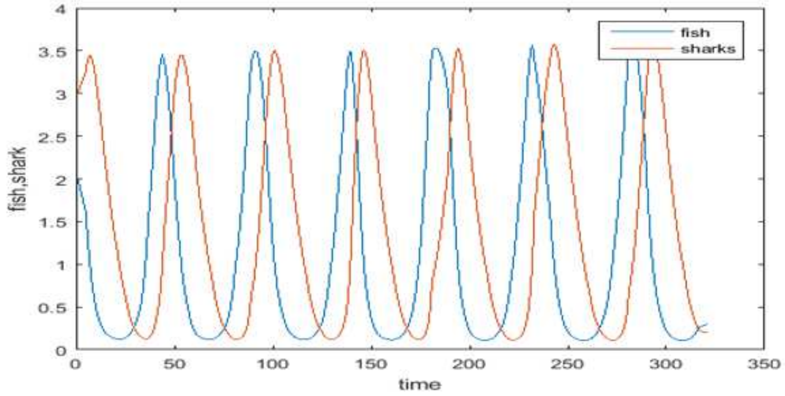

Fig. 1: Plot of shark and fish over time of the systen (1) whrere, $a = 6, b = 4, c = 3, d = 5, r = 2, \delta = 8$.

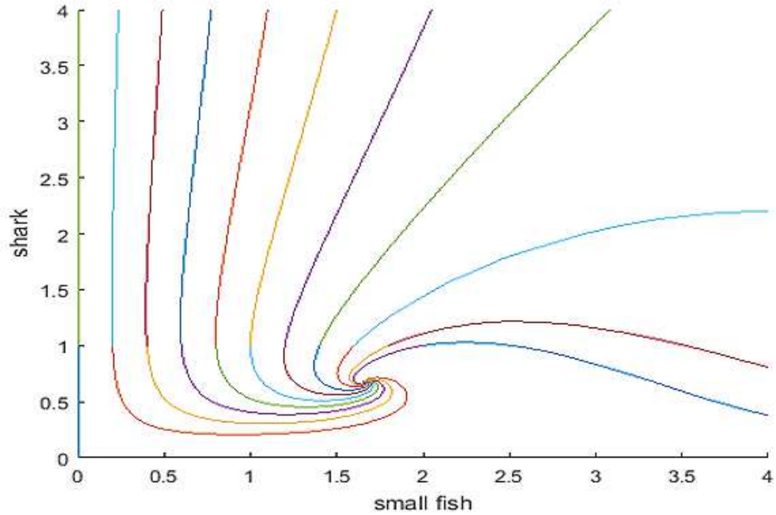

Fig. 2: Phase Plane of shark versus fish of the systen (1) whrere, $a = 6, b = 4, c = 3, d = 5, r = 2, \delta = 8$.

the Eigen values are positive(see Figure 1 and Figure 2) and this fixed point is a sink. The imaginary number imply that it will be periodic (see Figure 1 and figure 10). Both of the Figure shows that the prey population living with predator at the critical point. From Figure 3 and Figure 4, we observe that the critical point is saddle point because the Eigen values are opposite sign and from Figure 5 and Figure 6 observe that the critical point become(1,0) which is on the y-axis and identical to the stable point for the prey population in predator-free world. So here the predator dies out again. Also, From Figure 7 and Figure 8, we observe that we get an ellipse almost the above critical point. So, in the above cyclic pattern both the prey and predator population increase and decrease. Also, we observe that the equilibrium point is stable because both of the real part is negative and the imaginary number imply that it will be periodic (see Figure 9 and Figure 10).

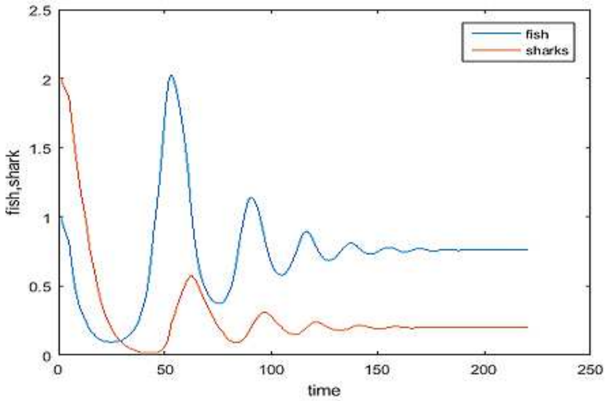

Fig. 3: Plot of shark and fish over time of the system(1)whrere, $a = 2, b = 4, c = 3, d = 5, r = 6, \delta = 4$.

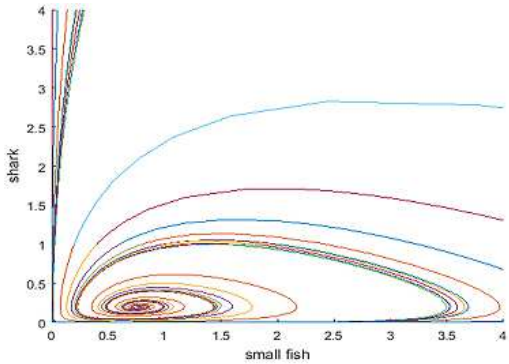

Fig. 4: Phase Plane of shark versus fish of the system(1)whrere, $a = 2, b = 4, c = 3, d = 5, r = 6, \delta = 4$.

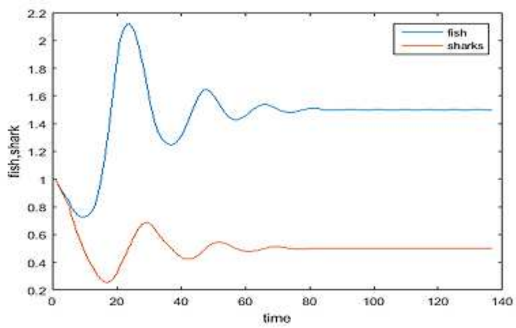

Fig. 5: Plot of shark and fish over time of the system(1)whrere, $a = 1, b = 1, c = 1, d = 1, r = 1, \delta = 1$.

5 Conclusion

In this paper, we consider a dynamical behavior of predator prey model then we establish a modiefied Lotka-Volterra predator prey model then we can polish the equations by adding more variables and getting a better content of the ecology. we cheek boundedness of the system and present the existence of equilibrium points. We have shown that this model has

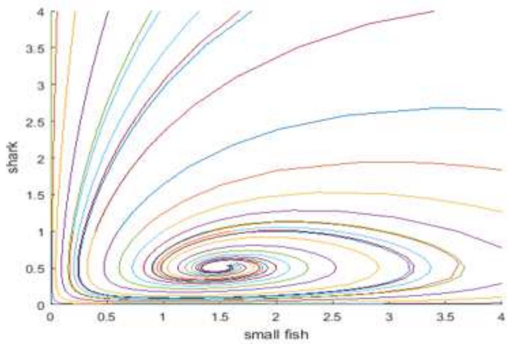

Fig. 6: Phase Plane of shark versus fish of the system(1)whrere, $a = 1, b = 1, c = 1, d = 1, r = 1, \delta = 1$.

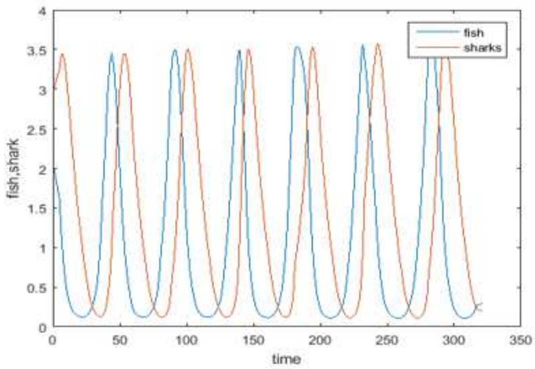

Fig. 7: Plot of shark and fish over time of the system(1)whrere, $a = 2, b = 1, c = 1, d = 1, r = 0, \delta = 0$.

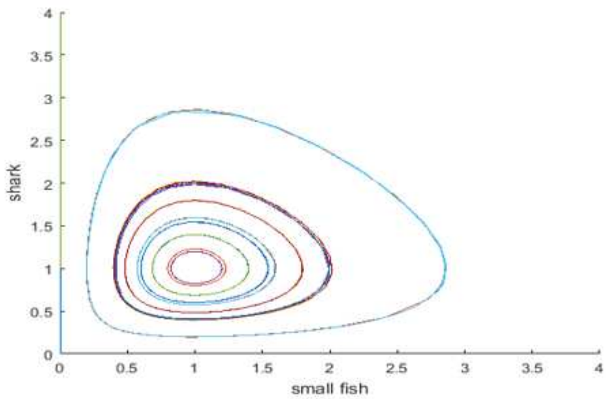

Fig. 8: Phase Plane of shark versus fish of the system(1)whrere, $a = 2, b = 1, c = 1, d = 1, r = 0, \delta = 0$.

four equilibrium points of which two are saddle point, one is locally unstable point and one is locally asymptotically stable point (see Theorem 2). We also analyze the stability of the proposed predator prey model. Finally, it is graphically shown that the analytic result using different parameter values.

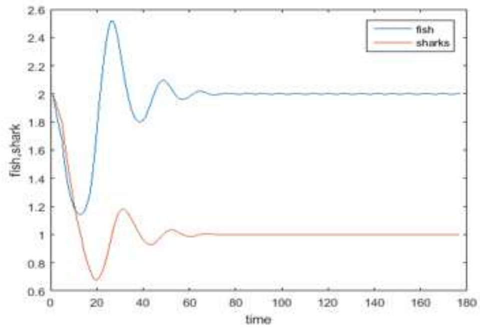

Fig. 9: Plot of shark and fish over time of the system(1)whrere, $a = 2, b = 1, c = 1, d = 1, r = 1, \delta = 1$.

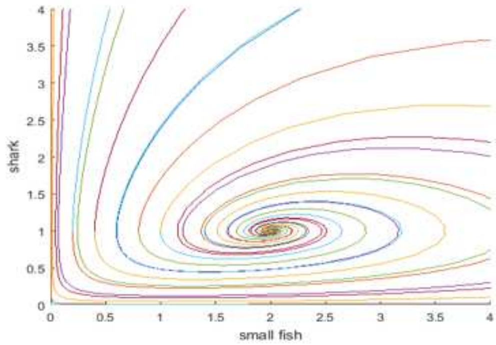

Fig. 10: Phase Plane of shark versus fish of the system(1)whrere, $a = 2, b = 1, c = 1, d = 1, r = 1, \delta = 1$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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